ЖЭТФ, 2004, том 126, вып. 1 (7), стр. 14–25

MODULATIONAL INSTABILITIES IN NEUTRINO–ANTINEUTRINO INTERACTIONS

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Submitted 13 January 2004

Using a semiclassical approach, we analyze the collective behavior of neutrinos and antineutrinos in a dense background. Applying the Wigner transform technique, we show that the interaction can be modeled by a coupled system of nonlinear Vlasov-like equations. From these equations, we derive a dispersion relation for neutrino–antineutrino interactions on a general background. The dispersion relation admits a novel modulational instability. Moreover, we investigate the modifications of the instability due to thermal effects. The results are examined, together with a numerical example, and we discuss the induced density inhomogeneities using parameters relevant to the early Universe.

PACS: 13.15.+g, 14.60.Lm, 97.10.Cv, 97.60.Bw

1. INTRODUCTION

Neutrinos have fascinated people ever since they were first introduced by Pauli in 1931. Since then, neutrinos have gone from hypothetical to an extremely promising tool for analyzing astrophysical events, and neutrino cosmology is one of the hottest topics in modern time due to the discovery that neutrinos may be massive [1]. Because of its weak interaction with other particles, neutrinos can travel great distances without being affected appreciably by material obstacles. They can therefore give us detailed information about events taking place deep within, e.g., supernovæ. Furthermore, because the neutrinos decoupled from matter at a redshift z of the order 10^{10}, as compared to $z \approx 10^3$ for photons, it is possible that neutrinos could give us a detailed understanding of the early Universe, if such a signal could be detected [2]. Massive neutrinos have also been a possible candidate for hot dark matter necessary for explaining certain cosmological observations, such as rotation curves of spiral galaxies [3]. Therefore, massive neutrinos could have a profound influence on the evolution of our Universe. Unfortunately, due to the Tremaine–Gunn bound [4], the necessary mass of the missing particles (if these are fermions) for explaining the formation of dwarf galaxies seems to make neutrinos of any species unlikely single candidates for dark matter. As a remedy to this problem, interacting hot dark matter has been suggested [5, 6], because the interaction prevents free-streaming smoothing of small-scale neutrino inhomogeneities. Thus, dark matter in astrophysics not only is a mystery but also plays an essential role in determining the dynamics of the Universe, its large-scale structures, the galaxies and superclusters. However, so far, the suggested «sticky» neu-

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trino models have not been successful in dealing with the dwarf galaxy problem [5].

A first successful indication that neutrinos have a nonzero mass came in 1998 through laboratory experiments of atmospheric neutrinos and their oscillations [7]. Although the allowed neutrino masses encompass a wide range[1)], it is currently believed that neutrinos have masses below 2 eV. This conclusion is further supported by independent cosmological observations (see, e.g., [9]). Thus, the masses of neutrinos are indeed very small, and the classical analysis by Tremaine and Gunn would therefore indicate that neutrinos can in no way be considered a sole candidate for dark matter. This conclusion is reanalyzed in this paper within the electro-weak framework, where neutrino–neutrino interactions occur as a natural consequence of the theory.

We thus consider the nonlinear interaction between neutrinos and antineutrinos in the lepton plasma of the early Universe, adopting a semiclassical model. Neutrinos and antineutrinos interact with dense plasmas through the charged and neutral weak currents arising from the Fermi weak nuclear interaction forces. Charged weak currents involve the exchange of the charged vector bosons associated with the processes involving interactions between leptons and neutrinos of the same flavor, while neutrino weak currents involve the exchange of the neutral vector bosons associated with processes involving neutrinos of all types interacting with arbitrary charged and neutral particles. Asymmetric flows of neutrinos and antineutrinos in the early Universe plasma may be created by the ponderomotive force of nonuniform intense photon beams or by shock waves. Here, using an effective field theory approach, a system of coupled Wigner–Moyal equations for nonlinearly interacting neutrinos and antineutrinos is derived, and it is shown that these equations admit a modulational instability. We then discuss the relevance of our results in the context of the dark matter problem, and it is moreover suggested that the nonlinearly excited fluctuations could be used as a starting point for obtaining a better understanding of the process of galaxy formation. It turns out that the short-time evolution of the primordial neutrino plasma medium in the temperature range $1\,\mathrm{MeV} < T < 10\,\mathrm{MeV}$ is governed by collisionless collective effects involving relativistic neutrinos and antineutrinos.

2. DISPERSION RELATION AND THE MOTION OF NEUTRINO BUNCHES

As a primer, we study the implication of the known dispersion of neutrinos on a thermal neutrino/antineutrino background, using the eikonal representation and the WKBJ approximation.

We suppose that a single neutrino (or antineutrino) moves in a fermionic sea composed of neutrino–antineutrino mixture. The energy E of the neutrino (antineutrino) is then given by (see, e.g., [10, 11])

$$E = \sqrt{p^2c^2 + m^2c^4} + V_\pm(\mathbf{r}, t), \tag{1}$$

where $\mathbf{p}$ is the neutrino (antineutrino) momentum, c the speed of light in vacuum, and m is the neutrino mass. The effective potential for a neutrino moving on a background of its own flavor and in thermal equilibrium is given by [10] (see also [12–15])[2)]

$$V_\pm(\mathbf{r}, t) = \pm 2\sqrt{2} G_F (n - \bar{n}), \tag{2a}$$

while the potential for a neutrino moving on a background of a different flavor is

$$V_\pm(\mathbf{r}, t) = \pm \sqrt{2} G_F (n - \bar{n}), \tag{2b}$$

where

$$\frac{G_F}{(\hbar c)^3} \approx 1.2 \cdot 10^{-5}\,\mathrm{GeV}^{-2},$$

G_F is the Fermi constant, n ($\bar{n}$) is the density of the background neutrinos (antineutrinos), and $+$ ($-$) represents the propagating neutrino (antineutrino). Expressions (2) are valid in the rest frame of the background. As seen from (1) and (2), while neutrinos moving in a background of neutrinos and antineutrinos change their energy by an amount $\sim G_F(n-\bar{n})$, the antineutrinos change their energy by $\sim -G_F(n-\bar{n})$ [16]. The extra factor of 2 in (2a) compared to (2b) comes from exchange effects between identical particles [13].

Relation (1) can be interpreted as a dispersion relation for relativistic and nonrelativistic neutrinos, with the identifications $E = \hbar\omega$ and $\mathbf{p} = \hbar\mathbf{k}$, i.e.,

$$\omega = c\sqrt{k^2 + \frac{m^2c^2}{\hbar^2}} + \frac{V_\pm}{\hbar}, \tag{3}$$

where $\hbar$ is the Planck constant divided by 2π. From Eq. (3), using the eikonal representation

$$E \to \hbar\omega_0 - \frac{i\hbar\partial}{\partial t}, \quad \mathbf{p} \to \hbar\mathbf{k}_0 + i\hbar\nabla$$

[1)] Some estimates even support the notion that neutrinos may contribute up to 20 % of the matter density of the Universe [8].

[2)] For a more detailed description of the potential, see the next section.

and the WKBJ approximation [17, 18]

$$\left|\frac{\partial\Psi}{\partial t}\right| \ll \omega_0|\Psi|, \quad |\nabla\Psi| \ll |\mathbf{k}_0||\Psi|,$$

we obtain a Schrödinger equation for slowly varying neutrino (antineutrino) wave packets $\Psi(\mathbf{r}, t)$ modulated by long-scale density fluctuations (i.e., neutrino bunches)[3)]

$$i\left(\frac{\partial}{\partial t} + \mathbf{v}_g\cdot\nabla\right)\Psi + \\ +\frac{\hbar c^2}{2E_0}\left[\nabla^2 - \left(1 - \frac{m^2c^4}{E_0^2}\right)(\mathbf{n}_0\cdot\nabla)^2\right]\Psi - \frac{V_\pm}{\hbar}\Psi = 0, \quad (4)$$

where

$$\mathbf{v}_g = c\mathbf{k}_0(k_0^2 + m^2c^2/\hbar^2)^{-1/2}$$

is the group velocity[4)] of relativistic neutrinos and antineutrinos, which have similar energy spectra, E_0 is the neutrino energy in the absence of interactions, $\mathbf{n}_0 = \mathbf{k}_0/|k_0|$, and $\mathbf{k}_0$ is the vacuum wavevector. We now suppose that the neutrino bunches themselves are nearly in thermal equilibrium (to be quantified in the next section). Then, we have the case of self-interacting neutrinos and antineutrinos, and the densities in the potential $V_\pm$ are given in terms of the sums

$$n = \sum_{i=1}^{M} n_i = \sum_{i=1}^{M}\langle|\Psi_{i+}|^2\rangle, \\ \bar{n} = \sum_{i=1}^{N}\bar{n}_i = \sum_{i=1}^{N}\langle|\Psi_{i-}|^2\rangle, \quad (5)$$

where Ψ_{i+} and Ψ_{i-} are the neutrino and antineutrino wave packets respectively (with i numbering the wave packets) and the angular bracket denotes the ensemble average. In this case, the relativistic neutrino and antineutrino wave packets are comoving with the background, and Eq. (4) thus yields

$$i\frac{\partial\Psi_{i\pm}}{\partial t} + \frac{\hbar c^2}{2E_0}\left(\nabla_\perp^2 + \frac{m^2c^4}{E_0^2}\nabla_{||}^2\right)\Psi_{i\pm} - \\ - \frac{V_\pm}{\hbar}\Psi_{i\pm} = 0, \quad (6)$$

where

$$\nabla_\perp^2 = \nabla^2 - (\mathbf{n}_0\cdot\nabla)^2 \quad \text{and} \quad \nabla_{||}^2 = (\mathbf{n}_0\cdot\nabla)^2.$$

Expressions (2a) and (5) reveal that self-interactions between relativistic neutrinos and antineutrinos produce a nonlinear asymmetric potential in Eq. (6). By further rescaling the coordinate along $\mathbf{n}_0$, Eq. (4) can finally be written as the coupled system

$$i\frac{\partial\Psi_{i\pm}}{\partial t} + \frac{\alpha}{2}\nabla^2\Psi_{i\pm} \mp \beta(n - \bar{n})\Psi_{i\pm} = 0, \quad (7)$$

where

$$\alpha = \frac{\hbar c^2}{E_0}, \quad \beta = \frac{2\sqrt{2}G_F}{\hbar}$$

for neutrinos moving on the same flavor background.

Equation (7) shows that this approach can lead to some interesting effects. The case of a single self-interacting neutrino bunch shows that the formation of dark solitary structures is possible. Furthermore, the slightly more complicated case of two interacting bunches, of either the neutrino–neutrino or neutrino–antineutrino type, can result in splitting and focusing the wave packets [19].

3. KINETIC DESCRIPTION

In the preceding section, we investigated the case of a neutrino bunch close to thermal equilibrium. In general, this may of course not be the case, and Eq. (2) must be modified. The more precise form of the potential $V_\pm$ for equal species due to neutrino forward scattering is given by [20]

$$V_\pm(t, \mathbf{r}, \mathbf{p}; f_{i\pm}) = \pm 2\sqrt{2}G_F\int d\mathbf{q}\,(1 - \hat{\mathbf{p}}\cdot\hat{\mathbf{q}}) \times \\ \times\left[\sum_{i=1}^{M} f_{i+}(t, \mathbf{r}, \mathbf{q}) - \sum_{i=1}^{N} f_{i-}(t, \mathbf{r}, \mathbf{q})\right], \quad (8)$$

where hatted quantities denote the corresponding unit vectors and $f_{i+}(t, \mathbf{r}, \mathbf{q})$ ($f_{i-}(t, \mathbf{r}, \mathbf{q})$) is the neutrino (antineutrino) distribution function corresponding to bunch i. The distribution functions are taken to be normalized such that

$$n_i(t, \mathbf{r}) = \int d\mathbf{q}\, f_{i+}(t, \mathbf{r}, \mathbf{q}), \\ \bar{n}_i(t, \mathbf{r}) = \int d\mathbf{q}\, f_{i-}(t, \mathbf{r}, \mathbf{q}), \quad (9)$$

3) See also Ref. [16] for a similar treatment of neutrino–electron interactions.

4) We note that when the scalelength of the density inhomogeneity is comparable to the wavelength of the modulated neutrino wave packets, we must modify the coupled Schrödinger equations to account for differing group velocities of neutrinos and antineutrinos in a fermionic sea. We expect a shift in the momentum of Eq. (13) and a slower growth rate of the modulational instability of neutrino quasiparticles involving short-scale density inhomogeneities.

where n_i ($\bar{n}_i$) is the number density of the ith neutrino (antineutrino) bunch.

We first note that when the distribution is thermal, potential (8) reduces exactly to (2a). Second, when the neutrinos have an almost thermal distribution, i.e., the corresponding distribution function can be expressed as (dropping the indices for notational simplicity)

$$f(t,\mathbf{r},\mathbf{p}) = f_0(p) + \delta f(t,\mathbf{r},\mathbf{p}), \quad |\delta f| \ll |f_0|,$$

we obtain the following form of the potential:

$$V_\pm(t,\mathbf{r},\mathbf{p};f_{i\pm}) = \pm 2\sqrt{2}\,G_F\Bigg[(n-\bar{n}) - \int d\mathbf{q}\,(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})\left(\sum_{i=1}^{M}\delta f_{i+} - \sum_{i=1}^{N}\delta f_{i-}\right)\Bigg]. \quad (10)$$

The last term is small and may therefore be neglected, and we obtain

$$V_\pm(t,\mathbf{r}) \approx \pm 2\sqrt{2}\,G_F(n-\bar{n}),$$

in accordance with expressions (2a), thus justifying equation of motion (7).

Now, we define a distribution function for the collective neutrino states by Fourier transforming the two-point correlation function for $\Psi_\pm$, according to [21]

$$f_{i\pm}(t,\mathbf{r},\mathbf{p}) = \frac{1}{(2\pi\hbar)^3}\int d\mathbf{y}\,\exp(i\mathbf{p}\cdot\mathbf{y}/\hbar)\times \langle\Psi_{i\pm}^*(t,\mathbf{r}+\mathbf{y}/2)\Psi_{i\pm}(t,\mathbf{r}-\mathbf{y}/2)\rangle, \quad (11)$$

where $\mathbf{p}$ represents the momentum of the neutrino (antineutrino) quasiparticles (we note that the ensemble average was not present in the original definition [21], but has important consequences when the wave packet has a random phase). With definition (11), it follows that

$$\langle|\Psi_{i\pm}(t,\mathbf{r})|^2\rangle = \int d\mathbf{p}\,f_{i\pm}(t,\mathbf{r},\mathbf{p}). \quad (12)$$

Thus, using (11) and (6) together with potential (8), we obtain the generalized Wigner–Moyal equation for $f_{i\pm}$,

$$\frac{\partial f_{i\pm}}{\partial t} + \frac{c^2\mathbf{p}}{E_0}\cdot\frac{\partial f_{i\pm}}{\partial \mathbf{r}} - \frac{2V_\pm}{\hbar}\sin\left[\frac{\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial\mathbf{r}}\cdot\frac{\overrightarrow{\partial}}{\partial\mathbf{p}} - \frac{\overleftarrow{\partial}}{\partial\mathbf{p}}\cdot\frac{\overrightarrow{\partial}}{\partial\mathbf{r}}\right)\right]f_{i\pm} = 0, \quad (13)$$

where the sin operator is defined in terms of its Taylor expansion, and the arrows denote the direction of operation. In the case of potential (2a), the last term in the sin operator drops out, and Eq. (13) reduces to the standard Wigner–Moyal equation [21]. Equation (13) was obtained in Ref. [20] using the density matrix approach.

Retaining only the lowest-order terms in $\hbar$ (i.e., taking the long-wavelength limit), we obtain the coupled Vlasov equations

$$\left[\frac{\partial}{\partial t} + \left(\frac{c^2\mathbf{p}}{E_0} + \frac{\partial V_\pm}{\partial\mathbf{p}}\right)\cdot\frac{\partial}{\partial\mathbf{r}}\right]f_{i\pm} - \frac{\partial V_\pm}{\partial\mathbf{r}}\cdot\frac{\partial f_{i\pm}}{\partial\mathbf{p}} = 0. \quad (14)$$

The term $\partial V_\pm/\partial\mathbf{p}$ represents the group velocity. While higher-order group velocity dispersion is present in (13), this is not the case in (14). Thus, information is partially lost by using Eq. (14). Furthermore, while Eq. (14) preserves the number of quasiparticles, Eq. (13) shows that this conclusion is in general not true, i.e., the particle number in a phase-space volume is not constant, and the higher-order terms $\partial^n V_\pm/\partial\mathbf{r}^n$ may moreover contain vital short-wavelength information. Equations similar to (14) have been used to study neutrino–electron interactions in astrophysical contexts [11].

We now suppose that we have small amplitude perturbations on a background of constant neutrino and antineutrino densities $n_i = n_{i0}$ and $\bar{n}_i = \bar{n}_{i0}$, respectively,

$$f_{i\pm}(t,\mathbf{r},\mathbf{p}) = f_{i0\pm}(\mathbf{p}) + \delta f_{i\pm}(\mathbf{p})\exp[i(\mathbf{K}\cdot\mathbf{r} - \Omega t)], \quad (15)$$

and $|\delta f_{i\pm}| \ll |f_{i0\pm}|$, where $\mathbf{K}$ and Ω are the perturbation wavevector and frequency, respectively. Thus, Eqs. (13) give

$$i\left[\Omega - \frac{c^2\mathbf{p}\cdot\mathbf{K}}{E_0} - \frac{2i}{\hbar}V_{0\pm}\sin\left(-\frac{i\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\mathbf{p}}\cdot\mathbf{K}\right)\right]\delta f_{i\pm} + \frac{2}{\hbar}\delta V_\pm\sin\left(\frac{i\hbar}{2}\mathbf{K}\cdot\frac{\overrightarrow{\partial}}{\partial\mathbf{p}}\right)f_{i0\pm} = 0, \quad (16)$$

where $\delta V_\pm = V_\pm(t,r,\mathbf{p};\delta f_{i\pm})$ and $V_{0\pm} = V_\pm(t,r,\mathbf{p};f_{i0\pm})$ from Eq. (8). Eliminating $\delta f_{i\pm}$ from (16), using $\delta V_- = -\delta V_+$, we have

$$\delta V_+(\mathbf{p}) = \frac{4\sqrt{2}\,iG_F}{\hbar}\int d\mathbf{q}\,(1-\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})\,\delta V_+(\mathbf{q})\,\times$$
$$\times\left[\sum_{i=1}^{M}\frac{\sin\left(\dfrac{i\hbar}{2}\mathbf{K}\cdot\overrightarrow{\dfrac{\partial}{\partial\mathbf{q}}}\right)f_{i0+}(\mathbf{q})}{\Omega-\dfrac{c^2\mathbf{q}\cdot\mathbf{K}}{E_0}-\dfrac{2i}{\hbar}V_{0+}(\mathbf{q})\sin\left(\dfrac{i\hbar}{2}\overleftarrow{\dfrac{\partial}{\partial\mathbf{q}}}\cdot\mathbf{K}\right)}+\sum_{i=1}^{N}\frac{\sin\left(\dfrac{i\hbar}{2}\mathbf{K}\cdot\overrightarrow{\dfrac{\partial}{\partial\mathbf{q}}}\right)f_{i0-}(\mathbf{q})}{\Omega-\dfrac{c^2\mathbf{q}\cdot\mathbf{K}}{E_0}-\dfrac{2i}{\hbar}V_{0-}(\mathbf{q})\sin\left(\dfrac{i\hbar}{2}\overleftarrow{\dfrac{\partial}{\partial\mathbf{q}}}\cdot\mathbf{K}\right)}\right]. \quad (17)$$

Assuming that $\delta f_{i\pm}(\mathbf{p})$ is a symmetric function of $\mathbf{p}$, which is a reasonable physical restriction, implies that $\delta V_\pm$ is independent of $\mathbf{p}$, and Eq. (17) simplifies to the dispersion relation

$$1 = \frac{4\sqrt{2}\,iG_F}{\hbar}\int d\mathbf{q}\left[\sum_{i=1}^{M}\frac{\sin\left(\dfrac{i\hbar}{2}\mathbf{K}\cdot\dfrac{\partial}{\partial\mathbf{q}}\right)f_{i0+}(\mathbf{q})}{\Omega-\dfrac{c^2\mathbf{q}\cdot\mathbf{K}}{E_0}-\dfrac{2i}{\hbar}\sin\left(\dfrac{i\hbar}{2}\mathbf{K}\cdot\dfrac{\partial}{\partial\mathbf{q}}\right)V_{0+}(\mathbf{q})}+\right.$$
$$\left.+\sum_{i=1}^{N}\frac{\sin\left(\dfrac{i\hbar}{2}\mathbf{K}\cdot\dfrac{\partial}{\partial\mathbf{q}}\right)f_{i0-}(\mathbf{q})}{\Omega-\dfrac{c^2\mathbf{q}\cdot\mathbf{K}}{E_0}-\dfrac{2i}{\hbar}\sin\left(\dfrac{i\hbar}{2}\mathbf{K}\cdot\dfrac{\partial}{\partial\mathbf{q}}\right)V_{0-}(\mathbf{q})}\right], \quad (18)$$

where we have dropped the arrows indicating the direction of operation. We note that if the background distribution is thermal, $V_{0\pm}$ is independent of $\mathbf{p}$, and the last term in the denominators of Eq. (18) vanishes.

3.1. The one-dimensional case

The simplest way to analyze dispersion relation (18) is to reduce the dimensionality of the problem. We therefore first consider the one-dimensional case, where we may use the identity

$$2\sin\left(\frac{i\hbar K}{2}\frac{\partial}{\partial p}\right)h(p) = i\left[h\left(p+\frac{\hbar K}{2}\right)-h\left(p-\frac{\hbar K}{2}\right)\right]$$

in order to rewrite dispersion relation (18) as

$$1 = -\frac{2\sqrt{2}\,G_F}{\hbar}\int dq\left\{\sum_{i=1}^{M}\frac{f_{i0+}(q+\hbar K/2)-f_{i0+}(q-\hbar K/2)}{\Omega-c^2qK/E_0+\Delta_+(q)}+\sum_{i=1}^{N}\frac{f_{i0-}(q+\hbar K/2)-f_{i0-}(q-\hbar K/2)}{\Omega-c^2qK/E_0+\Delta_-(q)}\right\}, \quad (19)$$

where we have introduced

$$\Delta_\pm(q) \equiv \frac{1}{\hbar}\left[V_{0\pm}\left(q+\frac{\hbar K}{2}\right)-V_{0\pm}\left(q-\frac{\hbar K}{2}\right)\right].$$

In the case of monoenergetic beams, i.e.,

$$f_{i0+}(p) = n_{i0}\delta(p-p_{i0}), \quad f_{i0-}(p) = \bar{n}_{i0}\delta(p-\bar{p}_{i0}),$$

Eq. (19) reduces to

$$1 = -\frac{2\sqrt{2}G_F}{\hbar}\left\{\sum_{i=1}^{M} n_{i0}\left[-\frac{\hbar c^2K^2}{E_0} + \Delta_+\left(p_{i0}+\frac{\hbar K}{2}\right) - \Delta_+\left(p_{i0}-\frac{\hbar K}{2}\right)\right]\times\right.$$
$$\times\left[\left(\Omega - \frac{c^2p_{i0}K}{E_0}\right)^2 - \left(\frac{\hbar c^2K^2}{2E_0}\right)^2 + \left(\Omega - \frac{c^2p_{i0}K}{E_0}\right)\left[\Delta_+\left(p_{i0}+\frac{\hbar K}{2}\right) + \Delta_+\left(p_{i0}-\frac{\hbar K}{2}\right)\right] + \right.$$
$$\left. + \frac{\hbar c^2K^2}{2E_0}\left[\Delta_+\left(p_{i0}+\frac{\hbar K}{2}\right) - \Delta_+\left(p_{i0}-\frac{\hbar K}{2}\right)\right] + \Delta_+\left(p_{i0}+\frac{\hbar K}{2}\right)\Delta_+\left(p_{i0}-\frac{\hbar K}{2}\right)\right]^{-1} +$$
$$+\sum_{i=1}^{N}\bar{n}_{i0}\left[-\frac{\hbar c^2K^2}{E_0} + \Delta_-\left(\bar{p}_{i0}+\frac{\hbar K}{2}\right) - \Delta_-\left(\bar{p}_{i0}-\frac{\hbar K}{2}\right)\right]\times$$
$$\times\left[\left(\Omega - \frac{c^2\bar{p}_{i0}K}{E_0}\right)^2 - \left(\frac{\hbar c^2K^2}{2E_0}\right)^2 + \left(\Omega - \frac{c^2\bar{p}_{i0}K}{E_0}\right)\left[\Delta_-\left(\bar{p}_{i0}+\frac{\hbar K}{2}\right) + \Delta_-\left(\bar{p}_{i0}-\frac{\hbar K}{2}\right)\right] + \right.$$
$$\left.\left. \frac{\hbar c^2K^2}{2E_0}\left[\Delta_-\left(\bar{p}_{i0}+\frac{\hbar K}{2}\right) - \Delta_-\left(\bar{p}_{i0}-\frac{\hbar K}{2}\right)\right] + \Delta_-\left(\bar{p}_{i0}+\frac{\hbar K}{2}\right)\Delta_-\left(\bar{p}_{i0}-\frac{\hbar K}{2}\right)\right]^{-1}\right\}, \quad (20)$$

where

$$V_{0\pm}(p) = \pm 2\sqrt{2}G_F\left[(n_0 - \bar{n}_0) - \mathrm{sgn}p\left(\sum_{i=1}^{M} n_{i0}\,\mathrm{sgn}p_{i0} - \sum_{i=1}^{N}\bar{n}_{i0}\,\mathrm{sgn}\bar{p}_{i0}\right)\right] \quad (21)$$

by Eq. (8).

We consider the simplest case of interacting neutrinos and antineutrinos with $M = N = 1$. We assume that they have equal densities $n_0 = \bar{n}_0$ and are counter-propagating, i.e., $p_0 = -\bar{p}_0 > 0$. From (21), we then obtain the potential

$$V_{0\pm}(p) = \mp 4\sqrt{2}G_F\,\mathrm{sgn}p n_0, \quad (22)$$

while Eq. (20) yields

$$\sigma\left(\frac{\hbar c^2K^2}{E_0} - 2\sigma\varepsilon\right)\left\{\left[\left(\Omega - \frac{c^2p_0K}{E_0}\right)^2 - \left(\frac{\hbar c^2K^2}{2E_0}\right)^2 - 2\sigma\varepsilon\left(\left(\Omega - \frac{c^2p_0K}{E_0}\right) - \frac{\hbar c^2K^2}{2E_0}\right)\right]^{-1} + \right.$$
$$\left. + \left[\left(\Omega + \frac{c^2p_0K}{E_0}\right)^2 - \left(\frac{\hbar c^2K^2}{2E_0}\right)^2 + 2\sigma\varepsilon\left(\left(\Omega + \frac{c^2p_0K}{E_0}\right) + \frac{\hbar c^2K^2}{2E_0}\right)\right]^{-1}\right\} = 1, \quad (23)$$

where

$$\sigma = \frac{2\sqrt{2}G_F n_0}{\hbar},$$

$$\varepsilon = 1 - \mathrm{sgn}(p_0 - \hbar K) = \begin{cases} 0, & p_0 > \hbar K, \\ 1, & p_0 = \hbar K, \\ 2, & p_0 < \hbar K. \end{cases}$$

Thus, for $\varepsilon = 0$, the growth rate is given by (see Figs. 1 and 2)

$$\frac{\Gamma^2}{K^2} = \sqrt{4v^2\left(\frac{\hbar c^2K}{2E_0}\right)^2 + 4v^2v_F^2 + v_F^4} - v^2 - v_F^2 - \left(\frac{\hbar c^2K}{2E_0}\right)^2, \quad (24)$$

where $\Gamma = -i\Omega$ is the instability growth rate and

$$v_F^2 \equiv 2\sqrt{2}\, G_F n_0 c^2/E_0.$$

We also note that as expected, the instability disappears in the limit $v \to 0$, just stating the well-known fact that there must be a nonzero relative velocity between the beams in order for the instability to occur. Because Γ^2 is positive, we have

$$\left(\frac{v}{v_F}\right)^2 - 2 < \left(\frac{\hbar c^2 K}{2E_0 v_F}\right)^2 < \left(\frac{v}{v_F}\right)^2, \qquad (25)$$

i.e.,

$$\ell_0 < \ell < \ell_0(1 - 2v_F^2/v^2)^{-1/2}, \qquad (26)$$

where we have introduced the length scales

$$\ell = \frac{2\pi}{K}, \quad \ell_0 = \frac{\hbar c^2}{2E_0 v}.$$

Thus, a higher neutrino momentum can retain a smaller instability length scale. It is clear from (24) that (i) the instability remains for arbitrary velocities (see Figs. 1 and 2), and (ii) the higher the neutrino velocity, the smaller the corresponding instability length scale ℓ.

3.2. Partial incoherence and thermal effects

Partial incoherence can in general lead to a lower growth rate, similar to the inverse Landau damping. We consider the following example of the results of stochastic effects (e.g., thermal fluctuations). Let the indeterminacy of the neutrino collective state manifest itself in a random phase $\varphi(x)$ of the background wave packet, with the width Δp defined according to

$$\langle \exp(-i\varphi(x+y/2)) \exp(i\varphi(x-y/2)) \rangle = \exp(-\Delta p|y|/\hbar).$$

Due to this random spread, the modulational instability is damped, as we show below. The Wigner function corresponding to the random phase assumption is given by the Lorentz distribution

$$f_0(p) = \frac{n_0}{\pi} \frac{\Delta p}{(p-p_0)^2 + \Delta p^2}. \qquad (27)$$

With this, we obtain Eq. (24) with

$$\Gamma \to \Gamma_D + \Delta c^2 pK/E_0,$$

where Γ_D is the reduced growth rate. Thus, we see that the broadening tends to suppress the growth. Moreover, a positive growth rate Γ_D requires that

$$\frac{2v\Delta p}{\hbar\Gamma} < \frac{\ell}{\ell_0}, \qquad (28)$$

where Γ is given by Eq. (24). Hence, the general property of a spread in momentum space, here exemplified by a random phase, is to put bounds on the modulational instability length scale ℓ.

Incoherent effects among the neutrinos and antineutrinos can also be approached for a background obeying the Fermi–Dirac statistics, i.e.,

$$f_{0\pm}(p) = \frac{cn_0}{\ln 4k_B T_\pm} \left[1 + \exp\left(\frac{c|p|}{k_B T_\pm}\right)\right]^{-1}, \qquad (29)$$

where we set $M = N = 1$ and assume $n_0 = \bar{n}_0$. Here, we have neglected the mass of the neutrinos (which leads to the correct result to the lowest order). For simplicity, we assume that $T_\pm = T$, and therefore dispersion relation (19) takes the form

$$1 = -\frac{4\sqrt{2}\, cG_F n_0}{\ln 4\hbar k_B T} \int_{-\infty}^{\infty} dp \left(\Omega - \frac{c^2 pK}{E_0}\right)^{-1} \times \left[\left(1 + \exp\left(\frac{c|p + \hbar K/2|}{k_B T}\right)\right)^{-1} + \left(1 + \exp\left(\frac{c|p - \hbar K/2|}{k_B T}\right)\right)^{-1}\right]. \qquad (30)$$

Dispersion relation (30) cannot be solved analytically, but it can be expressed as

$$1 = -Q\left[\mathrm{P}(I(\Omega_n, K_n)) + i\pi g(\Omega_n, K_n)\right], \qquad (31)$$

where $\mathrm{P}(I(\Omega_n, K_n))$ is the principal value of the integral

$$I = \int_0^{\infty} dx\, (1 + e^x)^{-1} \times \left[\frac{\Omega_n + K_n^2}{(\Omega_n + K_n^2)^2 - K_n^2 x^2} + \frac{\Omega_n - K_n^2}{(\Omega_n - K_n^2)^2 - K_n^2 x^2}\right], \qquad (32)$$

and $g = g_+ + g_-$, where

$$g_\pm(\Omega_n, K_n) = \frac{\Omega_n \pm K_n^2}{1 + \exp(|\Omega_n \pm K_n^2|/\sqrt{2}K_n)}, \qquad (33)$$

$$Q \equiv \frac{4}{\ln 4} \frac{2\sqrt{2}\, G_F n_0}{k_B T} \frac{E_0}{k_B T},$$

and we have introduced the dimensionless variables

$$\Omega_n \equiv \frac{\hbar E_0}{(k_B T)^2}\Omega, \quad K_n \equiv \frac{\hbar c}{\sqrt{2}\, k_B T} K.$$

The constant Q gives the ratio of the potential energy contribution of the background and the individual neutrino energy to the thermal energy of the background.

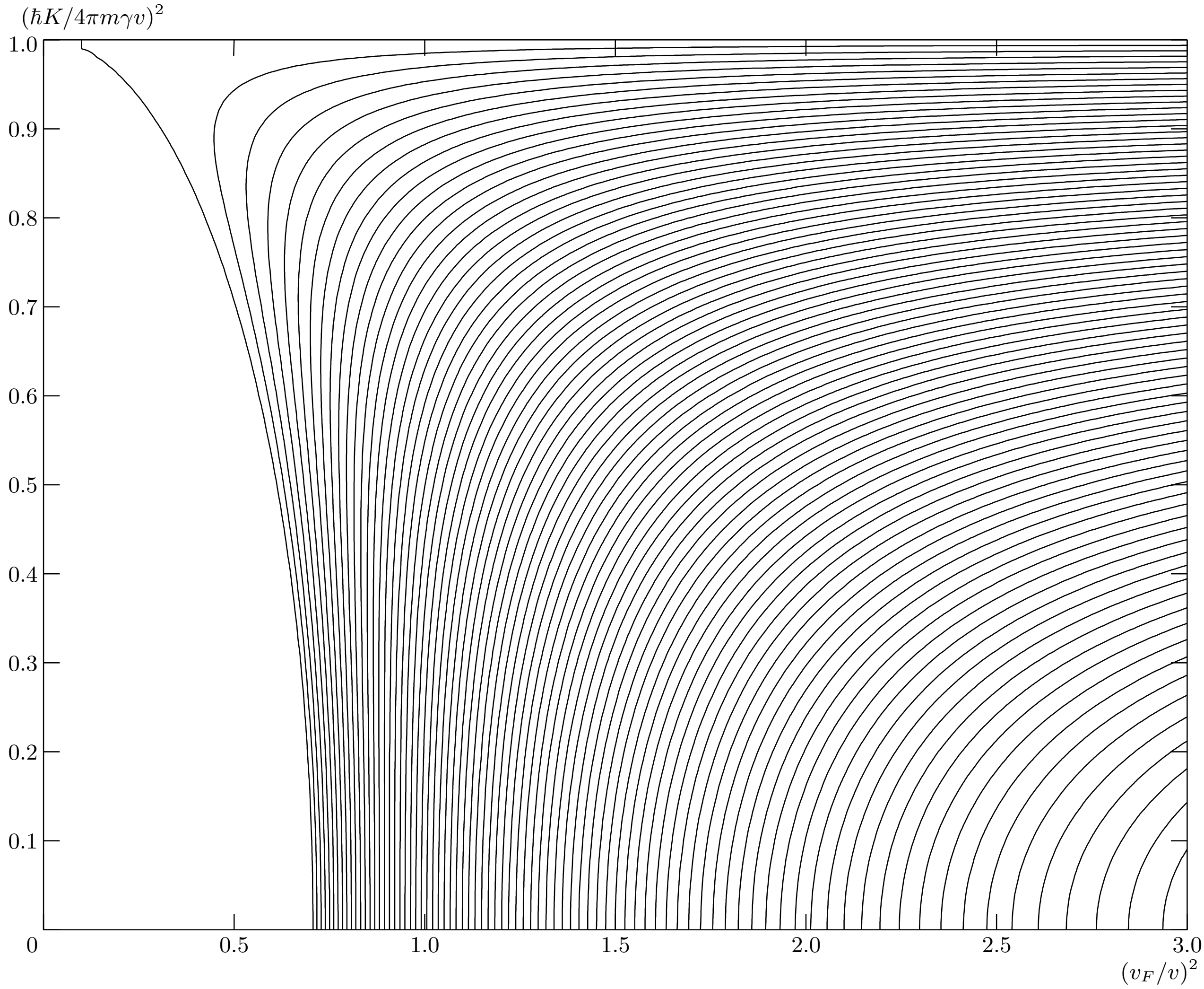

Fig. 1. A contour plot of the values of $(\Gamma/Kv)^2$, when $\varepsilon = 0$, for which the instability occurs. The function $(\Gamma/Kv)^2$ is constant along the contours and is plotted in terms of the variables $(v_F/v)^2$ and $(\hbar K/4\pi m\gamma v)^2$. Outside the contours, $\Gamma^2 < 0$

Furthermore, $E_0 \approx k_B T$, thus simplifying the expression for Q. The contributions from real and imaginary parts to the dispersion relation are plotted in Figs. 3 and 4, respectively. We note that for very short length scales, i.e., large K, the quantity $\Omega_n - K_n^2$ becomes negative, and the imaginary part in Eq. (31) changes sign, which cannot be seen from the long-wavelength limit equation (14). This behavior can in principle lead to growth instead of damping of the perturbations (see Ref. [22] for a general discussion of this behavior). We can obtain a quantitative measure of the growth/damping rate as follows. For any fixed K_{n0}, the dimensionless growth/damping rate

$$\Gamma_n = -i \operatorname{Im} \Omega_n$$

can be expressed as (with the value at Ω_{n0} denoted by 0)

$$\Gamma_n = \pi \frac{\left(Q^{-1} + \mathrm{P}(I_0)\right)(\partial g/\partial\Omega_{n0}) - g_0(\partial \mathrm{P}(I)/\partial\Omega_{n0})}{\pi^2 \left(\partial g/\partial\Omega_{n0}\right)^2 + \left(\partial \mathrm{P}(I)/\partial\Omega_{n0}\right)^2} \qquad (34)$$

to the first order around (Ω_{n0}, K_{n0}). Therefore, $\Gamma_n > 0$ if

$$\left(Q^{-1} + \mathrm{P}(I_0)\right)\left(\frac{\partial \ln g}{\partial\Omega_{n0}}\right) > \frac{\partial \mathrm{P}(I)}{\partial\Omega_{n0}}.$$

Moreover, using values given in Sec. 4, one can show that $Q^{-1} \approx 3\cdot 10^9$. Thus, Q^{-1} dominates the contribution to the growth/damping rate over a wide range of

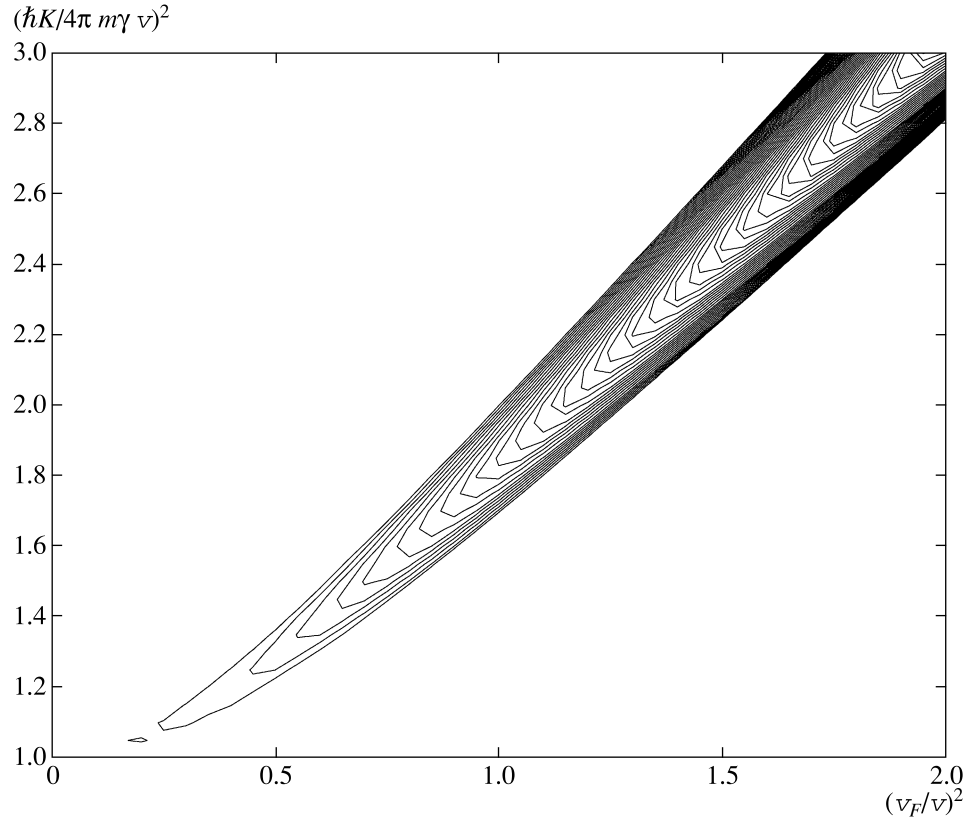

Fig. 2. The same plot as in Fig. 1, but for $\varepsilon = 2$

(Ω_n, K_n), and a positive growth rate is implied as long as $\partial g/\partial\Omega_{n0} > 0$.

4. APPLICATIONS

As a model for hot dark matter, massive neutrinos have for some time been one of the prime candidates, but as such they have faced the problem of the scale of the inhomogeneities that they can support. Due to the conservation of phase-space density, the Tremaine–Gunn limit constrains the neutrino mass for isothermal spheres of a given size. For dwarf galaxies, for which there is ample evidence of dark matter [23], the necessary mass of the neutrino is uncomfortably large [4, 24]. On the other hand, as pointed out in [5], interacting dark matter can in principle change this picture. Here we see from Eq. (26) that as the neutrino momentum increases, the typical length scale ℓ of the inhomogeneity that can be supported by the modulational instability decrease. From the definition of ℓ_0, we note that as v tends to c, $\ell_0 \to 0$, and due to Eq. (26), the allowed scale of inhomogeneity becomes squeezed between two small values. On the other hand, if $v \sim v_F$ (a condition stating that the neutrino number density must reach extreme values), the upper inhomogeneity scale limit diverges. A minimum requirement for the effect to be of importance is that the instability growth rate is larger than the Hubble parameter H. An estimate of the growth rate can be obtained as follows. At the onset of «free streaming» of neutrinos (i.e., their decoupling from matter and radiation) at $z \sim 10^{10}$, the neutrino number density can be estimated as $n_0 \approx 2.1 \cdot 10^{38}\,\mathrm{m}^{-3}$ (see, e.g., [25]). Furthermore, we assume that the neutrino mass is in the range $m \sim 1\,\mathrm{eV}$, and find $v_F \approx 9 \cdot 10^{-4} E_0^{-1/2}\,\mathrm{m/s}$. The temperature of the neutrinos, given by

$$T_\nu = (4/11)^{1/3} T_0 (1+z)$$

at neutrino decoupling (with T_0 being the present day CMB temperature) [25] is $T_\nu \approx 2 \cdot 10^{10}\,\mathrm{K}$. Thus, the thermal energy is roughly five orders of magnitude greater than the assumed rest mass of the neutrino, and in this sense the neutrinos can be well approximated as ultra-relativistic. In this case, using values of

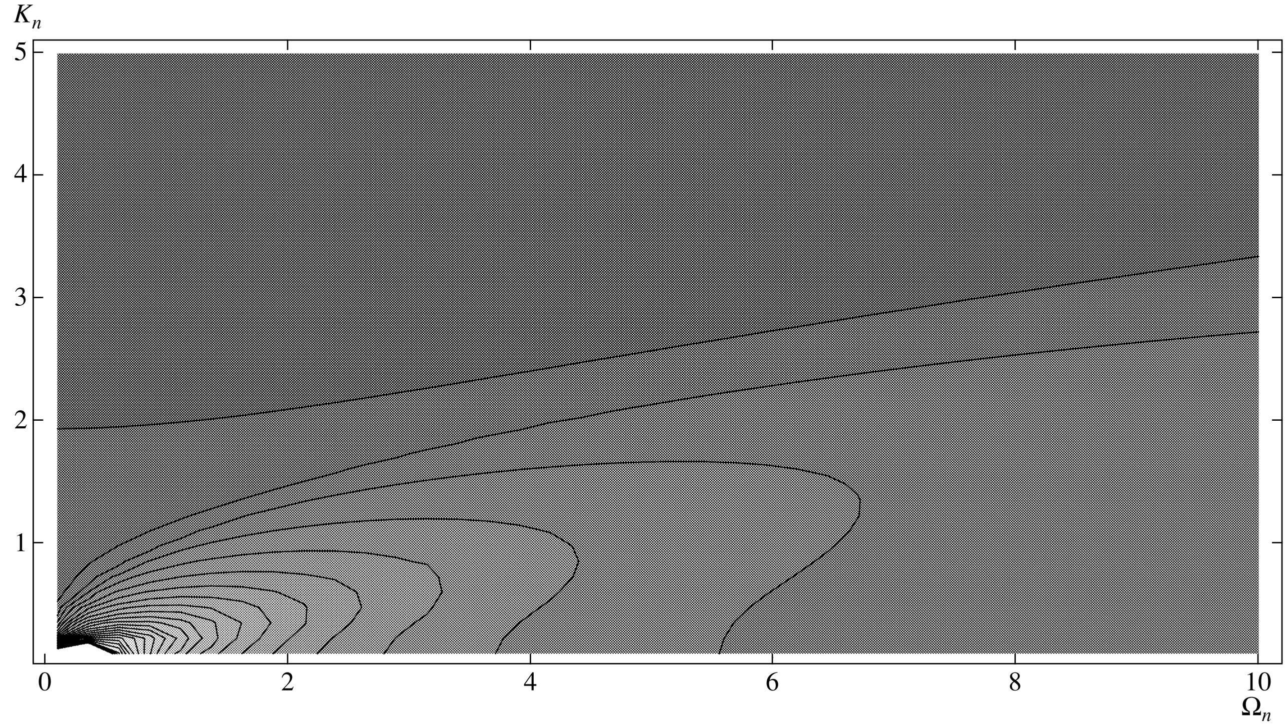

Fig. 3. A contour plot of the Cauchy principal value $\mathrm{P}(I(\Omega_n, K_n))$ as a function of the dimensionless variables Ω_n and K_n. We note that $\mathrm{P}(I(\Omega_n, K_n)) \geq 0$, being largest for small Ω_n and K_n, and approaching zero at infinity. The uppermost contour has $\mathrm{P}(I(\Omega_n, K_n)) = 0$

$(\hbar c^2 K/2E_0 v_F)^2$ in the middle range of inequality (25), we obtain from Eq. (24) that

$$\Gamma \approx \frac{2\sqrt{2} G_F n_0}{\hbar} \sim 16 \cdot 10^{10}\ \mathrm{s}^{-1}$$

for the values specified above. With the critical density assumed for the Universe, the Hubble time becomes

$$H^{-1} \approx H_0^{-1}(1+z)^{-3/2} \sim 5 \cdot 10^2\ \mathrm{s}$$

at the redshift 10^{10}, and therefore $\Gamma/H \gg 1$.

Although the two-stream instability may seem contrived as a cosmological application, the important issue displayed by this example is the nongravitational growth of inhomogeneities, given a small perturbation of a homogeneous, although anisotropic, background. The criticism of neutrinos as dark matter candidates is in particular based on the fact that they are ultra-relativistic for long times, with free-streaming smoothing as a result [4]. According to this, we would have to accept a top-down scenario for structure formation, if neutrinos would indeed be the missing dark matter [15]. These arguments are presented with the prerequisite that only gravitational instabilities are of importance after neutrino decoupling. Other instabilities, such as the one presented in this work, could in principle alter this picture, loosening the bounds set by the Tremaine–Gunn limit. The fact that the growth rate exceeds the inverse of the Hubble time by many orders of magnitude makes it clear that the mechanism may be of some importance. Moreover, the analogous estimate for the Fermi–Dirac background, although done in a simplistic manner, indicates that the growth of the large-K perturbations may be of importance. Furthermore, although we here have used parameters relevant to a cosmological setting, it could also be of interest to use the current formalism as a tool to investigate neutrino interactions within supernovæ, where the two-stream instability scenario may occur as a more natural ingredient than perhaps within cosmology.

5. CONCLUSION

We have considered the nonlinear coupling between neutrinos and antineutrinos in a dense plasma. We have found that their interactions are governed by a system of Wigner–Moyal equations, which admit a

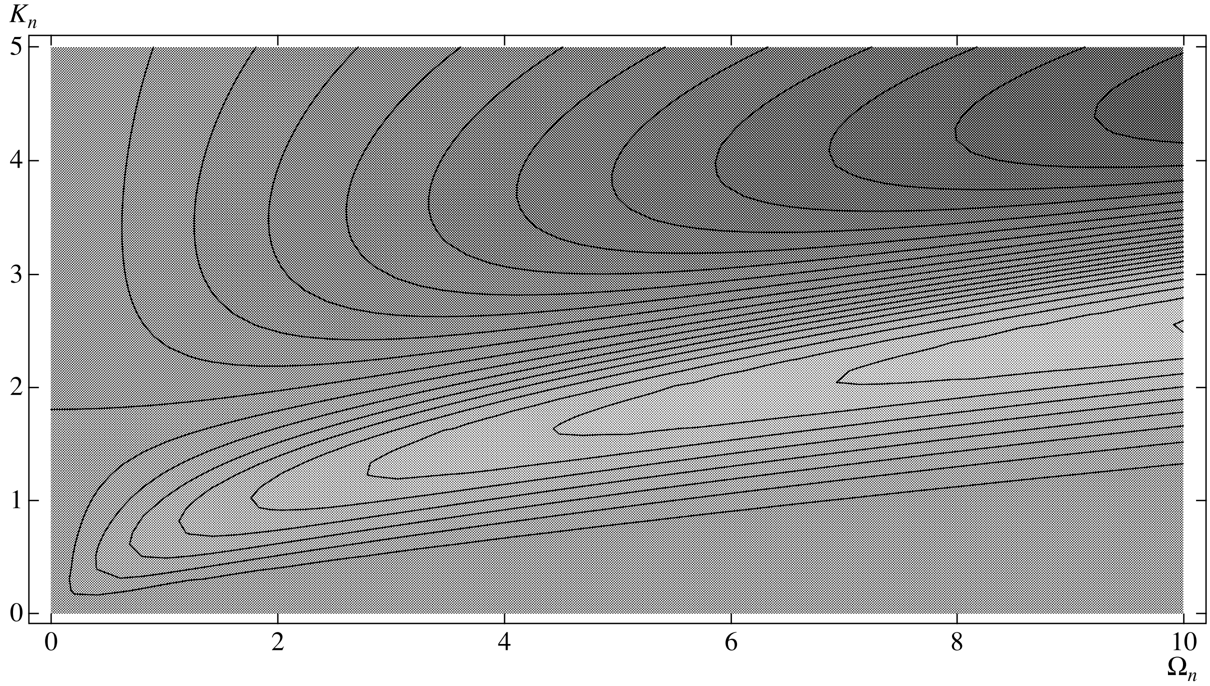

Fig. 4. A contour plot of the contribution $g(\Omega_n, K_n)$ of the poles to integral (30) as a function of the dimensionless variables Ω_n and K_n. The darker areas represent negative values, the lighter positive values, and g is zero on the contour emanating from $(\Omega_n, K_n) = (0, 1.8)$

modulational instability of the neutrino/antineutrino beams against large-scale (in comparison with the neutrino wavelength) density fluctuations. Physically, the instability arises because interpenetrating neutrino and antineutrino beams are like quasiparticles, carrying free energy that can be coupled to inhomogeneities due to a resonant quasiparticle–wave interaction that is similar to the Cherenkov interaction. Nonlinearly excited density fluctuations can be associated with the background inhomogeneity of the early Universe, and possibly counteract the free-streaming smoothing of the small-scale primordial fluctuations, thus making massive neutrinos plausible as a candidate for hot dark matter.

This research was partially supported by the Sida/NRF grant № SRP-2000-041, the Swedish Research Council through Contract № 621-2001-2274, and the Deutsche Forschungsgemeinschaft through the Sonderforschungsbereich 591.

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