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NEUTRON–ANTINEUTRON OSCILLATIONS IN A TRAP REVISITED

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We reexamine the problem of $n-\bar{n}$ oscillations for ultra-cold neutrons confined within a trap. We show that for up to 10^3 collisions with the walls, the process can be described in terms of wave packets. The $\bar{n}$ component grows linearly with time with the enhancement factor depending on the reflection properties of the walls.

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1. INTRODUCTION

For quite a long time, physics beyond the Standard Model continues to be an intriguing subject. Several reactions that may serve as signatures for the new physics have been discussed. One of the most elegant proposals is to look for $n-\bar{n}$ oscillations [1] (see also [2]). There are three possible experimental settings aimed at observation of this process. The first is to establish a limit on nuclear instability because $\bar{n}$ produced inside a nucleus will blow it up. The second is to use a neutron beam from a reactor. This beam propagates a long distance to the target in which the possible $\bar{n}$ component would annihilate and thus be detected. The third option, which we discuss in the present paper, is to use ultra-cold neutrons (UCN) confined in a trap. The main question is to what extent generation of the $\bar{n}$ component is reduced by the interaction with the trap walls. This subject was addressed by several authors [3–8]. In our opinion, a thorough investigation of the problem is still lacking.

First of all, a clear formulation of the problem of $n-\bar{n}$ oscillations in a cavity has been hitherto missing. Two different approaches were used without presenting sound arguments in favor of their applicability and without tracing connections between them.

In the first approach [4, 5], $n-\bar{n}$ oscillations are considered in the basis of the discrete eigenstates of the trap potential, with the splitting between n and $\bar{n}$ levels and $\bar{n}$ annihilation taken into account. The density of the trap eigenstates, which is proportional to the macroscopic trap volume, is huge and the states cluster together extremely thickly. But these arguments do not suffice to discard the discrete-state approach because the $n-\bar{n}$ mixing parameter is much smaller than the distance between adjacent levels (see below). The true reason due to which the above treatment is of little physical relevance is as follows. The spectrum of the neutrons provided to the trap by the source is continuous and certain time is needed for rearrangement of the initial wave function into standing waves corresponding to the trap eigenstates. As is shown below, this time interval appears to be of the order of the β-decay time, and therefore the standing wave regime, being interesting by itself, can hardly be reached in the real physical situation.

The second approach [3, 6, 7] treats the neutrons and antineutrons inside a trap as freely moving particles that undergo reflections from the trap walls. Collisions with the walls result in a reduction of the $\bar{n}$ component compared to the case of the free-space evolution. This suppression is due to two factors. The first is the annihilation inside the walls. The second factor is the phase decoherence of the n and $\bar{n}$ components induced by the difference of the wall potentials acting on n and $\bar{n}$. Reflections of antineutrons from the trap walls were first considered in [3]. The purpose of that paper was to investigate the principal possibility to observe $n-\bar{n}$ oscillations in a trap, and the authors estimated

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the reflection coefficient for antineutrons without paying attention to the decoherence phenomena. Only a single collision with the trap wall was considered in [3]. A comprehensive study of $n-\bar{n}$ oscillations in a trap was presented in [6, 7]. Decoherence and multiple reflections and the influence of gravitational and magnetic fields were included. The approximate equation for the annihilation probability after N collisions obtained in [7, Eq. (3.8)] coincides with the exact formula (59) in the present paper when $N \gg 1$. As we show below, the N-independent asymptotic regime settles at $N \gtrsim 10$.

Derivation of the exact equation for the annihilation probability with an arbitrary number of collisions is not the only purpose of the present work. We already mentioned the problem of the relation between the eigenvalue and the wave-packet approaches. Within the wave-packet approach, some basic notions such as the time between successive collisions and the collision time itself can be defined in a clear and rigorous way. Another question within the wave-packet formalism is the independence of the reflection coefficient from the width of the wave packet and the applicability of the stationary formalism to calculate reflections from the trap walls. These and some other principal points are for the first time considered in detail in the present paper.

We also mention that an alternative approach to the evaluation of the reflection coefficients for n and $\bar{n}$ was outlined in [8]. It is based on the time-dependent Hamilton formalism for the interaction of n and $\bar{n}$ with the trap walls. This subject remains outside the scope of the present paper.

The paper is organized as follows. In Sec. 2, we recall the basic equations describing $n-\bar{n}$ oscillations in free space. Section 3 is devoted to the optical potential approach to the interaction of n and $\bar{n}$ with the trap walls. In Sec. 4, we analyze the two formalisms proposed to treat $n-\bar{n}$ oscillations in the cavity, namely box eigenstates and wave packets. In Sec. 5, reflection from the trap walls is considered. Section 6 contains the main result in this work, the time dependence of the $\bar{n}$ component production probability. In Sec. 7, conclusions are formulated and problems to be solved outlined.

2. OSCILLATIONS IN FREE SPACE

We start by recalling the basic equations describing $n-\bar{n}$ oscillations in free space. The phenomenological Hamiltonian is a 2×2 matrix in the basis of the two-component $n-\bar{n}$ wave function (we set $\hbar = 1$),

$$H_{jl} = \left(H_j - i\frac{\Gamma_\beta}{2}\right)\delta_{jl} + \epsilon(\sigma_x)_{jl}, \tag{1}$$

where $j,l = n,\bar{n}$, $H_j = k^2/2m-\mu_j B$, μ_j is the magnetic moment, B is the external (e.g., the Earth) magnetic field, Γ_β is the β-decay width, ϵ is the $n-\bar{n}$ mixing parameter (see below), and σ_x is the Pauli matrix. Assuming the n and $\bar{n}$ wave functions to be plane waves, we write the two-component wave function of the $n-\bar{n}$ system as

$$\hat{\Psi}(x,t) = \begin{pmatrix}\psi_n(t)\\ \psi_{\bar{n}}(t)\end{pmatrix} e^{ikx}. \tag{2}$$

Evolution of the time-dependent part of $\hat{\Psi}(x,t)$ is then described by the equation

$$i\frac{\partial}{\partial t}\begin{pmatrix}\psi_n(t)\\ \psi_{\bar{n}}(t)\end{pmatrix} = \begin{pmatrix} E_n - i\dfrac{\Gamma_\beta}{2} & \epsilon \\ \epsilon & E_{\bar{n}} - i\dfrac{\Gamma_\beta}{2}\end{pmatrix} \times \\ \times \begin{pmatrix}\psi_n(t)\\ \psi_{\bar{n}}(t).\end{pmatrix} \tag{3}$$

The difference between E_n and $E_{\bar{n}}$ due to the Earth magnetic field is

$$\omega = E_{\bar{n}} - E_n = 2|\mu_n|B \approx 6\cdot 10^{-12}\ \text{eV}. \tag{4}$$

Diagonalizing the matrix in (3), we find $\psi_n(t)$ and $\psi_{\bar{n}}(t)$ in terms of their values at $t=0$,

$$\psi_n(t) = \\ = \left(\psi_n(0)\left(\cos\nu t + \frac{i\omega}{2\nu}\sin\nu t\right) - \psi_{\bar{n}}(0)\frac{i\epsilon}{\nu}\sin\nu t\right)\times \\ \times\exp\left[-\frac{1}{2}(i\Omega+\Gamma_\beta)t\right], \tag{5}$$

$$\psi_{\bar{n}}(t) = \\ = \left(-\psi_n(0)\frac{i\epsilon}{\nu}\sin\nu t + \psi_{\bar{n}}(0)\left(\cos\nu t - \frac{i\omega}{2\nu}\sin\nu t\right)\right)\times \\ \times\exp\left[-\frac{1}{2}(i\Omega+\Gamma_\beta)t\right], \tag{6}$$

where $\Omega = E_n + E_{\bar{n}}$, $\nu = (\omega^2/4+\epsilon^2)^{1/2}$, and $\omega = E_{\bar{n}} - E_n$. In particular, if $\psi_n(0)=1$ and $\psi_{\bar{n}}(0) = 0$, we have

$$|\psi_{\bar{n}}(t)|^2 = \frac{4\epsilon^2}{\omega^2+4\epsilon^2}\exp(-\Gamma_\beta t)\times \\ \times\sin^2\left(\frac{1}{2}\sqrt{\omega^2+4\epsilon^2}\,t\right). \tag{7}$$

The use of this equation to test fundamental symmetries is discussed in [9].

Without the magnetic field, i.e., for $\omega = 0$, and for $t \ll \epsilon^{-1}$, Eq. (7) yields

$$|\psi_{\bar{n}}(t)|^2 \approx \epsilon^2 t^2 \exp(-\Gamma_\beta t). \quad (8)$$

This law (for $t \ll \Gamma_\beta^{-1}$) has been used to establish the lower limit on the oscillation time $\tau = \epsilon^{-1}$. According to the ILL-Grenoble experiment [10],

$$\tau > 0.86 \cdot 10^8 \text{ s}. \quad (9)$$

The corresponding value of the mixing parameter is $\epsilon \approx 10^{-23}$ eV. This number is used in obtaining numerical results presented below.

The Earth magnetic field leads to a strong suppression of the $n-\bar{n}$ oscillations. With the value of ω given by (4), Eq. (7) leads to

$$|\psi_{\bar{n}}(t)|^2 \approx \frac{4\epsilon^2}{\omega^2} \exp(-\Gamma_\beta t) \sin^2 t/\tau_B \approx \approx 10^{-23} \sin^2 t/\tau_B, \quad (10)$$

where $\tau_B = (|\mu_n|B)^{-1} \approx 2 \cdot 10^{-4}$ s. In what follows, we assume that the magnetic field is screened.

For $\omega = 0$ but for arbitrary initial conditions, Eqs. (5) and (6) take the form

$$\psi_n(t) = (\psi_n(0) \cos \epsilon t - i\psi_{\bar{n}}(0) \sin \epsilon t) \times \times \exp\left[-\left(iE + \frac{\Gamma_\beta}{2}\right) t\right], \quad (11)$$

$$\psi_{\bar{n}}(t) = (-i\psi_n(0) \sin \epsilon t + \psi_{\bar{n}}(0) \cos \epsilon t) \times \times \exp\left[-\left(iE + \frac{\Gamma_\beta}{2}\right) t\right], \quad (12)$$

where $E = E_n = E_{\bar{n}}$.

3. OPTICAL POTENTIAL MODEL FOR THE TRAP WALL

We remind the reader that neutrons with the energy $E < 10^{-7}$ eV are called ultra-cold. An excellent review of UCN physics was given in [11] (see also [12]).

A useful relation connecting the neutron velocity v in cm/s and E in eV is given by

$$v[\text{cm/s}] = 10^2 \cdot \left(10^9 E[\text{eV}]/5.22\right)^{1/2}. \quad (13)$$

For $E = 10^{-7}$ eV, the velocity is $v \approx 4.4 \cdot 10^2$ cm/s.

A less formal definition of UCN involves the notion of the real part of the optical potential corresponding to the trap material (see below). Neutrons with energies less than the height of this potential are called ultra-cold. The two definitions are essentially equivalent because as we see in what follows, the real part of the optical potential is of the order 10^{-7} eV for most materials.

Our main interest is in strongly absorptive interaction of the $\bar{n}$ component with the trap walls. We therefore ignore very weak absorption of UCN on the walls [11, 12]. Due to complete reflection from the trap walls, UCN can be stored for about 10^3 s (β-decay time), as was first pointed out in [13].

To be specific, we consider UCN with $E = 0.8 \cdot 10^{-7}$ eV, which corresponds to $v = 3.9 \cdot 10^2$ cm/s (see (13)), $k = 12.3$ eV and de Broglie wave length $\lambda \approx 10^{-5}$ cm. In the next section, we describe UCN in terms of wave packets, and hence the above values must be attributed to the center of the packet.

We treat the interaction of n and $\bar{n}$ with the trap walls in terms of an energy-independent optical potential. The validity of this approach to UCN has been justified in a number of papers, see, e.g., [11, 12, 14]. There is still an open question concerning the discrepancy between theoretical prediction and experimental data on the UCN absorption. Interesting by itself, this problem is outside the scope of our work because, as already mentioned, absorption of neutrons may be ignored in the $n-\bar{n}$ oscillation process. The low-energy optical potential is given by

$$U_{jA} = \frac{2\pi}{m} N a_{j\text{A}}, \quad (14)$$

where $j = n, \bar{n}$; m is the neutron mass, N is the number of nuclei in a unit volume, and $a_{j\text{A}}$ is the j–A scattering length, which is real for n and complex for $\bar{n}$. For neutrons, the scattering lengths a_{nA} are accurately known for various materials [12]. For antineutrons, the situation is different. Experimental data on low-energy $\bar{n}$–A interaction are absent. Only some indirect information may be gained from level shifts in antiprotonic atoms, and therefore the values of $a_{\bar{n}A}$ used in [3, 6, 8, 15] as an input in the $n-\bar{n}$ oscillation problem are similar but not the same. We consider the set of $a_{\bar{n}\text{A}}$ calculated in [16] within the framework of internuclear cascade model as most reliable. Even this particular model leads to several solutions, and the one that we have chosen for ^{12}C (graphite and diamond) may be called «motivated» by Ref. [16]. To estimate the dependence on the material of the walls and to compare our results with those in [3], we also performed calculations for Cu. Scattering lengths for Cu are not given in [16] and we used the solution proposed in [3]. Our calculations were thus performed with the $\bar{n}$–A scattering lengths

$$a_{\bar{n}\mathrm{C}} = (3 - i1)\ \mathrm{fm}, \quad a_{\bar{n}\mathrm{Cu}} = (5 - i0.5)\ \mathrm{fm}. \qquad (15)$$

The scattering lengths for neutrons are [12]

$$a_{n\mathrm{C}} = 6.65\ \mathrm{fm}, \quad a_{n\mathrm{Cu}} = 7.6\ \mathrm{fm}. \qquad (16)$$

The concentrations of atoms N entering (14) are as follows:

$$N_{\mathrm{C(graphite)}} = 1.13 \cdot 10^{-16}\ \mathrm{fm}^{-3},$$

$$N_{\mathrm{C(diamond)}} = 1.63 \cdot 10^{-16}\ \mathrm{fm}^{-3},$$

$$N_{\mathrm{Cu}} = 0.84 \cdot 10^{-16}\ \mathrm{fm}^{-3}.$$

In accordance with (14), the optical potentials is then given by

$$\begin{aligned} U_{n\mathrm{C(gr)}} &= 1.95 \cdot 10^{-7}\ \mathrm{eV}, \\ U_{n\mathrm{C(diam)}} &= 2.8 \cdot 10^{-7}\ \mathrm{eV}, \\ U_{n\mathrm{Cu}} &= 1.66 \cdot 10^{-7}\ \mathrm{eV}; \end{aligned} \qquad (17)$$

$$\begin{aligned} U_{\bar{n}\mathrm{C(gr)}} &= (0.9 - i0.3) \cdot 10^{-7}\ \mathrm{eV}, \\ U_{\bar{n}\mathrm{C(diam)}} &= (1.3 - i0.4) \cdot 10^{-7}\ \mathrm{eV}, \\ U_{\bar{n}\mathrm{Cu}} &= (2 - i0.2) \cdot 10^{-7}\ \mathrm{eV}. \end{aligned} \qquad (18)$$

In this paper, we consider particles (n and $\bar{n}$) with energies below the potential barrier formed by the real part of the potential. For $\bar{n}$ and ^{12}C, the limiting velocity is $v = 4.15 \cdot 10^2$ cm/s.

4. WAVE PACKET VERSUS STANDING WAVES

It is convenient to use the short notation

$$U_j = V_j - iW_j\delta_{j\bar{n}} \qquad (19)$$

for optical potentials (17) and (18), where $j = n, \bar{n}$ and the wall material is not indicated explicitly. We consider the following model for the trap in which n–$\bar{n}$ oscillations may possibly be observed. We imagine two walls of type (19) separated by a distance $L \sim 10^2$ cm, i.e., the one-dimensional potential well of the form

$$U_j(x) = \{\theta(-x - L) + \theta(x)\}\{V_j - iW_j\delta_{j\bar{n}}\}, \qquad (20)$$

with $\theta(x)$ being the step function. Our goal is to follow the time evolution of the $\bar{n}$ component in such a trap assuming that the initial state is a pure n one.

The first question to be answered is how to describe the wave function of the system. Two different approaches seem to be feasible and both were discussed in the literature [4, 6, 8]. The first is to consider oscillations occuring in the wave packet and to investigate to what extent reflections from the walls distort the picture compared to the free-space regime. The second approach is to consider the eigenvalue problem in potential well (20), to find energy levels for n and $\bar{n}$, and to consider oscillations in this basis. Because of different interactions with the walls, the levels of n and $\bar{n}$ are splitted and the $\bar{n}$ levels acquire annihilation widths.

At first glance, this approach might seem inadequate because in a trap with $L \sim 10^2$ cm, the density of states is very high, the characteristic quantum number corresponding to the UCN energy is very large, and the splitting δE between adjacent n-levels (or between the levels of the n and $\bar{n}$ spectra) is extremely small. The values of all these quantities are given below, and it follows that $\delta E < 10^{-14}$ eV. However, this approach cannot be discarded without further analysis because the n–$\bar{n}$ mixing parameter $\epsilon \approx 10^{-23}$ eV is much smaller than δE.

To understand the relation between the two approaches, we note that the initial conditions correspond to a beam of UCN provided by a source. The momentum spectrum of UCN depends on the specific experimental conditions. In order to stay on general grounds and at the same time to simplify the problem, we assume that the UCN beam entering the trap has the form of a Gaussian wave packet. We suppose that at $t = 0$, the center of the wave packet is at $x = x_0$, and hence

$$\psi_k(x, t = 0) = (\pi a^2)^{-1/4} \times \\ \times \exp\left(-\frac{(x - x_0)^2}{2a^2} + ikx\right), \qquad (21)$$

where a is the width of the wave packet in coordinate space. The normalization of wave function (21) corresponds to one particle in the entire one-dimensional space,

$$\int_{-\infty}^{+\infty} dx |\psi_k(x, t = 0)|^2 = 1. \qquad (22)$$

For $E = 0.8 \cdot 10^{-7}$ eV and the beam resolution $\Delta E/E = 10^{-3}$, we have

$$k = 12.3\ \mathrm{eV}, \quad a = 3.2 \cdot 10^{-3}\ \mathrm{cm}. \qquad (23)$$

The width of wave packet (21) increases with time according to

$$a' = a\left[1+\left(\frac{t}{ma^2}\right)^2\right]^{1/2} \approx \frac{t}{ma} \qquad (24)$$

and becomes comparable with the trap size L for $t \sim 10^3$ s. For the wave hitting the wall and the reflected wave to be clearly resolved, the condititon $a'/v \ll \tau_L$, or $a' \ll L$ must be satisfied, where $\tau_L \sim 1$ s is the time between two consecutive collisions with the trap walls. Reflection of the wave packet from the walls is considered in detail in the next section. Here, we show that $t \sim 10^3$ s is the characteristic time needed for the rearrangement of the initial wave packet into stationary states of the trapping box.

We consider the eigenvalue problem for potential well (20). The parameters of potential (20) for neutrons are $V_n \approx 2 \cdot 10^{-7}$ eV and $L \approx 10^2$ cm. The number of levels is

$$M \approx \frac{L\sqrt{2mV}}{\pi} \approx \frac{10^8}{\pi}. \qquad (25)$$

According to (23), the center of wave packet (21) has the momentum $k = 12.3$ eV, which corresponds to a state with the number of nodes $j \approx 2 \cdot 10^7$ and $k_j L \approx 6 \cdot 10^7 \gg 1$. Positions of such highly excited levels in a finite-depth potential are indistinguishable from the spectrum in a potential box with infinite walls. Therefore,

$$\varphi_j(x) \approx \sqrt{\frac{2}{L}}\sin\omega_j x, \quad \omega_j = \frac{\pi j}{L}. \qquad (26)$$

Wave functions (26) describe semiclassical states with $j \gg 1$ in a potential well with sharp edges. The «frequency» ω_j is very high compared to the width of the wave packet in momentum space,

$$\omega_j \approx 6 \cdot 10^5 \text{ cm}^{-1} \gg \nu = \frac{1}{\sqrt{2}a} \approx 2 \cdot 10^2 \text{ cm}^{-1}.$$

This implies that the wave packet spans over a large number of levels. To determine this number, we note that the distance between adjacent levels around the center of the wave packet is

$$\delta E = E_{j+1} - E_j \sim 10^{-14} \text{ eV}.$$

The highly excited levels within the energy band

$$\Delta E = 10^{-3} E \sim 10^{-10} \text{ eV}$$

corresponding to wave packet (21) are to a high accuracy equidistant, as they should be in the semiclassical regime. The number of states within ΔE is

$$\Delta j = \Delta E/\delta E \sim 10^4$$

and their density in momentum space is

$$\rho(\omega) = a\Delta j \approx L/\pi \sim 10^6 \text{ eV}^{-1}. \qquad (27)$$

We can now answer the question formulated at the beginning of this section, namely whether the $n-\bar{n}$ oscillations in the trap should be described in terms of the wave packet or in terms of the stationary eigenfunctions. At $t = 0$, the wave function has the form of the wave packet (21) provided by the UCN source. Due to collisions with the trap walls, transitions from the initial state (21) into discrete (or quasi-discrete for $\bar{n}$) eigenstates (26) occur.

The time evolution of the initial wave function (21) proceeds according to

$$\psi(x,t) = \int dx' G(x,t;x',0)\psi_k(x',0), \qquad (28)$$

where $G(x,t;x',0)$ is the time-dependent Green's function for potential well (20). Using the spectral representation for G, we can write

$$\psi(x,t) = \sum_j e^{-iE_j t}\varphi_j(x)\int dx'\varphi_j^*(x')\psi_k(x',0). \qquad (29)$$

In the semiclassical approximation, the distance between the adjacent levels is $\delta E = \pi/\tau_L$, and therefore one may think that at $t = \tau_L$, i.e., already at the first collision, the neighboring terms in (29) would cancel each other. But this is not the case. Indeed,

$$\varphi_{j+1}(x)\exp(-iE_{j+1}t) + \varphi_j(x)\exp(-iE_j t) =$$
$$= \frac{\exp(-iE_j t)}{i\sqrt{2L}}\left[\exp(i\omega_j x)\left(1+\exp\left(i\frac{\pi}{L}(x-vt)\right)\right) - \right.$$
$$\left. - \exp(-i\omega_j x)\left(1+\exp\left(-i\frac{\pi}{L}(x+vt)\right)\right)\right].$$

Therefore, there is a constructive interference at $x = \pm vt$ either in the first or in the second term respectively. This is true with the whole sum of terms in (29) taken into account, and hence we can pass from summation to integration in (29). The overlap of the wave functions entering (29) can be easily evaluated provided the center of the wave packet x_0 is not within the bandwidth distance a' from the trap walls. The overlap is given by the integral

$$\int dx' \varphi_j^*(x')\psi_k(x',0) \approx \frac{i}{(2\sqrt{\pi}La)^{1/2}} \times$$
$$\times \int\limits_{-L}^{0} dx' \exp\left(-\frac{(x'-x_0)^2}{2a^2} + i(k-\omega_j)\,x'\right) =$$
$$= \frac{i}{(2\sqrt{\pi}La)^{1/2}} \times$$
$$\times \int\limits_{0}^{L} dx'' \exp\left(-\frac{(x''+x_0)^2}{2a^2} - i(k-\omega_j)x''\right). \quad (30)$$

At this step, we have omitted the exponential with the high frequency $(k+\omega_j)$. We next take $(x''+x_0)/(\sqrt{2}a)$ as a new variable and assume that $|x_0| \gg a$, $L-|x_0| \gg a$ (we recall that x_0 is negative because $-L < x < 0$). The result is that

$$\int dx' \varphi_j^*(x')\psi_k(x',0) \approx$$
$$\approx i\left(\frac{\sqrt{\pi}a}{L}\right)^{1/2} \exp\left(-\frac{a^2}{2}(k-\omega_j)^2 + i(k-\omega_j)x_0\right). \quad (31)$$

Corrections to (31) are of the order of a/L. We now consider frequency summation in (29). This summation can be replaced by integration over ω because the density of semiclassical states $\rho(\omega)$ is very high. We thus arrive at

$$\psi(x,t) = \frac{1}{\left[\sqrt{\pi}a\left(1+i\dfrac{t}{ma^2}\right)\right]^{1/2}} \times$$
$$\times \left(\exp\left[-\frac{\alpha(x,t)}{2a^2\left(1+\dfrac{t^2}{m^2a^4}\right)}\right] + \right.$$
$$\left. + \exp\left[-\frac{\alpha(-x,t)}{2a^2\left(1+\dfrac{t^2}{m^2a^4}\right)}\right]\right), \quad (32)$$

$$\alpha(x,t) = (x-x_0-v_0t)^2 - it\frac{(x-x_0)^2}{ma^2} -$$
$$-2ik_0a^2(x-x_0) + i\frac{k_0^2a^2}{m}t + 2ik_0x_0\left(a^2+\frac{t^2}{m^2a^2}\right). \quad (33)$$

The second term in Eq. (32) describes the reflected wave packet (see the next section). According to (21), (28), and (32), all that happens to the wave packet in the trap is broadening and reflections. This is true during some initial period of its life history at least. How long does this period last? The answer to this question may be obtained by estimating the accuracy of performing frequency integration instead of summation over discrete states in (29).

To estimate the time scale for the rearrangement of initial wave packet (21) into trap standing waves (26), it is convenient to introduce the difference

$$\delta\psi(x,t) = \psi_{sum}(x,t) - \psi_{int}(x,t)$$

between the «exact» wave function (29) and the approximate integral representation (32). Whenever

$$\delta w(t) = \int dx\left(|\psi_{sum}|^2 - |\psi_{int}|^2\right) =$$
$$= 2\int dx\, \Re(\psi_{int}\delta\psi) \ll 1, \quad (34)$$

we can consider oscillations as proceeding in the wave packet basis. With

$$f(\omega) = \sqrt{\frac{\sqrt{\pi}a}{2L^2}} \times$$
$$\times \exp\left(-\frac{a^2}{2}(k_0-\omega)^2 - i\frac{\omega^2}{2m}t + i\omega(x-x_0) + ikx_0\right), \quad (35)$$

we have the estimate

$$\delta\psi(x,t) = \sum_n f(\omega_n) - \int d\omega \rho(\omega) f(\omega) =$$
$$= -\sum_n \int\limits_{\omega_n}^{\omega_{n+1}} d\omega\rho(\omega)(f(\omega)-f(\omega_n)) \approx$$
$$\approx -\sum_n \int\limits_{\omega_n}^{\omega_{n+1}} d\omega\rho(\omega) f'(\omega_n)(\omega-\omega_n) =$$
$$= -\frac{1}{2}\sum_n f'(\omega_n)(\omega_{n+1}-\omega_n). \quad (36)$$

From (35), we obtain that

$$f'(\omega) = g(\omega)f(\omega),$$
$$g(\omega) = i(x-x_0-vt) - (k_0-\omega)a^2. \quad (37)$$

Because $f(\omega)$ is a narrow Gaussian peak, we can substitute $g(\omega)$ by $g(k_0)$, and then (36) results in

$$\delta\psi(x,t) \approx \frac{\pi}{2L}(x-x_0-v_0t)\psi_{int}(x,t). \quad (38)$$

From (34) and (38), we have

$$\delta w \approx \frac{\pi}{2L}\int\limits_{-\infty}^{+\infty} dx|x-x_0-v_0t||\psi_{int}(x,t)|^2 \propto$$
$$\propto \frac{a'}{L} \sim \frac{t}{maL} \sim \frac{t}{10^3\ \text{s}}, \quad (39)$$

where a' is given by (23).

Roughly speaking, the time $t \sim 10^3$ s needed for the neutron wave function to rearrange into the trap eigenstate is comparable to the neutron life-time, and the neutron would rather «die» than adjust to the new boundary conditions. The wave packet formalism is therefore used in what follows. Some additional subtleties arising from the quantization of levels in the trapping box are discussed in Sec. 7.

5. REFLECTION FROM THE TRAP WALLS

We return to one-dimensional trap (20). Let the particle moving from $x = -\infty$ enter the trap at $t = 0$ through the window at $x = -L$. At $t = \tau_L$, it reaches the wall at $x = 0$, the n component is reflected from the wall and the $\bar{n}$ component is partly reflected and partly absorbed. The wave packet describing the interaction with the wall has the form

$$\psi(x,t) = \pi^{-3/4}\sqrt{\frac{a}{2}}\int\limits_{-\infty}^{+\infty} dk\psi_j(k,x) \times \\ \times \exp\left(-\frac{a^2}{2}(k-k_0)^2 + iL(k-k_0) - i\frac{t}{2m}k^2\right), \quad (40)$$

where $j = n, \bar{n}$ and

$$\psi_j(k,x) = e^{ikx} + R(k)e^{-ikx} = \\ = e^{ikx} + \rho_j(k)e^{\phi_j(k)}e^{-ikx}. \quad (41)$$

For the n component, $\rho_n(k) = 1$ because we neglect very weak absorption of neutrons at the surface. The integral (40) with the first term in (41) is trivial. To integrate the second term in (41), we note that due to the Gaussian form factor with $ak_0 \sim 10^3 \gg 1$, the dominant contribution to integral (40) comes from a narrow interval of k around k_0. Expanding $R_j(k)$ at $k - k_0$ and keeping the leading term, we obtain

$$R_j(k) \approx \rho_j(k_0)\exp(i\phi_j(k_0)) \times \\ \times \left[1 + i\phi_j'(k_0)(k-k_0) + \delta_{j\bar{n}}\frac{\rho_j'(k_0)}{\rho_j(k_0)}(k-k_0)\right] \approx \\ \approx \rho_j(k_0)\exp\left(i\phi_j(k_0) + i\phi_j'(k_0)(k-k_0)\right). \quad (42)$$

The validity of the last step for $\bar{n}$ becomes clear from the explicit expressions for $\rho_{\bar{n}}(k)$ and $\phi_{\bar{n}}(k)$ presented below.

Integration in (40) can now be easily performed, with the result [17]

$$\psi_j(x,t) = \frac{1}{\left[\sqrt{\pi}a\left(1 + i\dfrac{t}{ma^2}\right)\right]^{1/2}} \times \\ \times \left(\exp\left[-\frac{\alpha_{inc}(x,t)}{2a^2\left(1 + \dfrac{t^2}{m^2a^4}\right)}\right] + \right. \\ \left. + R_j(k_0)\exp\left[-\frac{\alpha_{refl}(x,t)}{2a^2\left(1 + \dfrac{t^2}{m^2a^4}\right)}\right]\right), \quad (43)$$

$$\alpha_{inc}(x,t) = (x + L - v_0t)^2 - it\frac{(x+L)^2}{ma^2} - \\ -2ik_0a^2(x+L) + i\frac{k_0^2a^2}{m}t + 2ik_0L\left(a^2 + \frac{t^2}{m^2a^2}\right), \quad (44)$$

$$\alpha_{refl}(x,t) = \alpha_{inc}(-x+\phi',t) + \\ + 2ik_0\phi'\left(a^2 + \frac{t^2}{m^2a^2}\right). \quad (45)$$

From (43)–(45), we see that the essence of $R(k)$ in the wave packet formalism is the same as in the time-independent approach. Therefore, imposing standard boundary conditions at $x = 0$, we obtain the reflection coefficients

$$R_j(k) = \rho_j(k)\exp(i\phi_j(k)) = \frac{k - i\kappa_j}{k + i\kappa_j}, \quad (46)$$

$$\kappa_n = [2m(V_n - E)]^{1/2}, \\ \kappa_{\bar{n}} = [2m(V_{\bar{n}} - iW_{\bar{n}} - E)]^{1/2} = \kappa_{\bar{n}}' - i\kappa_{\bar{n}}'', \quad (47)$$

$$\operatorname{tg}\phi_n = \frac{-2k\kappa_n}{k^2 - \kappa_n^2}, \quad \operatorname{tg}\phi_{\bar{n}} = \frac{-2k\kappa_{\bar{n}}'}{k^2 - (\kappa_{\bar{n}}')^2 - (\kappa_{\bar{n}}'')^2}, \quad (48)$$

$$\rho_n = 1, \quad \rho_{\bar{n}}^2 = 1 - \frac{4k\kappa_{\bar{n}}''}{(k + \kappa_{\bar{n}}'')^2 + (\kappa_{\bar{n}}')^2}. \quad (49)$$

For ^{12}C (graphite), in particular,

$$\rho = 0.56, \quad \theta \equiv \phi_{\bar{n}} - \phi_n = 0.72. \quad (50)$$

The first term in the right-hand side of (45) can be written as $[-x + L - v_0(t - \phi'/v_0)]^2$. Hence the collision time or time delay is [17, 18]

$$\tau_{j,coll} = \frac{\phi_j'(k_0)}{v_0} = \operatorname{Re}\frac{2m}{k\kappa_j}. \quad (51)$$

For neutrons, i.e., for real κ_n, Eq. (51) gives the well-known result

$$\tau_{n,coll} = [E(V_n - E)]^{-1/2}.$$

This result is in line with the naive estimate $\tau_{n,coll} \sim l/v_0 \sim 10^{-8}$ s [8], where $l \lesssim \lambda$ is the penetration depth.

For ^{12}C (graphite), Eq. (51) yields

$$\tau_{n,coll} = 0.7 \cdot 10^{-8}\ \mathrm{s}, \quad \tau_{\bar{n},coll} = 1.1 \cdot 10^{-8}\ \mathrm{s}. \tag{52}$$

Equations (43)–(45) supplemented by the above inequality allow following the time evolution of the beam inside the trap. We imagine an observer placed at the bandwidth distance from the wall, i.e., at $x = -a$. According to (43)–(45), such an observer concludes that the incident wave (the first term in (43)) dominates at times $t \le \tau_L - \tau_a$, while the reflected wave prevails at $t \ge \tau_L + \tau_a$. With this splitting of the time interval around $N\tau_L$, $N = 1, 2, \ldots$, we use the notation $(N\tau_L-)$ and $(N\tau_L+)$ for the moments before and after the N_{th} collision. Thus, we can calculate the $\bar{n}$ production rate because we have rigorous definitions of the collision time and the time interval between the two subsequent collisions.

6. ANNIHILATION RATE IN A TRAP

We can now inquire into the problem of time-dependence of the $\bar{n}$ production probability. In free space, it is given by $|\psi_{\bar{n}}(t)|^2 = \epsilon^2 t^2$ (see (2)), while in a trap with the complete annihilation or total loss of coherence at each collision, it has a linear time dependence $|\psi_{\bar{n}}(t)|^2 = \epsilon^2 \tau_L t$ [8].

To avoid cumbersome equations and because we consider the time interval $t \ll \Gamma_\beta^{-1}$, we omit $\exp(-\Gamma_\beta t)$ factors. Production of $\bar{n}$ during the collision can also be neglected [8]. The difference in collision times (52) for n and $\bar{n}$ may also be ignored. In the previous section, we have seen that the interaction of the wave packet with the wall is described in terms of reflection coefficients (46) [1)].

We assume that at $t = 0$, a pure-n beam enters the trap at $x = -L$. After crossing the trap, i.e., at $t = (\tau_L-)$, the time-dependent parts of the wave functions are given by (12)[2)],

$$\begin{gathered} \psi_n(\tau_L-) = \cos(\epsilon\tau_L)\exp(-iE\tau_L), \\ \psi_{\bar{n}}(\tau_L-) = \sin(\epsilon\tau_L)\exp[-i(E\tau_L + \pi/2)]. \end{gathered} \tag{53}$$

After the first reflection at $t = (\tau_L+)$, we have

$$\begin{gathered} \psi_n(\tau_L+) = \cos(\epsilon\tau_L)\exp[-i(E\tau_L - \phi_n)], \\ \psi_{\bar{n}}(\tau_L+) = \rho_{\bar{n}}\sin(\epsilon\tau_L)\exp[-i(E\tau_L - \phi_{\bar{n}} + \pi/2)]. \end{gathered} \tag{54}$$

Evolution from $t = (\tau_L+)$ to $t = (2\tau_L-)$ again proceeds in accordance with (12),

$$\begin{aligned} \psi_{\bar{n}} = \frac{1}{2}\sin(2\epsilon\tau_L)\left(1 + \rho e^{i\theta}\right) \times \\ \times \exp[-i(2E\tau_L - \phi_{\bar{n}} + \pi/2)] \approx \\ \approx \epsilon\tau_L\left(1 + \rho e^{i\theta}\right)\exp[-i(2E\tau_L - \phi_{\bar{n}} + \pi/2)], \end{aligned} \tag{55}$$

where $\theta = \phi_{\bar{n}} - \phi_n$ is the decoherence phase and $\rho \equiv \rho_{\bar{n}}$. The answer for $\psi(N\tau_L-)$ now seems evident:

$$\begin{aligned} \psi_{\bar{n}}(N\tau_L-) = \epsilon\tau_L \frac{1 - \rho^N e^{iN\theta}}{1 - \rho e^{i\theta}} \times \\ \times \exp[-i(NE\tau_L - \phi_n + \pi/2)]. \end{aligned} \tag{56}$$

This conjecture is easy to verify by induction. For $t = (2\tau_L-)$, the result was derived explicitly in (55). Evolving (56) through one reflection at $t = N\tau_L$ and free propagation from $t = (N\tau_L+)$ to $t = ((N+1)\tau_L-)$, we arrive at (56) with $(N+1)$ instead of N. This completes the proof.

Therefore, the admixture of $\bar{n}$ before the N^{th} collision, i.e., at $t = N\tau_L-$ is

$$|\psi_{\bar{n}}(N\tau_L-)|^2 = \epsilon^2\tau_L^2 \frac{1 + \rho^{2N} - 2\rho^N\cos N\theta}{1 + \rho^2 - 2\rho\cos\theta}. \tag{57}$$

The annihilation probability at the jth collision is

$$P_a(j) = (1 - \rho^2)|\psi_{\bar{n}}(j\tau_L-)|^2. \tag{58}$$

The total annihilation probability after N collisions is therefore given by

$$\begin{aligned} P_a(N) = (1 - \rho^2)\sum_{k=1}^{N}|\psi_{\bar{n}}(k\tau_L)|^2 = \\ = \frac{\epsilon^2\tau_L^2(1 - \rho^2)}{1 + \rho^2 - 2\rho\cos\theta}\left(N + \frac{\rho^2(1 - \rho^{2N})}{1 - \rho^2} - \right. \\ \left. -2\rho\frac{\cos\theta - \rho - \rho^N[\cos(N+1)\theta + \rho\cos N\theta]}{1 + \rho^2 - 2\rho\cos\theta}\right). \end{aligned} \tag{59}$$

1) An alternative description using time-evolution operators was proposed in [8].

2) We state this although the Gaussian form factor in (43) also depends on time, the corresponding terms in the time-dependent Schrödinger equation are of the order of $1/ak_0$ compared to the derivative of the exponent $\exp(-iEt)$; we also note that the form factors are the same for n and $\bar{n}$ up to a constant multiplier.

After several collisions, the terms proportional to ρ^N, ρ^{2N}, and ρ^{N+1} may be dropped because $\rho \sim 0.5$ (see (50)). Then (59) takes the form

$$P_a(N) \approx \frac{\epsilon^2\tau_L^2}{1+\rho^2-2\rho\cos\theta} \times \times \left(N(1-\rho^2) + 1 - \frac{(1-\rho^2)^2}{1+\rho^2-2\rho\cos\theta}\right). \quad (60)$$

Three different regimes may be inferred from (60). For a very strong annihilation, i.e., $\rho \ll 1$,

$$P_a(N) = \epsilon^2\tau_L^2 N = \epsilon^2\tau_L t. \quad (61)$$

For the complete decoherence at each collision, i.e., for $\theta = \pi$,

$$P_a(N) = \epsilon^2\tau_L^2 \left(N\frac{1-\rho}{1+\rho} + \frac{\rho(2-\rho)}{(1+\rho)^2}\right) \approx \frac{1-\rho}{1+\rho}\epsilon^2\tau_L t. \quad (62)$$

For the (unrealistic) situation where $\theta = 0$,

$$P_a(N) = \epsilon^2\tau_L^2 \left(N\frac{1+\rho}{1-\rho} - \frac{\rho(2+\rho)}{(1-\rho)^2}\right) \approx \frac{1+\rho}{1-\rho}\epsilon^2\tau_L t. \quad (63)$$

For the values of ρ and θ corresponding to optical potentials (17) and (18), the quantity

$$Q_a(N) = (\epsilon^2\tau_L^2 N)^{-1} = (\epsilon^2\tau_L t)^{-1} P_a(N)$$

calculated in accordance with the exact equation (59) is displayed in Fig. 1. This figure shows that the linear time dependence settles after about 10 collisions with the trap walls. The asymptotic value of $Q_a(N)$, which may be called the enhancement factor, is 1.5–2 depending on the wall material.

Proposals have been discussed in the literature [6, 19] to compensate the decoherence phase θ by applying the external magnetic field. Assuming the ideal situation that the regime $\theta = 0$ may be achieved in such a way and also assuming that the reflection coefficient ρ can be varied in the whole range by varying the trap material, we plot the quantity $N_{eff}(\rho)$ defined as

$$P_a(N) = \epsilon^2\tau_L^2 N_{eff}(\rho) \quad (64)$$

in Fig. 2. Thus defined, $N_{eff}(\rho)$ obviously depends also on the number of collisions N; the results for $N = 10$ and $N = 50$ are presented in Fig. 2. This figure shows what can be expected from the trap experiments in the most favorable, although hardly realistic scenario.

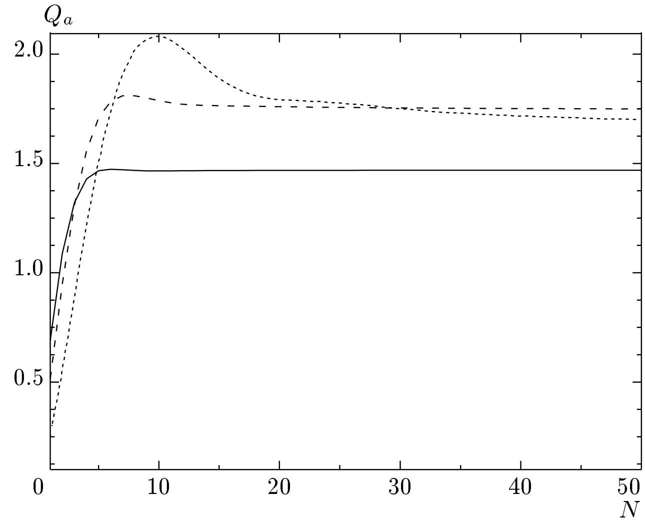

Fig. 1. Plot of the $Q_a(N) = (\epsilon^2\tau_L t)^{-1}P_a(N)$ dependence versus N. The solid line corresponds to ^{12}C (graphite), the dashed one to ^{12}C (diamond), and the dotted one to Cu

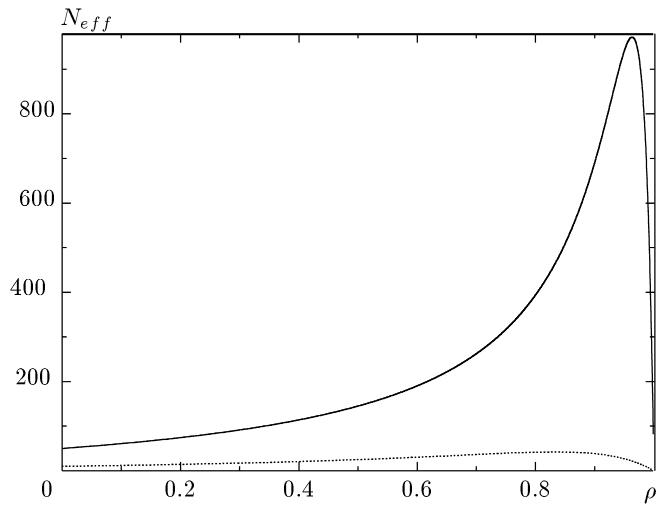

Fig. 2. Plot of the N_{eff} dependence versus ρ at $\theta = 0$. The solid line is for the number of collisions $N = 50$, the dashed line corresponds to $N = 10$

7. CONCLUDING REMARKS

We have reexamined the problem of $n-\bar{n}$ oscillations for UCN in a trap. Our aim was to present a clear formulation of the problem, to calculate the amplitude of the $\bar{n}$ component for an arbitrary observation time and for any given reflection properties of the trap walls. We have shown that for the physically relevant observation time (i.e., for the time interval less than the β-decay time), the process of $n-\bar{n}$ oscillations is described in terms of wave packets, while the standing-wave regime may settle only at later times. By calculating the difference between the n and $\bar{n}$ collision times, the new light has been shed on the decoherence phenomena. For the first time, an exact equation has been derived for the annihilation probability for an arbitrary number of collisions with the trap walls. In line with the conclusions of the previous authors on the subject, this probability grows linearly with time. We have calculated the enhancement factor entering this linear time dependence and found this factor to be 1.5–2 depending on the reflection properties of the wall material.

Despite the extensive investigations reviewed in this article and the results of the present paper, the list of problems for further work is large. The central and most difficult task is to obtain reliable parameters of the optical potential for antineutrons. The beam of $\bar{n}$ with the energy in the range of 10^{-7} eV will be hardly accessible in the near future. Therefore, work has to be continued along the two lines mentioned above: to deduce the parameters of the optical potential from the level shifts in antiprotonic atoms and to construct reliable optical models that can be confronted with the available experimental data on $\bar{n}$–nuclear interaction at higher energies. In a forthcoming publication, we plan to present numerical calculation of the time evolution of a wave packet into standing waves and to discuss some features of $n-\bar{n}$ oscillations in the eigenfunction basis, which were not discussed in Ref. [4]. Another task is to perform calculation for the specific geometry of the trap and a realistic spectrum of the neutron beam. This requires an input corresponding to a specific experimental setting.

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