ЖЭТФ, 2004, том 126, вып. 2 (8), стр. 259–267

VECTOR MESON DOMINANCE PION ELECTROMAGNETIC FORM FACTOR WITH THE σ-MODEL LOOP CORRECTIONS

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Submitted 18 December 2003

A model is developed for electromagnetic form factor of the pion. One-loop corrections are included in the linear σ-model. The ρ-meson contribution is added in an extended VMD model. The form factor, calculated without fitting parameters, is in a good agreement with experiment for space-like and time-like photon momenta. Loop corrections to the two-pion hadronic contribution $a_\mu^{(had,\pi)}$ to the muon anomalous magnetic moment are calculated. The optimal value of the σ-meson mass appears to be very close to the ρ-meson mass.

PACS: 12.39.Fe, 12.40.Vv, 13.40.Gp

1. INTRODUCTION

It has recently been understood that the pion electromagnetic form factor is a very important physical quantity that plays a key role in the test of the Standard model at the electroweak precision level. The reason is that at low energies, the production cross section

$$\sigma(e^+e^- \to \pi^+\pi^-) = \frac{\pi\alpha^2}{3s}\left(1-\frac{4m_\pi^2}{s}\right)^{3/2}|F_\pi(s)|^2, \quad (1)$$

where s is the squared total energy in center-of-mass system, α is the fine structure constant, and m_π is the pion mass, dominates over the other hadronic channels and accounts for more than 70 % of the total hadronic contribution to the muon anomalous magnetic moment (AMM)

$$a_\mu = \frac{g_\mu - 2}{2}.$$

The recent measurement of a_μ from Brookhaven E821 experiment [1] has boosted the interest in a renewed theoretical calculation of this quantity [2].

The main ingredient of the theoretical prediction of a_μ, which is responsible for the bulk of the theoretical error, is the hadronic contribution to the vacuum polarization. The contribution of the $\pi^+\pi^-$ channel to the electron–positron annihilation process can be written in terms of the form factor $F_\pi(s)$ via the dispersion integral [3]

$$a_\mu^{(had,\pi)} = \frac{1}{4}\int\limits_{4m_\pi^2}^{\infty} K(s)\left(1-\frac{4m_\pi^2}{s}\right)^{3/2}|F_\pi(s)|^2 ds,$$
$$K(s) = \frac{\alpha^2}{3\pi^2 s}\int\limits_0^1 \frac{x^2(1-x)}{x^2+(1-x)s/m_\mu^2}dx, \quad (2)$$

where m_μ is the muon mass.

Conventional strategy of the model-independent evaluation of this integral consists in utilization of precise experimental data (at least at low energies, where perturbative QCD cannot be reliably applied). However, the announced accuracy, which is to be reached soon in E821 experiment, requires calculation of electromagnetic radiative corrections to cross section (1) [4]. Apart from the pure $\pi^+\pi^-$ events, electromagnetic radiative corrections include the $\pi^+\pi^-\gamma$ process where the photon is radiated from the final pions. In the current experiments at Φ and B factories, based on the radiative return method [5], this contribution

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cannot be extracted in a model-independent way[1)] and the corresponding procedure requires model-dependent approaches. This, in turn, stimulates development and study of different theoretical models of the pion–photon interaction. The simplest one is the point-like scalar quantum electrodynamics (*sQED*) [7] joined with the standard vector meson dominance (VMD) model (see, e.g., [8]) for description of the $\gamma^* \to \pi^+\pi^-$ transition form factor in the ρ-resonance region. Such a model was used in Ref. [4] for construction of the Monte Carlo event generator.

In the present paper, we consider a modification of the pion electromagnetic form factor in the linear σ-model [9] with spontaneously broken chiral symmetry, which includes the nucleon sector. The ρ-meson contribution is added following Refs. [10, 11]. In particular, the ρ coupling to the pion and nucleon is introduced through gauge-covariant derivatives, while the direct $\gamma\rho$ coupling has an explicitly gauge-invariant form. We calculate the pion form factor in the one-loop approximation in the strong interaction and compare $F_\pi(s)$ with the precise data obtained from elastic $e^-\pi^+$ scattering and e^+e^- annihilation in the pion pair.

We take the loop corrections to $a_\mu^{(had,\pi)}$ into account. In general, because the σ-model Lagrangian contains the sQED Lagrangian as a constituent part, one can expect that the difference between the predictions of σ-model+VMD and sQED + VMD is small. Indeed, it follows from our calculation that the loop corrections increase the low-energy part of the right-hand side of Eq. (2) by about 2 per cent, as compared with sQED + VMD.

2. FORMALISM

2.1. Lagrangian

The Lagrangian of the model consists of two parts,

$$\mathcal{L} = \mathcal{L}^{(1)} + \mathcal{L}^{(2)}.$$

The first one is the Lagrangian of the chiral linear σ-model [9] with an explicit symmetry-breaking term $c\phi$. After spontaneous breaking of chiral symmetry and redefinition of the scalar field via

$$\phi = \sigma + v,$$

where $v = \langle\phi\rangle$ is the vacuum expectation value, the Lagrangian takes the form

$$\begin{aligned}\mathcal{L}^{(1)} = \bar{N}(i\partial\!\!\!/ - m_N)N + \frac{1}{2}\left[(\partial\sigma)^2 - m_\sigma^2\sigma^2\right] + \\ + \frac{1}{2}\left[(\partial\vec{\pi})^2 - m_\pi^2\vec{\pi}^{\,2}\right] - \\ - g_\pi\bar{N}\left(\sigma + i\gamma_5\vec{\tau}\vec{\pi}\right)N - \lambda\left(\sigma^2 + \vec{\pi}^{\,2}\right)\times \\ \times\left[v\sigma + \frac{1}{4}\left(\sigma^2+\vec{\pi}^{\,2}\right)\right] + \text{const},\end{aligned} \quad (3)$$

where N, $\vec{\pi}$, and σ are the respective fields of the nucleon, pion, and meson with vacuum quantum numbers, g_π is the coupling constant, λ is parameter of the meson potential, and

$$\partial\!\!\!/ \equiv \partial^\mu\gamma_\mu, \quad (\partial\sigma)^2 = \partial^\mu\sigma\partial_\mu\sigma,$$

etc. All parameters of the model are related via

$$m_N = g_\pi v, \quad m_\sigma^2 = 2\lambda v^2 + m_\pi^2, \quad m_\pi^2 = \frac{c}{v}. \quad (4)$$

Moreover, in the tree-level approximation, $v = f_\pi$, where $f_\pi = 93.2$ MeV is the pion weak decay constant. More details on the σ-model can be found, e.g., in [12, Ch. 5, Sec. 2.6].

The second part of the Lagrangian includes coupling to the electromagnetic field A^μ and the field ρ^μ of the *neutral* ρ-meson. This coupling can be obtained using the minimal substitutions

$$\begin{aligned}&\partial^\mu N \to \left(\partial^\mu + ie\frac{1+\tau_3}{2}A^\mu + ig_\rho\frac{\tau_3}{2}\rho^\mu\right)N,\\ &\partial^\mu\pi^a \to \partial^\mu\pi^a + (eA^\mu + g_\rho\rho^\mu)\varepsilon^{3ab}\pi^b,\\ &\qquad a,b = 1,2,3,\\ &\partial^\mu\sigma \to \partial^\mu\sigma,\end{aligned} \quad (5)$$

where e is the proton charge, g_ρ is the coupling constant, and τ_3 is the third component of the Pauli vector $\vec{\tau} = (\tau_1,\tau_2,\tau_3)$. In addition, we include the direct coupling of the photon to the ρ-meson in the version of VMD model from Refs. [10, 11]. We thus obtain

$$\begin{aligned}\mathcal{L}^{(2)} = \frac{1}{2}m_\rho^2\rho_\mu\rho^\mu - \frac{1}{4}\rho_{\mu\nu}\rho^{\mu\nu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \\ - (eA_\mu + g_\rho\rho_\mu)(\vec{\pi}\times\partial^\mu\vec{\pi})_3 + (eA_\mu + g_\rho\rho_\mu)^2(\vec{\pi}^{\,2} - \pi_3^2) - \\ - g_\rho\bar{N}\gamma^\mu\frac{\tau_3}{2}N\rho_\mu - e\bar{N}\gamma^\mu\frac{1+\tau_3}{2}NA_\mu - \frac{e}{2f_\rho}\rho_{\mu\nu}F^{\mu\nu},\end{aligned} \quad (6)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad \rho_{\mu\nu} = \partial_\mu\rho_\nu - \partial_\nu\rho_\mu,$$

and f_ρ determines the $\gamma\rho$ coupling. In Eq. (6), we assume equal coupling constants of ρ to the pion and the nucleon in accordance with the universality hypothesis

[1)] Even in direct scanning experiments, a model-independent treatment of the $\pi^+\pi^-\gamma$ events suggested in [6] seems too complicated to be used in the near future.

of Sakurai (see, e.g., [12, Ch. 5, Sec. 4]. At the same time, the $\gamma\rho$ coupling constant f_ρ does not necessarily coincide with g_ρ. Lagrangian (6) is gauge-invariant because of the form of the $\gamma\rho$ coupling. We mention that the nucleon contribution is also included in Lagrangian (6), contrary to [10, 11].

2.2. Counterterms and renormalization

Because one of the purposes of the present paper is to take loop corrections to the pion electromagnetic vertex into account, we need to specify the way of renormalization of the parameters. We use the conventional approach and assume that Lagrangians (3) and (6) involve the «bare» fields, coupling constants and masses, to be marked by the subscript «0». The bare fields require rescaling,

$$(\vec{\pi}_0, \sigma_0) = \sqrt{Z_\pi}(\vec{\pi}, \sigma), \quad N_0 = \sqrt{Z_N}N,$$
$$\rho_0^\mu = \sqrt{Z_\rho}\rho^\mu, \quad A_0^\mu = \sqrt{Z_A}A^\mu,$$

where Z_π, Z_N, Z_ρ, and Z_A are the respective wave-function renormalization constants for the pion (or sigma), nucleon, rho, and photon. The procedure for obtaining the counterterm Lagrangian is known (see, e.g., [13, Ch. 10]). For $\mathcal{L}^{(1)}$, the corresponding counterterm Lagrangian is given by

$$\begin{aligned}\mathcal{L}_{ct}^{(1)} = \delta_{Z_N}\bar{N}i\not{\partial}N - \delta_{g_\pi}v\bar{N}N - \delta_{g_\pi}\bar{N}\left(\sigma + i\gamma_5\vec{\tau}\vec{\pi}\right)N + \\ + \frac{1}{2}\delta_{Z_\pi}\left[(\partial\vec{\pi})^2 + (\partial\sigma)^2\right] - \\ - \frac{1}{2}\left(\delta_\mu + 3\delta_\lambda v^2\right)\sigma^2 - \frac{1}{2}\left(\delta_\mu + \delta_\lambda v^2\right)\vec{\pi}^2 - \\ - \frac{1}{4}\delta_\lambda\left(\vec{\pi}^2 + \sigma^2\right)^2 - \delta_\lambda v\left(\sigma\vec{\pi}^2 + \sigma^3\right) - \\ - \left[\left(\delta_\mu + \delta_\lambda v^2\right)v - \delta_c\right]\sigma + \text{const.}\end{aligned} \quad (7)$$

There are six constants $\delta_{Z_\pi}, \delta_{Z_N}, \delta_\mu, \delta_\lambda, \delta_{g_\pi}, \delta_c$, which can be fixed by imposing six conditions on the Green's functions in general. In the calculation of the pion electromagnetic vertex, only one constant δ_{Z_π} is needed (see Sect. 2.3).

For Lagrangian (6), we can first define the physical values of the electric charge

$$e = e_0\sqrt{Z_A}$$

and the ρ coupling

$$g_\rho = g_{\rho 0}\sqrt{Z_\rho}.$$

It is also convenient to introduce

$$\hat{m}_\rho = \sqrt{Z_\rho}m_{\rho 0}$$

(the ρ-meson mass in the absence of coupling to pions). The counterterm Lagrangian can be written as

$$\begin{aligned}\mathcal{L}_{ct}^{(2)} = -\delta_{Z_A}\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \delta_{Z_\rho}\frac{1}{4}\rho_{\mu\nu}\rho^{\mu\nu} - \\ - \delta_{Z_\pi}\left(eA_\mu + g_\rho\rho_\mu\right)\left(\vec{\pi}\times\partial^\mu\vec{\pi}\right)_3 + \\ + \delta_{Z_\pi}\left(eA_\mu + g_\rho\rho_\mu\right)^2\left(\vec{\pi}^{\,2} - \pi_3^2\right) - \\ - \delta_{Z_N}g_\rho\bar{N}\gamma^\mu\frac{\tau_3}{2}N\rho_\mu - \delta_{Z_N}e\bar{N}\gamma^\mu\frac{1+\tau_3}{2}NA_\mu - \\ - \delta_{f_\rho}\frac{e}{2}\rho_{\mu\nu}F^{\mu\nu}.\end{aligned} \quad (8)$$

It follows that we in general need three additional constants δ_{Z_A}, δ_{Z_ρ}, and δ_{f_ρ}, once δ_{Z_π} and δ_{Z_N} are fixed. Finally, the total Lagrangian is the sum[2)]

$$\mathcal{L} = \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \mathcal{L}_{ct}^{(1)} + \mathcal{L}_{ct}^{(2)}. \quad (9)$$

2.3. Contribution to the pion electromagnetic form factor from the σ-model sector

Feynman rules for Lagrangian (9) are obtained according to the standard prescriptions [14]. The counterterm constants can be found by imposing the following constraints on the respective self-energy operators of the pion, sigma-meson, and nucleon:

$$\begin{aligned}\Sigma_\pi(m_\pi^2) = \frac{d}{dp^2}\Sigma_\pi(p^2)\Big|_{p^2=m_\pi^2} = \Sigma_\sigma(m_\sigma^2) = 0, \\ \Sigma_N(\not{p})\Big|_{\not{p}=m_N} = \frac{d}{d\not{p}}\Sigma_N(\not{p})\Big|_{\not{p}=m_N} = 0.\end{aligned} \quad (10)$$

These conditions imply that the respective pole positions of the pion, nucleon, and sigma propagators are located at the physical mass of the pion, nucleon, and sigma. In addition, the residue of the pion and nucleon propagators is unity, ensuring the absence of renormalization for the external pion and nucleon (but not for the external sigma-meson). We also impose the condition $\langle\sigma\rangle$, which is ensured by requiring that the so-called tadpole diagrams vanish. Correspondingly, the tandpole diagrams do not contribute to the quantities calculated below.

In calculation of the loop integrals, we use the dimensional regularization method (see, e.g., [13, Appendix A. 4]. Exploiting conditions (10), we find the constant δ_{Z_π}:

[2)] The mass m_ρ in $\mathcal{L}^{(2)}$ is replaced by $\hat{m}_\rho$.

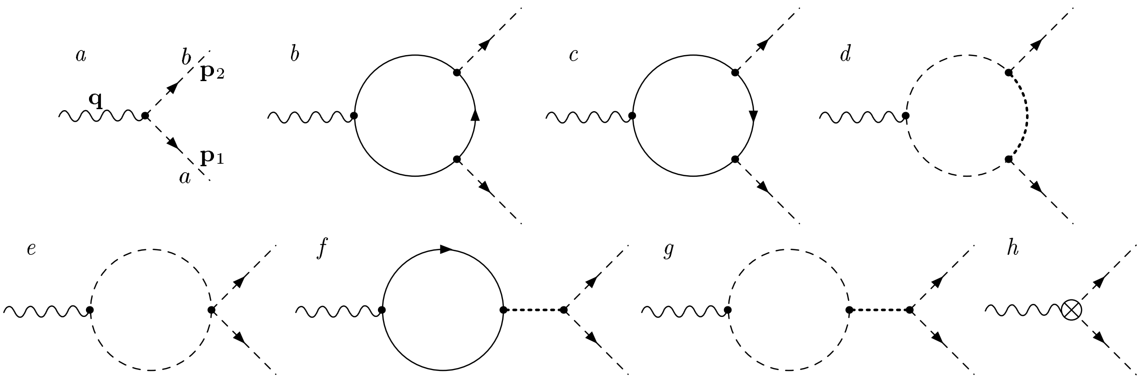

Fig. 1. One-loop diagrams contributing to the pion electromagnetic form factor in the σ-model. Dashed lines depict pion, dotted lines — sigma, solid lines — nucleon, and wavy lines — photon. Small crossed circle denotes the counterterm. Diagram a corresponds to the pion form factor in sQED

$$\delta_{Z_\pi} = -\frac{g_\pi^2}{4\pi^2}\int_0^1 \left[I_\epsilon - \ln\frac{\tilde\Delta_{N\pi}}{\Lambda^2} + \right. \\ \left. + \frac{m_\pi^2 x(1-x)}{\tilde\Delta_{N\pi}} + \frac{\left(m_\sigma^2 - m_\pi^2\right)^2 x(1-x)}{4m_N^2\tilde\Delta_{\pi\sigma}} \right] dx, \quad (11)$$

$$\begin{aligned} \tilde\Delta_{N\pi} &= m_N^2 - m_\pi^2 x(1-x), \\ \tilde\Delta_{\pi\sigma} &= m_\sigma^2 x + m_\pi^2 (1-x)^2. \end{aligned} \quad (12)$$

In these equations,

$$I_\epsilon = \frac{2}{\epsilon} - \gamma_E + \ln 4\pi, \\ \epsilon = 4 - D \to 0,$$

where D is the space–time dimension, $\gamma_E \approx 0.5772$ is the Euler constant, and Λ is an arbitrary scale mass, which drops out in the physical observables.

The one-loop contributions to the pion electromagnetic vertex coming from the σ-model are shown in Fig. 1. Using the isospin structure of the vertices, or negative charge-conjugation parity of the photon, one can show that diagrams e, f, and g vanish. Counterterm h cancels the divergences coming from loop contributions b and c, while contribution d is finite.

In the general case of the off-mass-shell pions, the electromagnetic vertex $\Gamma^\mu_{ab}(p_1,p_2,q)$ for the process

$$\gamma^*(q) \to \pi^a(p_1) + \pi^b(p_2)$$

has the form

$$-i\Gamma^\mu_{ab}(p_1,p_2,q) = \varepsilon^{3ab}\left[F(p_1^2,p_2^2,q^2)(p_2-p_1)^\mu + \right. \\ \left. + G(p_1^2,p_2^2,q^2)(p_2+p_1)^\mu\right], \quad (13)$$

with scalar functions $F(p_1^2,p_2^2,q^2)$ and $G(p_1^2,p_2^2,q^2)$. On the mass shell, $p_1^2 = p_2^2 = m_\pi^2$, the function $G(m_\pi^2,m_\pi^2,q^2)$ drops out, while $F(m_\pi^2,m_\pi^2,q^2)$ becomes equal to the pion form factor $F_\pi(q^2)$. With the loop corrections denoted by $\Delta F^{(\sigma)}(q^2)$, we find

$$F_\pi^{(\sigma)}(q^2) = 1 + \Delta F^{(\sigma)}(q^2) + \delta_{Z_\pi}, \quad (14)$$

where the total correction is finite and is given by

$$\Delta F^{(\sigma)}(q^2) + \delta_{Z_\pi} = \frac{g_\pi^2}{4\pi^2}\left[\int_0^1 \left(\ln\frac{m_N^2 - m_\pi^2 x(1-x)}{m_N^2 - q^2 x(1-x)} - \right.\right. \\ \left. - x(1-x)\left[\frac{m_\pi^2}{\tilde\Delta_{N\pi}} + \frac{\left(m_\sigma^2 - m_\pi^2\right)^2}{4m_N^2\tilde\Delta_{\pi\sigma}}\right]\right) dx + \\ + \int_0^1\int_0^1 \left(\frac{y^2 m_\pi^2}{\Delta_{N\pi}(y,x)} + \right. \\ \left.\left. + \frac{y(1-y)\left(m_\sigma^2 - m_\pi^2\right)^2}{4m_N^2\Delta_{\pi\sigma}(y,x)}\right) dx\,dy\right], \quad (15)$$

$$\begin{aligned} \Delta_{N\pi}(y,x) &= m_N^2 - q^2 y^2 x(1-x) - m_\pi^2 y(1-y), \\ \Delta_{\pi\sigma}(y,x) &= m_\sigma^2 (1-y) - y^2 (q^2 x(1-x) - m_\pi^2), \end{aligned} \quad (16)$$

with $\tilde\Delta_{N\pi}$ and $\tilde\Delta_{\pi\sigma}$ defined in Eq. (12).

2.4. Contribution to the pion electromagnetic form factor from the ρ-meson

The contribution to the pion electromagnetic form factor from the ρ-meson can be written in the compact form

$$F_\pi^{(\rho)}(q^2) = = -\frac{g_\rho(q^2)}{f_\rho(q^2)}\frac{q^2}{q^2-\hat{m}_\rho^2-\Pi_\rho(q^2)} + \Delta F_\pi^{(\rho\omega)}(q^2). \quad (17)$$

This expression includes several effects coming from the loop corrections shown in Fig. 2.

1) The q^2-dependent vertex $g_\rho(q^2)$ describes loop corrections to the $\rho\pi\pi$ coupling that originate from the σ-model (Fig. 2, diagrams a). These corrections have not been included in Ref. [11]. We can write

$$g_\rho(q^2) = g_\rho[1+\Delta F^{(\sigma)}(q^2)+\delta_{Z_\pi}]. \quad (18)$$

It is seen that the expression in the square brackets is the same as in Eq. (14) and is finite. From Eq. (18), we obtain

$$g_\rho(q^2) = g_\rho(m_\rho^2)\frac{1+\Delta F^{(\sigma)}(q^2)+\delta_{Z_\pi}}{1+\Delta F^{(\sigma)}(m_\rho^2)+\delta_{Z_\pi}} \quad (19)$$

in terms of the constant $g_\rho(m_\rho^2)$. From the experimental width of the $\rho\to\pi\pi$ decay, $\Gamma_{\rho\to\pi\pi} = 150.7$ MeV [15], we have $|g_\rho(m_\rho^2)| = 6.05$. To find the real and imaginary part of $g_\rho(m_\rho^2)$, we can use the relations

$$\begin{aligned} &\mathrm{Re}\, g_\rho(m_\rho^2) = \frac{1}{\sqrt{1+\lambda^2}}|g_\rho(m_\rho^2)|, \\ &\mathrm{Im}\, g_\rho(m_\rho^2) = \frac{\lambda}{\sqrt{1+\lambda^2}}|g_\rho(m_\rho^2)|, \\ &\lambda = \mathrm{Im}\,\Delta F^{(\sigma)}(m_\rho^2)[1+\mathrm{Re}\,\Delta F^{(\sigma)}(m_\rho^2)+\delta_{Z_\pi}]^{-1}. \end{aligned} \quad (20)$$

2) The ρ-meson self-energy has the structure

$$\Pi_\rho^{\mu\nu}(q) = (g^{\mu\nu}-q^\mu q^\nu/q^2)\Pi_\rho(q^2)$$

and corresponds to the diagrams shown in Fig. 2b. It leads to the following exact propagator of the ρ-meson (Fig. 2, diagrams c):

$$G_\rho^{\mu\nu}(q) = -i\frac{g^{\mu\nu}-q^\mu q^\nu/q^2}{q^2-\hat{m}_\rho^2-\Pi_\rho(q^2)} + i\frac{q^\mu q^\nu}{q^2\hat{m}_\rho^2}. \quad (21)$$

Calculation of the loop integrals in Fig. 2b results in

$$\begin{aligned} \Pi_\rho(q^2) &= q^2[\omega(q^2)-\delta_{Z_\rho}], \\ \omega(q^2) &= \frac{g_\rho^2}{16\pi^2}[-I_\epsilon + I(q^2)], \end{aligned} \quad (22)$$

$$I(q^2) = 2\int_0^1\left[2x\ln\frac{\Delta_N}{\Lambda^2}+(1-2x)\ln\frac{\Delta_\pi}{\Lambda^2}\right]\times \times(1-x)\,dx, \quad (23)$$

with

$$\Delta_N = m_N^2 - q^2x(1-x), \quad \Delta_\pi = m_\pi^2 - q^2x(1-x).$$

The self-energy has a logarithmic divergence and requires renormalization. The authors of Ref. [11] renormalized the self-energy by applying a dispersion relation with two subtractions. We prefer an alternative method of counterterms, which is expressed in Eq. (22) through the constant δ_{Z_ρ}. We can fix the latter from the constraint on the self-energy at the physical mass m_ρ of the ρ-meson,

$$\frac{d}{dq^2}\mathrm{Re}\,\Pi_\rho(q^2)|_{q^2=m_\rho^2} = 0 \quad \Rightarrow \quad (24)$$

$$\delta_{Z_\rho} = Z_\rho - 1 = \mathrm{Re}\,\omega(m_\rho^2) + m_\rho^2\,\mathrm{Re}\,\omega'(m_\rho^2), \quad (25)$$

where

$$\mathrm{Re}\,\omega'(q^2) \equiv \frac{d}{dq^2}\mathrm{Re}\,\omega(q^2).$$

It is seen from Eqs. (22) and (25) that the self-energy

$$\Pi_\rho(q^2) = q^2[\omega(q^2) - \mathrm{Re}\,\omega(m_\rho^2) - m_\rho^2\,\mathrm{Re}\,\omega'(m_\rho^2)]$$

is finite. Near the physical mass, it has the expansion

$$\mathrm{Re}\,\Pi_\rho(q^2) = -m_\rho^4\,\mathrm{Re}\,\omega'(m_\rho^2) + O((q^2-m_\rho^2)^2), \quad (26)$$

and therefore the coupling g_ρ is not renormalized due to self-energy loops [11]. There is also a finite mass shift

$$m_\rho^2 - \hat{m}_\rho^2 = -m_\rho^4\,\mathrm{Re}\,\omega'(m_\rho^2). \quad (27)$$

For the definition of $\hat{m}_\rho$, see the paragraph before Eq. (8).

Above the two-pion threshold, the self-energy acquires an imaginary part coming from the pion loop (the third diagram in Fig. 2b). Namely, at $q^2 < 4m_N^2$, the imaginary part and the q^2-dependent $\rho\to\pi\pi$ decay width are given by the respective expressions

$$\begin{aligned} &\mathrm{Im}\,\Pi(q^2) = -\frac{g_\rho^2 q^2}{48\pi}\left(1-\frac{4m_\pi^2}{q^2}\right)^{3/2}\theta(q^2-4m_\pi^2), \\ &\Gamma_\rho(q^2) = -\,\mathrm{Im}\,\Pi(q^2)/\sqrt{q^2}. \end{aligned} \quad (28)$$

3) Closely related to the self-energy are loop corrections to the $\gamma\rho$ coupling constant, shown in Fig. 2d. The sum of all contributions is proportional to the tensor $g^{\mu\nu}-q^\mu q^\nu/q^2$, similarly to the tree-level term. Introducing the q^2-dependent vertex, we obtain

$$\frac{1}{f_\rho(q^2)} = \frac{1}{f_\rho} - \frac{\omega(q^2)}{g_\rho} + \delta_{f_\rho}, \quad (29)$$

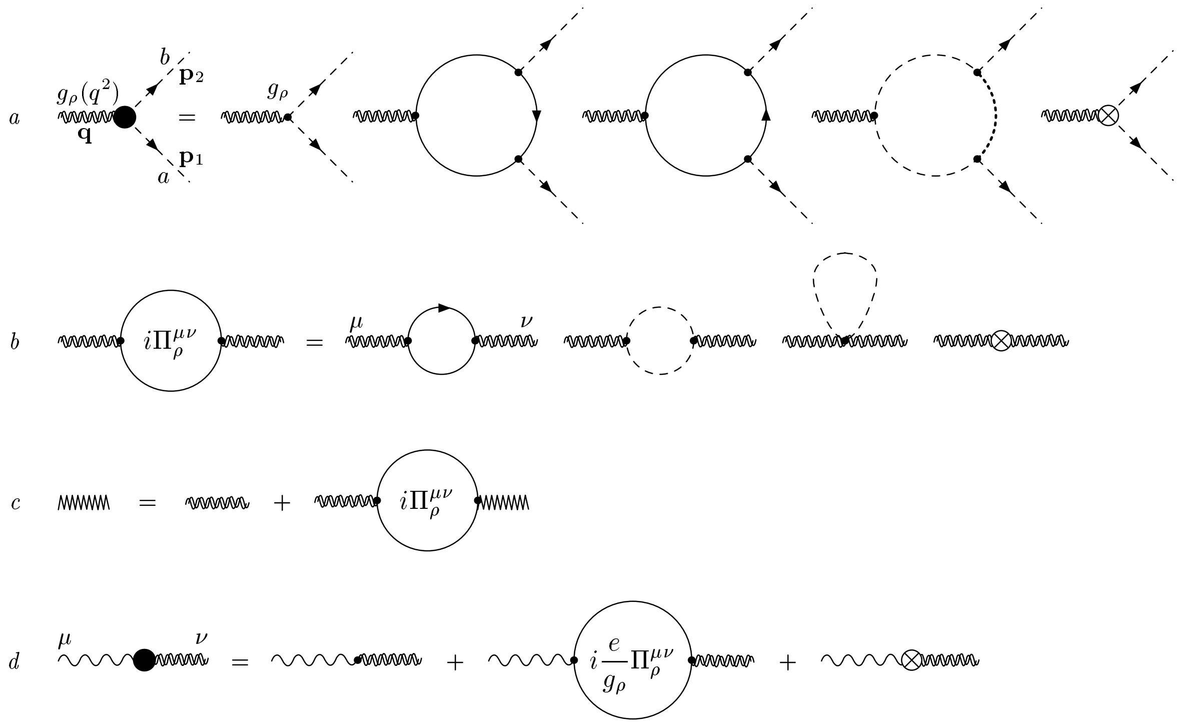

Fig. 2. Diagrams a show π, σ, and N loop corrections to the $\rho\pi\pi$ vertex; diagrams b show the π and N loop corrections to the self-energy of the ρ-meson; diagrams c graphically represent the equation for the exact ρ-meson propagator; diagrams d represent the π and N loop corrections to the $\gamma\rho$ vertex. Double-wavy lines depict the ρ-meson. The corresponding counterterms are indicated by crossed circles

where δ_{f_ρ} can be fixed by requiring that on the mass shell $q^2 = m_\rho^2$, the coupling $f_\rho(m_\rho^2)$ is related to the $\rho \to e^+e^-$ decay width. The experimental width $\Gamma_{\rho\to e^+e^-} = 6.77$ keV [15] is reproduced with $|f_\rho(m_\rho^2)| \approx 5.03$. From Eq. (29), we find

$$\frac{1}{f_\rho(q^2)} = \frac{1}{f_\rho(m_\rho^2)} + \frac{1}{g_\rho}[\omega(m_\rho^2) - \omega(q^2)], \tag{30}$$

where the real part of the constant $f_\rho(m_\rho^2)$ is determined from $|f_\rho(m_\rho^2)|$ and from the imaginary part,

$$\operatorname{Im} f_\rho(m_\rho^2) = |f_\rho(m_\rho^2)|^2 \operatorname{Im}\omega(m_\rho^2)/g_\rho.$$

It is seen from (30) that the effective $\gamma\rho$ vertex is finite. A similar procedure for this vertex was used in [11], although only the real part of $f_\rho(m_\rho^2)$ was taken from experiment.

We also mention that in calculating $\Pi_\rho(q^2)$ and $f_\rho(q^2)$, we used $|g_\rho(m_\rho^2)|$ instead of g_ρ in order to obtain the correct width of the ρ-meson.

4) The last term in Eq. (17) describes the ρ–ω interference due to electromagnetic effects [8]. The explicit form of the contribution to the pion form factor can be taken from Ref. [16],

$$\Delta F_\pi^{(\rho\omega)}(q^2) = -\varepsilon_{\rho\omega}\frac{g_\rho}{f_\omega}\frac{q^2}{q^2 - m_\omega^2 + im_\omega\Gamma_\omega}, \tag{31}$$

$$\varepsilon_{\rho\omega} = \frac{\Pi_{\rho\omega}}{m_\omega^2 - m_\rho^2 - i[m_\omega\Gamma_\omega - m_\rho\Gamma_\rho(q^2)]}, \tag{32}$$

where $\Gamma_\omega = 8.43$ MeV is the full decay width of the ω-meson with mass m_ω, $f_\omega = 17.05$ is the $\gamma\omega$ coupling constant, which is fixed from the $\omega \to e^+e^-$ decay width $\Gamma_{\omega\to e^+e^-} = 0.6$ keV [15], and $\Pi_{\rho\omega} \approx -3.8\cdot 10^{-3}$ GeV^2 is the mixed $\rho - \omega$ self-energy.

3. RESULTS AND DISCUSSION

We first specify the parameters of the model. The constant g_π is determined from the tree-level Goldberger–Treiman relation

$$g_\pi = m_N/f_\pi$$

(the first equation in (4)), while $g_\rho(m_\rho^2)$ and $f_\rho(m_\rho^2)$ are fixed from experiment, as described in Sec. 2.4.

Table 1. Two-pion contribution $a_\mu^{(had,\pi)}$ to the muon anomalous magnetic moment (in units 10^{-11}). The upper integration limit in Eq. (2) is 0.8 GeV^2

sQED	σ-model	sQED + VMD	σ-model + VMD	σ-model + VMD ($f_\rho = g_\rho$)	Ref. [2]
525	753	4667	4763	4745	4774 ± 51

Table 2. Dependence of $a_\mu^{(had,\pi)}$ on the mass of σ-meson

m_σ, GeV	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2
$a_\mu^{(had,\pi)}$, 10^{-11}	4546	4583	4640	4710	4788	4867	4946	5024	5099

The σ mass is chosen equal to the ρ mass, $m_\sigma = m_\rho$, in line with Ref. [17], where σ and ρ are assumed degenerate. Furthermore, $m_\rho = 768.5$ MeV and $m_\omega = 782.57$ MeV [15].

It is interesting to note that calculation of the self-energy of the ρ-meson gives $\hat{m}_\rho = 795$ MeV. This value is rather close to the physical mass m_ρ. In this connection, we note that the authors of [11] fitted $\hat{m}_\rho$ from the $\pi\pi$ scattering and obtained 810 MeV. The difference in the above values of $\hat{m}_\rho$ is partially due to our taking the nucleon loop into account, which was not considered in [11].

As mentioned in Sec. 2.4, both the real and imaginary parts of the coupling constants $g_\rho(m_\rho^2)$ and $f_\rho(m_\rho^2)$ have been included. The calculation yields

$$\operatorname{Re} g_\rho(m_\rho^2) = 6.036, \quad \operatorname{Im} g_\rho(m_\rho^2) = 0.405,$$

$$\operatorname{Re} f_\rho(m_\rho^2) = 4.96, \quad \operatorname{Im} f_\rho(m_\rho^2) = -0.82.$$

Taking the imaginary parts into account leads to a small correction to the results obtained with

$$\operatorname{Im} g_\rho(m_\rho^2) = \operatorname{Im} f_\rho(m_\rho^2) = 0.$$

Our main results are demonstrated in Fig. 3 and Tables 1 and 2. The calculated pion form factor $|F_\pi(q^2)|^2$ for space-like and time-like values of q^2 is presented in Fig. 3. Apparently, the agreement with the data [18] from elastic electron–pion scattering and the data [19] from e^-e^+ annihilation in two pions is quite good. We emphasize that there are no fitting or tuning parameters in our approach.

There is a strong interference of the two contributions, $F_\pi^{(\sigma)}$ and $F_\pi^{(\rho)}$, in the total form factor. In Fig. 3, we show the contribution $F_\pi^{(\sigma)}$ separately. It follows from Eqs. (17) and (18) that the ρ contribution also includes π, σ, and N loops coming from the σ-model. Switching off these corrections, i.e., putting

$$\Delta F^{(\sigma)}(q^2) + \delta_{Z_\pi} = 0,$$

we obtain

$$F_\pi^{(sQED+VMD)}(q^2) = 1 - \frac{g_\rho}{f_\rho(q^2)} \frac{q^2}{q^2 - \hat{m}_\rho^2 - \Pi_\rho(q^2)} + \Delta F_\pi^{(\rho\omega)}(q^2). \quad (33)$$

Here, the first term corresponds to sQED, and the second and the third terms are the ρ-meson contributions. We note that Eq. (33) corresponds to the extended version of VMD in Ref. [11]; in the «standard» VMD model [8], one has the dependence

$$\frac{m_\rho^2}{m_\rho^2 - q^2 - i m_\rho \Gamma_\rho(q^2)}.$$

Our calculation shows that the difference between the form factor calculated in σ-model + VMD (Eqs. (14) and (17)), and that in sQED + VMD (Eq. (33)) is small, and therefore the results for sQED + VMD are not plotted in Fig. 3. Nevertheless, the difference may show up in the integrated quantity $a_\mu^{(had,\pi)}$ for the muon AMM.

The calculated values of $a_\mu^{(had,\pi)}$ are shown in Table 1. In general, loop corrections are important in the σ-model (compare the first and the second columns). Their role is however diminished in the full calculation, which includes the dominant ρ-meson contribution (the third and the fourth columns in Table 1). The difference between σ-model+VMD and sQED + VMD calculations is about 2 %.

In this connection, our result can be used to estimate the size of radiative corrections due to final-state-photon radiation in the $e^+e^- \to \pi^+\pi^-\gamma$ process.

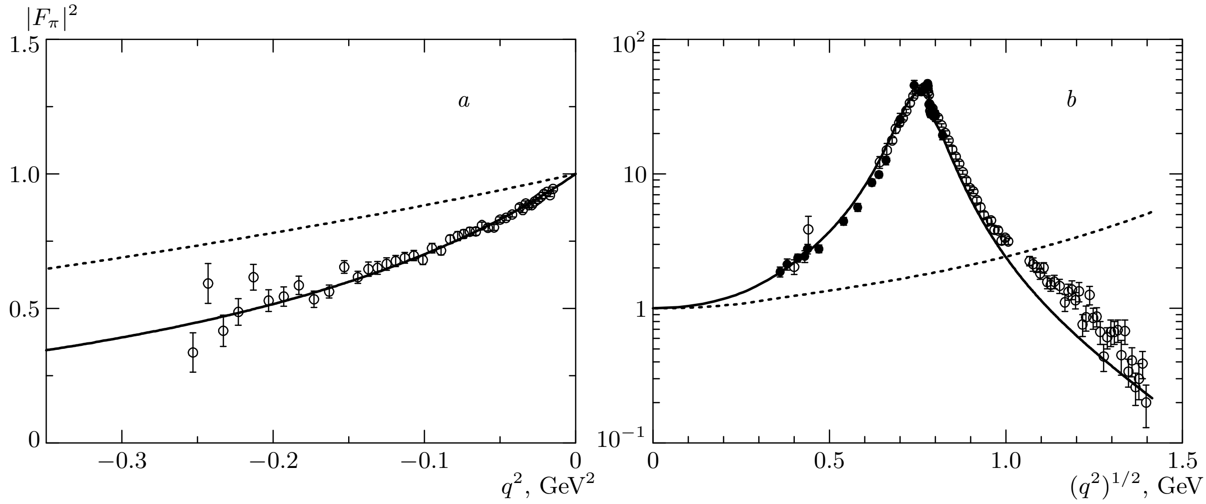

Fig. 3. The pion electromagnetic form factor for space-like q^2 (a) and time-like q^2 (b). Experimental data are from [18] and [19] respectively. Solid lines — σ-model + VMD, dotted lines — σ-model

The corresponding contribution to the muon AMM, $a_\mu^{(had,\pi\gamma)}$, is calculated in [4] in the sQED + VMD framework. In general, the ratio

$$\sigma(e^+e^- \to \pi^+\pi^-\gamma)/\sigma(e^+e^- \to \pi^+\pi^-)$$

is of the order α. We expect that the model dependence of $a_\mu^{(had,\pi\gamma)}$ is similar to that of $a_\mu^{(had,\pi)}$. Therefore, the deviation of $a_\mu^{(had,\pi\gamma)}$ calculated in sQED + VMD from $a_\mu^{(had,\pi\gamma)}$ in a more realistic model, for example, in σ-model + VMD, is about 2%. The overall model-dependent effect in the contribution $a_\mu^{(had,\pi\gamma)}$ to the muon AMM is of the order $\alpha \cdot 2\% \approx 0.015\%$ and is therefore negligible.

In the fifth column, we show the result obtained if we put $f_\rho = g_\rho$ in Lagrangian (6). This approximation corresponds to the full universality of Sakurai. In this case, the renormalization procedure for the $\gamma\rho$ vertex changes and δ_{f_ρ} in Eq. (8) is equal to δ_{Z_ρ}/g_ρ. The numerical results for the form factor and $a_\mu^{(had,\pi)}$, however, change very little, e.g., by about 0.4% for the integral.

As mentioned above, we chose the mass $m_\sigma = m_\rho$ for the σ-meson in the calculation. This particle is associated with the $f_0(400-1200)$-meson in [15]. In view of its undetermined status, we study the dependence of the calculated integral $a_\mu^{(had,\pi)}$ on m_σ in Table 2. As can be seen from Table 2, the integral varies considerably. We take the value 4774 ± 51 from [2] as a very accurate fit to the experimental integral. Then, for the indicated error bars, we obtain the mass of σ in the interval from 720 to 850 MeV. The central value 785 MeV is surprisingly close to the ρ-meson mass. Therefore, our calculation agrees with the hypothesis in Ref. [17] about degeneracy of σ- and ρ-mesons.

In the calculations, we took only the diagrams with the ρ-meson entering on the tree level into account. In particular, the loops with an intermediate ρ-meson for the $\gamma\pi^+\pi^-$ and $\rho\pi^+\pi^-$ vertices are left out. Such contributions can be consistently considered in the models in which the ρ-meson is included together with its chiral partner, the axial-vector a_1-meson, for example, in the so-called gauged σ-model [20] or chiral quantum hadrodynamics [21, 22]. This work requires further investigation.

4. CONCLUSIONS

We developed a model for the electromagnetic vertex of the pion. The model is based on the linear σ-model, which generates the loops with the intermediate pion, sigma, and nucleon. The ρ-meson is included in line with the extended VMD model [11]. The coupling of ρ to the pion and nucleon is introduced through gauge-covariant derivatives, and the direct $\gamma\rho$ coupling has a gauge-invariant form. The ρ-meson self-energy and the modified $\gamma\rho$ vertex are generated by the pion and nucleon loops. The renormalization is consistently performed using the method of counterterms without cut-off parameters.

The pion electromagnetic form factor calculated in the one-loop approximation in the strong interaction is in good agreement with the precise data obtained from elastic $e^-\pi^+$ scattering and e^+e^- annihilation into $\pi^+\pi^-$. The effect of the σ-model loops turns out to be small.

We calculated the contribution of the $e^+e^- \to \to \pi^+\pi^-$ process to the muon AMM, $a_\mu^{(had,\pi)}$. The calculation agrees quite well (by 0.15 %) with the recent very accurate fit in Ref. [2]. The contribution of the σ-model loops to $a_\mu^{(had,\pi)}$ is about 2 %.

We also estimated the size of the model-dependent effects in $a_\mu^{(had,\pi\gamma)}$, the contribution to the muon AMM from final-state-photon radiation in the $e^+e^- \to \pi^+\pi^-\gamma$ process. It is about $\alpha \cdot 2\,\% \approx 0.015\,\%$ and is therefore negligible. Hence, our calculation does not contradict the conclusion in Ref. [4] that the final-state radiative process $e^+e^- \to \pi^+\pi^-\gamma$ can be evaluated in scalar QED supplemented with the VMD model.

The only free parameter of the model, which is not fixed from the experiment, is the σ-meson mass. Comparison with the fit in Ref. [2] strongly indicates that the value of this mass is close to the mass of the ρ-meson. This conclusion is consistent with Ref. [17], where the mesons σ and ρ are assumed to be degenerate.

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