

QUASISTATIC WAVES OF HYDROGRAVITY GENERATED IN THE GALACTIC INTERSTELLAR MEDIUM BY A PULSATING NEUTRON STAR

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Based on principles of classical hydrodynamics and Newtonian gravity, the theory of hydrogravity, formulated in the manner of hydromagnetic theory, is developed to account for the gravitational effect of global pulsations of a star on the motions of the ambient gas–dust interstellar medium. Analytic derivation of the dispersion relation for canonical gravity waves at the free surface of an incompressible inviscid liquid is presented illustrating practical usefulness of the proposed approach, heavily relying on the concept of classical gravitational stress introduced long ago by Fock and Chandrasekhar, and accentuating the shear character of this mode. Particular attention is given to gas-dynamical oscillations of a similar physical nature generated by a pulsating neutron star in an unbounded spherical shell of gas and dust promoted by circumstellar gravitational stresses and damped by viscosity of the interstellar matter. Computed in the long-wavelength approximation, the periods of these gravity-driven shear modes, referred to as quasistatic modes of hydrogravity, are found to be proportional to periods of the gravity modes in the neutron star bulk. Given that collective oscillations of cosmic plasma in the wave under consideration should be accompanied by electromagnetic radiation and taking into account that only the radio waves of this radiation can freely travel through the galactic gas–dust clouds, it is conjectured that the considered effect of gravitational coupling between seismic vibrations of a neutron star and fluctuations of the galactic interstellar medium should manifest itself in the radio range of pulsar spectra. Some useful implications of the theory developed here to a number of current problems of asteroseismology are briefly discussed.

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1. INTRODUCTION

It has been realized long ago that the restless behavior of neutron stars, exhibited in the pulsar spectra by millisecond micropulses, owes its origin to seismic vibrations triggered either by implosion effects of supernova events or by starquakes [1–3], which may be connected with some short gamma-ray bursts [4, 5]. By now, there are tolerably coherent arguments showing

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that neutron stars (both pulsars and magnetars) can support long-lasting pulsations driven by bulk forces of elasticity, gravity, and magnetism of the neutron star matter [6–10]. At the same time, the influence of neutron star pulsations on a gas–dust interstellar medium (ISM), which serves as a fluid matrix mediating a vast variety of gas-dynamical processes (e.g., [11]), remains less studied. This work discusses the hydrodynamic mechanism of gravitational coupling between seismic vibrations of a neutron star and fluctuations of gas–dust flows in the ambient envelope. Specifically, we consider a model in which a pulsating neutron star embedded in a gas–dust spherical shell is regarded as a source of large-scale hydrodynamical wave motions promoted by circumstellar gravitational stresses and damped by viscous stresses. The oscillatory motions in question have the same physical nature as the gravity waves at the free surface of an incompressible viscous fluid caused by the presence of a constant field of Newtonian gravity (e.g., [12–14]), the wave process being well-known in the physics of planetary atmospheres [15].

This paper presents arguments that proper mathematical treatment of these gravity-driven wave motions of interstellar medium can be developed on the basis of self-consistent equations for variables of classical hydrodynamics and Newtonian gravity, which are formulated in a manner of governing equations of the hydromagnetic theory. In pursuit of this aim, we follow two different approaches, both relying on the key concept of Newtonian gravitational stress. The underlying idea of the first method, constituting the content of Secs. 2 and 3, is to include the Newtonian gravitational field in a set of gas-dynamical variables of circumstellar motions by considering this field on an equal footing with the standard hydrodynamical variables such as the density and velocity. The second method, formulated in the Appendix, is based on coupled equations involving the density, the velocity, and the gravitational stress tensor. Particular attention is drawn to the fact that both these methods yield analytically identical estimates for the frequency and lifetime of the gravity modes owing their existence to fluctuations in circumstellar gravitational stresses caused by pulsations of a neutron star. In the discussion, we point out some useful applications of the theory developed here.

2. GOVERNING EQUATIONS OF HYDROGRAVITY

The point of departure in our considerations is the representation of the body force of gravity

$$\mathbf{F} = -\rho\,\mathbf{g}, \quad \nabla\mathbf{g} = 4\pi G\rho, \quad \mathbf{g} = -\nabla U \tag{1}$$

through the tensor of gravitational stresses G_{ik}:

$$\begin{gathered} F_i = -\rho \mathrm{g}_i = -\frac{\partial G_{ik}}{\partial x_k}, \\ G_{ik} = \frac{1}{4\pi G}\left[\mathrm{g}_i\,\mathrm{g}_k - \frac{1}{2}(\mathrm{g}_j\,\mathrm{g}_j)\delta_{ik}\right]. \end{gathered} \tag{2}$$

To the best of our knowledge, this form of the gravitational force in the stationary material continuum of density ρ was first discussed long ago by Fock [17] and justified by Chandrasekhar [18]. Such a possibility is apparent from the identity

$$F_i = \rho\frac{\partial U}{\partial x_i} = -\frac{\partial}{\partial x_k} \times \times \left\{\frac{1}{4\pi G}\left[\frac{\partial U}{\partial x_i}\frac{\partial U}{\partial x_k} - \frac{1}{2}\left(\frac{\partial U}{\partial x_j}\frac{\partial U}{\partial x_j}\right)\delta_{ik}\right]\right\}. \tag{3}$$

Also, the discussion of Newtonian gravitational stresses can be found in [19, 20]. The matter of particular interest for our present discussion is Chandrasekhar's suggestion [18] to incorporate the above tensor representation for the static force of Newtonian gravity in the dynamical description of the gravity-driven motions of an inviscid fluid. Specifically, it is shown in [18] that replacement of the standard expression for the gravitational force,

$$F_i = -\rho \mathrm{g}_i,$$

in the Euler equation for an ideal fluid

$$\rho\frac{dV_i}{dt} = -\frac{\partial P}{\partial x_i} - \rho\mathrm{g}_i, \quad \frac{d}{dt} = \frac{\partial}{\partial t} + V_k\frac{\partial}{\partial x_k}, \tag{4}$$

by the above tensor representation

$$F_i = -\nabla_k G_{ik}$$

allows one to rewrite (4) in the form of a conservation law for the density of linear momentum ρV_i,

$$\frac{\partial(\rho V_i)}{\partial t} = -\frac{\partial \mathcal{P}_{ik}}{\partial x_k}, \quad \mathcal{P}_{ik} = \rho V_i V_k + P\delta_{ik} + G_{ik} \tag{5}$$

where $\mathcal{P}_{ik}$ is the total flux density.

We recall at this point that the key statement in the MHD theory is that the state of motion of a magnetoactive fluid can be uniquely specified by the density $\rho(\mathbf{r},t)$, the flow velocity $\mathbf{V}(\mathbf{r},t)$, and the magnetic flux density $\mathbf{B}(\mathbf{r},t)$, which are regarded on an equal footing as independent dynamical variables (see, e.g., [21–23]). The equations of dissipation-free MHD theory

$$\frac{\partial\rho}{\partial t} = -\frac{\partial\rho V_k}{\partial x_k},$$

$$\rho\frac{dV_i}{dt} = -\frac{\partial}{\partial x_i}\left(P+\frac{B^2}{8\pi}\right)+\frac{\partial}{\partial x_k}\left\{\frac{1}{8\pi}[B_iB_k+B_kB_i]\right\},$$

$$\frac{\partial B_i}{\partial t} = \frac{\partial}{\partial x_k}[V_iB_k - V_kB_i]$$

describe the fluid mechanics of a highly ionized (perfectly conducting) ISM threaded by a galactic magnetic field B.

Remarkably, Eq. (5) permits the equivalent representation

$$\rho\frac{dV_i}{dt} = -\frac{\partial}{\partial x_i}\left(P-\frac{g^2}{8\pi G}\right) - \frac{\partial}{\partial x_k}\left\{\frac{1}{8\pi G}[g_ig_k+g_kg_i]\right\}, \quad (6)$$

which in appearance is similar to the Euler equation of the hydromagnetic model for interstellar gas dynamics. This then indicates that the constructive treatment of the gravity-driven gas dynamics of the interstellar medium can be developed on a methodological footing similar to that lying at the base of magnetohydrodynamics. In particular, this suggests that the gravitational field $\mathbf{g}(\mathbf{r},t)$ can be regarded as an independent variable of the ISM motion on an equal footing with basic variables of interstellar gas dynamics, the density $\rho(\mathbf{r},t)$ and the flow velocity $\mathbf{V}(\mathbf{r},t)$. Then, adhering to this attitude, our next goal is to specify the form of the constitutive equation for the gravity–flow coupling, that is, an equation describing the kinematic relation between the vector field of classical gravity $\mathbf{g}(\mathbf{r},t)$ and the density of linear momentum $\rho(\mathbf{r},t)\mathbf{V}(\mathbf{r},t)$.

It is customarily taken for granted that the time evolution of the density governed by the continuity equation

$$\frac{\partial\rho}{\partial t} = -\frac{\partial(\rho V_k)}{\partial x_k} \quad (7)$$

does not affect the analytic form of the equation for the static gravitational field,

$$\frac{\partial g_k}{\partial x_k} = 4\pi G\rho \to \rho = \frac{1}{4\pi G}\frac{\partial g_k}{\partial x_k}. \quad (8)$$

The partial derivative with respect to time of the left-hand side of Eq. (8) should therefore be equal to the left-hand side of continuity equation (7),

$$\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial x_k}\left(\frac{1}{4\pi G}\frac{\partial g_k}{\partial t}\right) = -\frac{\partial}{\partial x_k}(\rho V_k) \to \to \frac{\partial}{\partial x_k}\left[\frac{\partial g_k}{\partial t} + 4\pi G\rho V_k\right] = 0. \quad (9)$$

It follows that the equation

$$\frac{\partial g_k}{\partial t} = -4\pi G\rho V_k \quad (10)$$

resulting from the last identity is in agreement with both the equation of static gravity and the continuity equation. As postulated by the above arguments, Eq. (10) should be regarded as the constitutive equation for kinematic gravity–flow coupling. This shows that the standard equation for static Newtonian gravity,

$$\nabla_k g_k = 4\pi G\rho,$$

preserves its validity at all times, if valid initially. With all the above reservations in mind, we arrive at self-consistent equations of hydrodynamics and gravity–flow coupling,

$$\frac{d\rho}{dt} + \rho\frac{\partial V_k}{\partial x_k} = 0, \quad (11)$$

$$\rho\frac{dV_i}{dt} + \frac{\partial}{\partial x_i}\left(P-\frac{g^2}{8\pi G}\right) + \frac{\partial}{\partial x_k}\left\{\frac{1}{8\pi G}[g_ig_k+g_kg_i]\right\} = 0, \quad (12)$$

$$\frac{\partial g_k}{\partial t} + 4\pi G\rho V_k = 0, \quad (13)$$

which in what follows are called the equations of hydrogravity for short.

The extension of these equations to the case of a viscous fluid is straightforward,

$$\rho\frac{dV_i}{dt} + \frac{\partial W}{\partial x_i} - \frac{\partial P_{ik}}{\partial x_k} = \frac{\partial\Pi_{ik}}{\partial x_k}, \quad \Pi_{ik} = 2\nu\rho V_{ik}, \quad V_{ik} = \frac{1}{2}\left(\frac{\partial V_i}{\partial x_k}+\frac{\partial V_k}{\partial x_i}\right), \quad (14)$$

$$W = P - \frac{g^2}{8\pi G}, \quad P_{ik} = -\frac{1}{8\pi G}[g_ig_k+g_kg_i], \quad (15)$$

where W is the total pressure and P_{ik} is the anisotropic gravitational stress tensor. We use Π_{ik} to denote the viscous stress tensor, V_{ik} the rate-of-strain tensor, and ν the kinematic viscosity of the gas–dust circumstellar medium.

3. HYDROGRAVITY MODES IN THE STELLAR COCOON MODEL

In the stellar cocoon model under consideration, a neutron star, embedded in a thick dusty shell of supernova debris, is regarded as a solid globe immersed in a spherical fluid matrix. To make the problem analytically tractable, we adopt the uniform density approximation for both the stellar matter ρ_s and the gas–dust medium ρ. It is presumed that the star and spherical dusty envelope are in hydrostatic equilibrium. This means that the characteristic times of the accretion processes are long compared to times of hydrodynamical fluctuations of gas–dust flow in the circumstellar shell. The purpose of this section is to delineate the equilibrium parameters of such an object, to wit, the spatial distribution of the static gravitational field $\mathbf{g}(r)$ and the hydrostatic pressure $P(r)$ in the regions of space relevant to the problem in question. In the absence of the accretion processes, the above characteristics are determined by the equations

$$\nabla \mathbf{g}_s(r) = 4\pi G\rho_s, \quad \nabla P_s(r) = -\rho_s \mathbf{g}_s(r), \quad r < R_s, \tag{16}$$

$$\nabla \mathbf{g}^0(r) = 4\pi G\rho, \quad \nabla P_0(r) = -\rho\, \mathbf{g}^0(r), \quad R_s < r < R. \tag{17}$$

Equations (16) describe the static gravitational field $\mathbf{g}_s$ and hydrostatic pressure P_s in the star bulk, R_s is the star radius, and

$$M_s = (4\pi/3)\rho_s R_s^3$$

is the star mass. Equations (17) determine the static gravitational field $\mathbf{g}^0$ and hydrostatic pressure P_0 in the dusty cocoon of radius R. In the star interior, the solutions of (16) are given by

$$\mathbf{g}_s(r) = \frac{4\pi}{3} G\rho_s \mathbf{r}, \quad P_s(r) = \frac{2\pi}{3}\rho_s^2 G(R_s^2 - r^2), \quad r < R_s. \tag{18}$$

In the stellar envelope, the static gravitational field $\mathbf{g}^0(r)$ and hydrostatic pressure $P_0(r)$ are given by the following solutions of Eqs. (17):

$$\mathbf{g}^0(r) = \frac{4\pi}{3} G\rho \left[1 + \frac{(\rho_s - \rho)}{\rho}\frac{R_s^3}{r^3}\right]\mathbf{r}, \quad R_s < r < R, \tag{19}$$

$$P_0(r) = \frac{2\pi}{3} G\rho^2(R^2 - r^2) + \frac{4\pi}{3} G\frac{R_s^3}{R}\rho(\rho_s - \rho)\frac{R - r}{r}, \quad R_s < r < R. \tag{20}$$

The detailed derivation of this latter equation, in a somewhat different context, can be found in [16]. Hereafter, superscript zero labels the static gravity field in the ambient gas–dust envelope.

3.1. Shear oscillations driven by gravitational stresses

In what follows, we consider small-amplitude disturbances in the dense gas–dust shell of a stellar envelope, generated by seismic vibrations of the central neutron star. We assume that these are not accompanied by fluctuations in density. While the model of incompressible viscous fluid is admittedly a highly idealized approximation, nevertheless we do not expect to lose any essential gas-dynamical features of gravity waves in the interstellar medium on this account. Under such disturbances, the quantities entering the equations of hydrogravity are infinitesimally perturbed as

$$V_i = V_i^0 + v_i, \quad V_i^0 = 0, \quad \mathrm{g}_i = \mathrm{g}_i^0 + \delta\mathrm{g}_i, \quad P = P_0 + p, \tag{21}$$

$$W = W_0 + w, \quad W_0 = P_0 - \frac{\mathrm{g}_0{}^2}{8\pi G}, \quad w = p - \frac{1}{4\pi}\mathrm{g}_k^0\delta\mathrm{g}_k, \tag{22}$$

$$P_{ik} = P_{ik}^0 + t_{ik}, \quad P_{ik}^0 = -\frac{1}{8\pi G}[\mathrm{g}_i^0\mathrm{g}_k^0 + \mathrm{g}_k^0\mathrm{g}_i^0], \quad t_{ik} = -\frac{1}{4\pi G}[\mathrm{g}_i^0\delta\mathrm{g}_k + \mathrm{g}_k^0\delta\mathrm{g}_i], \tag{23}$$

$$\Pi_{ik} = \Pi_{ik}^0 + \pi_{ik}, \quad \Pi_{ik}^0 = 0, \quad \pi_{ik} = 2\nu\rho\, v_{ik}, \quad v_{ik} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right), \tag{24}$$

where g_i^0 and P_0 are determined by Eqs. (19) and (20), respectively. Inserting (21)–(24) in (11)–(13), we arrive at the linearized equations of hydrogravity for heavy incompressible viscous fluid,

$$\rho\frac{\partial v_i}{\partial t} = -\frac{\partial w}{\partial x_i} + \frac{\partial t_{ik}}{\partial x_k} + \frac{\partial \pi_{ik}}{\partial x_k}, \quad \frac{\partial v_k}{\partial x_k} = 0, \tag{25}$$

$$\frac{\partial \delta\mathrm{g}_i}{\partial t} = -4\pi G\rho\, v_i. \tag{26}$$

Scalar multiplication of (25) with v_i and integration over the cocoon volume (ignoring the effects of surface stresses) yields

$$\frac{\partial}{\partial t}\int \frac{\rho\, v^2}{2}\, d\mathcal{V} = -\int [t_{ik} + \pi_{ik}]\, v_{ik}\, d\mathcal{V}. \tag{27}$$

This equation gives the rate at which the kinetic energy of the gas-dynamical motions changes. The most important point for our present discussion is that gravitational forces in the volume of the dusty shell do work, which is characterized by the inseparable link between anisotropic gravitational stresses and shear fluctuations of material flow, $t_{ik}v_{ik}$. This suggests that gravitational stresses impart to the ambient gas–dust matter a portion of mechanical rigidity generic to viscoelastic materials whose response to an external disturbance is accompanied by shear fluctuations.

In appearance, Eq. (27) is similar to the equation of energy balance for shear response of an isotropic viscoelastic material continuum,

$$\frac{\partial}{\partial t}\int \frac{\rho \dot{u}^2}{2} d\mathcal{V} = -\int [\sigma_{ik} + \pi_{ik}]\dot{u}_{ik} d\mathcal{V}, \quad \sigma_{ik} = 2\mu\, u_{ik},$$

$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right), \quad \frac{\partial u_k}{\partial x_k} = 0,$$

where u_i is the material displacement field (related to the velocity as $v_i = \dot{u}_i$, which implies $v_{ik} = \dot{u}_{ik}$ for the rate of strains), σ_{ik} is the Hookean shear stress, u_{ik} is the shear strain tensor, and μ is the shear modulus. This last equation is obtained by scalar multiplication with $\dot{u}_i$ of the basic equation of continuum mechanics of viscoelastic incompressible matter,

$$\rho \ddot{u}_i = \nabla_k\, \sigma_{ik} + \nabla_k \pi_{ik},$$

followed by integration over the volume.

It is expected, therefore, that a pulsating neutron star is to generate shear oscillations of gas–dust matter in the circumstellar envelope. Clearly, the only way of exploring this statement is to evaluate the frequency and lifetime of the mode promoted by gravitational stresses. In doing this, we use the energy variational principle. The procedure is as follows. The first step is to use the separable representation for fluctuating variables

$$\begin{gathered} v_i(\mathbf{r}, t) = a_i(\mathbf{r})\, \dot{\alpha}(t), \quad v_{ik}(\mathbf{r}, t) = a_{ik}\,(\mathbf{r})\, \dot{\alpha}(t), \\ a_{ik} = \frac{1}{2}[\nabla_i a_k + \nabla_k a_i]. \end{gathered} \tag{28}$$

Substituting this form for v_i in (26) and eliminating the time derivative, we obtain

$$\delta \mathrm{g}_i(\mathbf{r}, t) = -4\pi\, G\, \rho a_i(\mathbf{r})\, \alpha(t). \tag{29}$$

The analogous separable forms for tensors of gravitational t_{ik} and viscous π_{ik} stresses are

$$\begin{gathered} t_{ik}(\mathbf{r}, t) = \tau_{ik}(\mathbf{r})\alpha(t), \\ \tau_{ik}(\mathbf{r}) = \rho[\mathrm{g}_i^0(\mathbf{r}) a_k(\mathbf{r}) + \mathrm{g}_k^0(\mathbf{r}) a_i(\mathbf{r})], \\ \pi_{ik} = 2\,\rho\nu\, a_{ik}\, \dot{\alpha}. \end{gathered} \tag{30}$$

Hereafter, $a_i(\mathbf{r})$ is the instantaneous displacement field and $\alpha(t)$ is the temporal amplitude of the oscillations. Inserting (28) and (30) in (27), we arrive at the equation governing the time evolution of $\alpha(t)$ having the form of the standard equation for a damped harmonic oscillator,

$$M\ddot{\alpha}(t) + D\dot{\alpha}(t) + K\alpha(t) = 0, \tag{31}$$

where the parameters (inertia M, stiffness K, and viscous friction D) are given by

$$\begin{gathered} M = \int \rho\, a_i(\mathbf{r})\, a_i(\mathbf{r})\, d\mathcal{V}, \\ D = \frac{1}{2}\int \rho\nu[\nabla_i a_k + \nabla_k a_i]\,[\nabla_i a_k + \nabla_k a_i]\, d\mathcal{V}, \end{gathered} \tag{32}$$

$$K = \frac{1}{2}\int \rho[\mathrm{g}_i^0 a_k + \mathrm{g}_k^0 a_i]\,[\nabla_i a_k + \nabla_k a_i]\, d\mathcal{V}. \tag{33}$$

The well-known solution of (31) is

$$\alpha(t) = \alpha_0 \exp(-t/\tau) \cos\omega t,$$

where

$$\omega^2 = \omega_0^2[1 - (\omega_0\tau)^{-2}], \quad \omega_0^2 = \frac{K}{M}, \quad \tau = \frac{2M}{D}. \tag{34}$$

Here, ω_0 stands for the frequency of nondissipative free oscillations and τ is the viscous damping time. Long-lasting oscillations exist if $\omega_0\tau \gg 1$. Thus, to evaluate the frequency and lifetime of the gravity modes in the spherical gas–dust nebula surrounding a pulsating neutron star, we must specify the form of the instantaneous displacement $\mathbf{a}(\mathbf{r})$, which is solenoidal in view of our adopted approximation of incompressible fluid. In doing this, we consider the quasistatic regime of wave motions of a spherical envelope generated by seismic vibrations of a neutron star. In the case of quasistatic waves, the velocity field $\mathbf{v}(\mathbf{r}, t)$ is determined by solutions of the Laplace equation

$$\nabla^2\mathbf{v}(\mathbf{r}, t) = 0,$$

which is regarded as the long-wavelength limit of the Helmholtz equation of standing waves

$$\nabla^2\mathbf{v}(\mathbf{r}, t) + k^2\mathbf{v}(\mathbf{r}, t) = 0,$$

since in the limit of extremely long wavelengths ($\lambda \to \infty$), the wave vector $k = 2\pi/\lambda \to 0$. In view of the relation

$$\mathbf{v} = \mathbf{a}(\mathbf{r})\dot{\alpha}(t),$$

the instantaneous displacement field $\mathbf{a}(\mathbf{r})$ satisfies the equations

$$\nabla^2\mathbf{a}(\mathbf{r}) = 0, \quad \nabla\mathbf{a}(\mathbf{r}) = 0. \tag{35}$$

The poloidal solution of (35),

$$\mathbf{a}_s(\mathbf{r}) = (N/l)[\nabla \times [\nabla \times \mathbf{r}\, r^{-\ell-1} P_\ell(\mu)]] = \\ = -N\,\nabla\, r^{-(\ell+1)}\, P_\ell(\mu), \quad \mu = \cos\theta, \quad (36)$$

describes a spheroidal quasistatic wave (the long-wavelength limit of a standing wave) in the circumstellar envelope. Here, $P_\ell(\mu)$ is the Legendre polynomial of degree ℓ; the spherical polar coordinate system with fixed polar axis is used. This irrotational ($\nabla \times \mathbf{a}_s = 0$) field of displacement is generated by a spheroidally pulsating neutron star.

A neutron star executing global torsional vibrations produces fluctuations of gas–dust flows of substantially differentially-rotational character. The field of material displacements in this kind of oscillatory motions of the circumstellar shell is given by the toroidal vector field

$$\mathbf{a}_t(\mathbf{r}) = [\boldsymbol{\phi}(\mathbf{r}) \times \mathbf{r}], \quad \boldsymbol{\phi} = N\nabla\,(r^{-(\ell+1)}\,P_\ell(\mu)). \quad (37)$$

This field is also the general solution of (35). It is noteworthy that the parameters M, K, and D depend on arbitrary constant N as N^2, and hence ω_0 and τ are independent of the specific form of N.

3.2. Spheroidal hydrogravity mode

We assume that R, the radius of the circumstellar cloud, is much larger than the star radius R_s:

$$R_s/R \ll 1.$$

Therefore, the limits of integration along the radial variable r can be taken from the surface of the star, $r = R_s$, to the outer surface of the gas–dust shell removed to infinity, $r = R \to \infty$; a neutron star looks like an oscillating blob immersed in a spherical gas–dust matrix of infinitely large radius. For the parameters of inertia $M_s(\ell)$, internal friction $D_s(\ell)$, and the lifetime $\tau_s(\ell)$ of a spheroidal g-mode, computed as functions of the multipole degree of oscillations (ℓ), we obtain

$$\begin{aligned} M_s(\ell) &= 4\pi\rho\frac{N^2}{R_s^{2\ell+1}}\,\frac{\ell+1}{2\ell+1}, \\ D_s(\ell) &= 8\pi\eta\frac{N^2}{R_s^{2\ell+3}}\,\frac{\ell+1}{\ell+2}, \end{aligned} \quad (38)$$

$$\begin{aligned} \tau_s(\ell) &= \tau\,[(2\ell+1)(\ell+2)]^{-1}, \\ \tau &= \frac{R_s^2}{\nu}, \quad \nu = \frac{\eta}{\rho}, \end{aligned} \quad (39)$$

where η is the dynamical viscosity of the interstellar medium, τ is the time constant of exponential time decay due to viscous dissipation of the oscillatory motions. In somewhat different context, this last expression for $\tau_s(\ell)$ has first been established by Lamb [12]. More laborious are calculations of the restoring force parameter K of hydrogravity whose analytic form is given by expression (33). We omit discussion of the tedious integration procedure and only note that these calculations are appreciably facilitated by use of Maple symbolic algebra software. As a result, we obtain

$$K_s(\ell) = \frac{16\pi^2}{3}\,N^2\frac{G\,\rho^2}{R_s^{2\ell+1}}(\ell+1) \times \\ \times \left[\frac{2(\ell+2)}{2\ell+1} + \frac{\rho_s-\rho}{\rho}\right]. \quad (40)$$

Figure 1 illustrates circumstellar gravitational stresses generated by spheroidal quadrupole vibrations of a neutron star. The frequency of undamped spheroidal modes of hydrogravity in the galactic ISM is given by

$$\omega_{0s}^2(\ell) = \frac{4\pi}{3}G\,\rho\,(2\ell+1)\left[\frac{2(\ell+2)}{2\ell+1} + \frac{\rho_s-\rho}{\rho}\right]. \quad (41)$$

Because the density of the star ρ_s is much greater than that of the ambient interstellar medium ρ, that is, $\rho/\rho_s \ll 1$, the last formula (41) can be replaced by

$$\omega_{0s}^2(\ell) = \omega_G^2\,(2\ell+1), \quad \omega_G^2 = \frac{4\pi}{3}G\,\rho_s = \frac{GM_s}{R_s^3}, \quad (42)$$

where M_s and R_s are the mass and radius of the neutron star and ω_G is the natural unit of frequency of g-modes in the star bulk. This result is perhaps the most striking outcome of the considered models, which shows that the frequency of the hydrogravity mode in the ISM is independent of the density of the galactic interstellar matter. Formula (42) can be compared with that for the frequency of nonradial spheroidal g-modes in the neutron star bulk computed in [24] as a function of the multipole degree ℓ,

$$\omega_s^2({}_0G_\ell) = \omega_G^2\,[2(\ell-1)]. \quad (43)$$

We see that in the limit of very high overtones, $\ell \gg 1$, the frequency of spheroidal g-modes in the ISM coincides with that for g-modes in the neutron star bulk,

$$\omega_{0s}(\ell) = \omega_s({}_0G_\ell).$$

3.3. Toroidal hydrogravity mode

The inertia $M_t(\ell)$, internal friction $D_t(\ell)$, and lifetime $\tau_t(\ell)$ as functions of the multipole degree ℓ of the

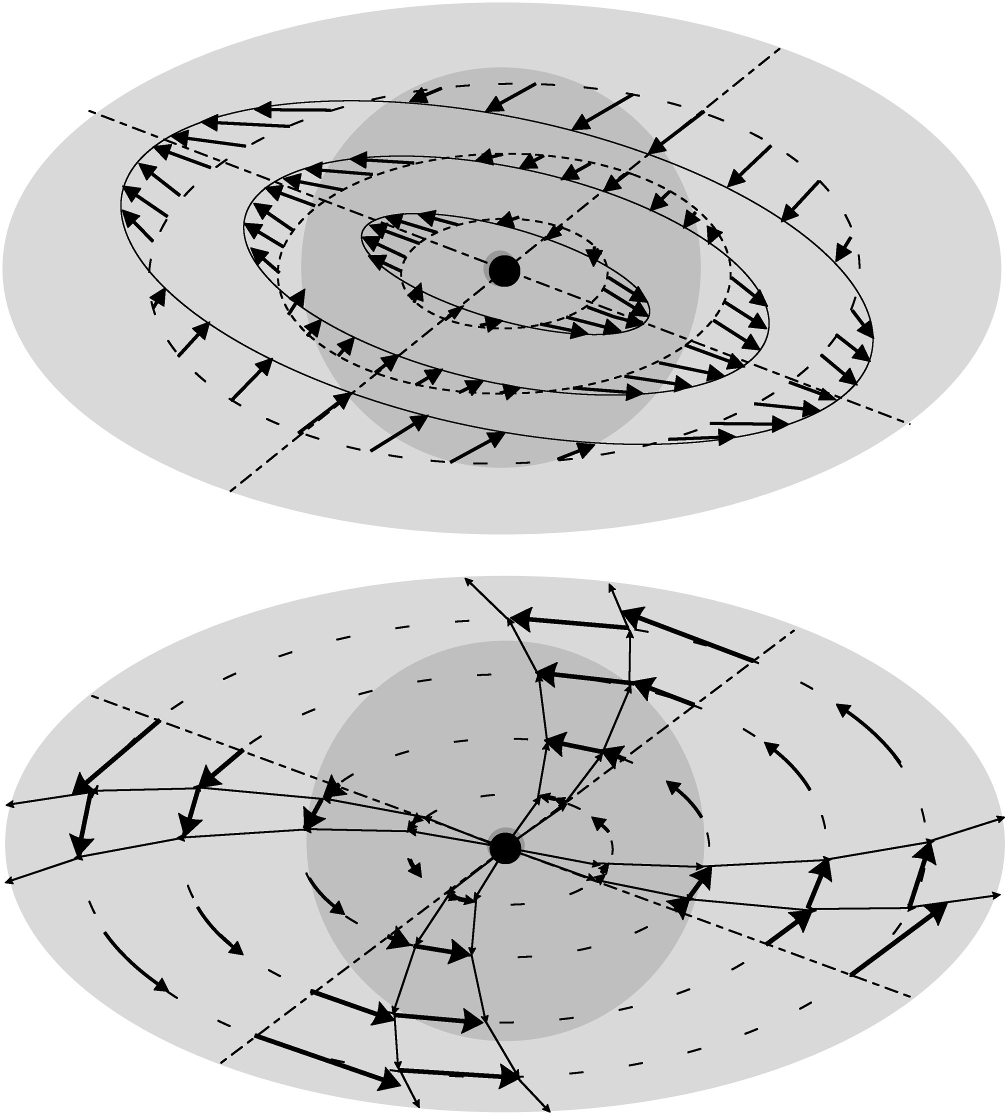

Fig. 1. Geometric illustration of material displacements in circumstellar hydrogravity waves generated by spheroidally (top picture) and torsionally (bottom picture) oscillating neutron star

toroidal hydrogravity mode are given by

$$M_t(\ell) = N^2 \frac{4\pi\rho}{R_s^{2\ell-1}} \frac{\ell(\ell+1)}{(2\ell+1)(2\ell-1)},$$
$$D_t(\ell) = N^2 \frac{4\pi\eta}{R_s^{2\ell+1}} \frac{\ell(\ell+1)(\ell+2)}{(2\ell+1)}, \tag{44}$$

$$\tau_t(\ell) = 2\tau \left[(2\ell-1)(\ell+2)\right]^{-1},$$
$$\tau = \frac{R_s^2}{\nu}, \quad \nu = \frac{\eta}{\rho}. \tag{45}$$

The obtained analytic estimate (45) for the time of viscous relaxation of rotational oscillations in the circumstellar envelope (pictured in the lower part of Fig. 1) has the same practical usefulness as Lamb's formula (39). It follows that high-multipole gravity modes decay faster then low-multipole ones, and this conclusion is independent of the adopted approximation of incompressible matter. A similar conclusion has been derived in [26] for the decay time of the toroidal mode in the neutron star bulk. In a cosmic hydrogen plasma,

$$\rho\nu = 2.2 \cdot 10^{-15} T^{5/2} / \ln\Lambda.$$

The numerical value of the factor $\Lambda \approx$ 10–15 and $\rho \approx 10^{-24}$ g$\cdot$cm^{-3} [15]. For $\rho\nu = 10^{-15}$ cm$^2\cdot$s, the time of viscous relaxation $\tau_\nu \approx 10^3$ s; for $\rho\nu = 10^{-20}$ cm$^2\cdot$s, the lifetime of interstellar mode of hydrogravity is of the order of 10 years. Another way of computing the time of viscous dissipation is based on the formula

$$\eta = nk_BT/\nu$$

(where ν is the collision frequency in the gas–dust interstellar matter, n is the particle density, and T is the temperature), which leads to lifetimes from 10^2 to 10^6 years. All this leaves no doubt about the existence of the hydrogravity mode in the galactic interstellar medium.

The restoring force parameter $K_t(\ell)$ is given by

$$K_t(\ell) = \frac{8\pi^2}{3}N^2\frac{G\rho^2}{R_s^{2\ell-1}}\frac{\ell(\ell+2)}{2\ell+1}\times \left[\frac{2(\ell+1)}{2\ell-1}+\frac{\rho_s-\rho}{\rho}\right]. \quad (46)$$

The frequency of the dissipationless toroidal hydrogravity mode in the ambient ISM is given by

$$\omega_{0t}^2(\ell) = \frac{2\pi}{3}G\rho\frac{(2\ell-1)(\ell+2)}{\ell+1}\times \left[\frac{2(\ell+1)}{2\ell-1}+\frac{\rho_s-\rho}{\rho}\right]. \quad (47)$$

In the natural limit $\rho/\rho_s \ll 1$, the latter formula is replaced by

$$\omega_{0t}^2(\ell) = \omega_G^2\frac{(2\ell-1)(\ell+2)}{2(\ell+1)}, \quad (48)$$

where ω_G stands for the frequency of g-modes in the neutron star bulk. This equation again highlights the fact that the frequency of the hydrogravity mode is independent of the material properties of the galactic interstellar matter. For the sake of comparison, the frequency of the torsional g-modes in the neutron star is given by [25]

$$\omega_t^2({}_0G_\ell) = \omega_G^2(\ell-1). \quad (49)$$

We see that in the limit $\ell \gg 1$, we have $\omega_{0t}(\ell) = \omega_t({}_0G_\ell)$. The difference between periods of hydrogravity modes in the ISM and g-modes in the neutron star bulk is illustrated in Fig. 2. Our expectation that considered kind of interstellar motions can be detected in an electromagnetic signal from a pulsating neutron star rests on the plausible assumption that collective oscillations of charged species in the interstellar gravity waves should be accompanied by emission of electromagnetic waves. That the radio range of such an emission is not extinguished by the gas–dust cloud of the ISM suggests that the interstellar gravity waves

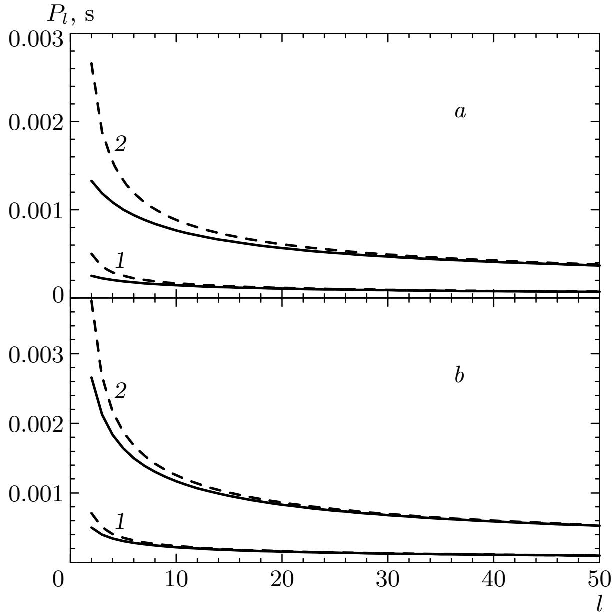

Fig. 2. Period P_ℓ, in seconds, as a function of multipole degree ℓ of spheroidal (a) and torsional (b) hydrogravity modes in the circumstellar envelope (solid line) and the corresponding g-modes in the neutron star bulk (dashed line) computed for two models of neutron stars. *1* — $M_s = 1.3M_\odot$, $R_s = 13$ km; *2* — $M_s = 0.1M_\odot$, $R_s = 18$ km

could manifest themselves by a periodic radio signal whose timing is determined by the frequency of the gravity mode in the central star. Also, it is noteworthy that in [27], it is shown that accounting for electromagnetic processes around a torsionally oscillating neutron star can provide understanding some peculiarities of gamma-ray bursts.

3.4. Canonical gravity waves from equations of hydrogravity

In this subsection, we show that the approach suggested above regains the well-known results of the classical fluid-dynamic theory regarding the dispersion relation for the free surface gravity waves in an incompressible inviscid liquid caused by the presence of a constant gravitational field (see, e.g., [12–14]). This is interesting in its own right because the developed treatment discloses the fact that the classical gravity waves are of a substantially shear character. From the energy balance equation (27) obtained above, it follows that the dissipative free fluctuations of an incompressible liquid promoted by anisotropic Newtonian gravitational stresses are governed by equations of the form

$$\rho\dot{v}_i = -(4\pi G)^{-1}\nabla_k[\mathrm{g}_i^0\delta\mathrm{g}_k + \mathrm{g}_k^0\delta\mathrm{g}_i], \quad \delta\dot{\mathrm{g}}_i = -4\pi G\rho\, v_i. \tag{50}$$

These equations should be supplemented by the incompressibility condition

$$\nabla\mathbf{v} = 0,$$

which implies that

$$\nabla\delta\mathbf{g} = 0.$$

Adhering to the treatment of hydrodynamic gravity waves in an incompressible fluid of infinite depth, given in [14], we take the plane $z = 0$ as corresponding to the equilibrium fluid surface. In this case, the constant gravitational field $\mathbf{g}^0$ has the components

$$\mathbf{g}^0 = [\mathrm{g}_x = 0, \quad \mathrm{g}_y = 0, \quad \mathrm{g}_z = -\mathrm{g}] = \mathrm{const}. \tag{51}$$

The motions are restricted to the xz plane, which means that the fluctuating field of the velocity is a function of x and z and therefore has just two nonzero components

$$v_x = v_x(x,z), \quad v_y = 0, \quad v_z = v_z(x,z). \tag{52}$$

Given (51) and (52), in the Cartesian coordinates, Eqs. (50) break up into the set of noncombining equations

$$\rho\frac{\partial v_x}{\partial t} = \frac{\mathrm{g}}{4\pi G}\frac{\partial\delta\mathrm{g}_x}{\partial z}, \quad \frac{\partial\delta\mathrm{g}_x}{\partial t} = -4\pi G\rho v_x, \tag{53}$$

$$\rho\frac{\partial v_z}{\partial t} = \frac{\mathrm{g}}{4\pi G}\frac{\partial\delta\mathrm{g}_z}{\partial z}, \quad \frac{\partial\delta\mathrm{g}_z}{\partial t} = -4\pi G\rho v_z, \tag{54}$$

exhibiting strong coupling between fluctuations in the velocity of hydrodynamical flow and gravitational field. Taking time derivative in Eqs. (53) and (54), we find that the resultant equations can easily be combined to give identical equations for each fluctuating variable,

$$\frac{\partial^2 v_x}{\partial t^2} = -\mathrm{g}\frac{\partial v_x}{\partial z}, \quad \frac{\partial^2\delta\mathrm{g}_x}{\partial t^2} = -\mathrm{g}\frac{\partial\delta\mathrm{g}_x}{\partial z}, \tag{55}$$

$$\frac{\partial^2 v_z}{\partial t^2} = -\mathrm{g}\frac{\partial v_z}{\partial z}, \quad \frac{\partial^2\delta\mathrm{g}_z}{\partial t^2} = -\mathrm{g}\frac{\partial\delta\mathrm{g}_z}{\partial z}. \tag{56}$$

We consider an incompressible liquid with a free surface. By «free gravity wave» we understand a disturbance traveling in this liquid whose amplitude is exponentially decreasing toward the depth, i.e., as $z \to -\infty$, and for which the velocity components are described by

$$v_x = -Ake^{kz}\sin(kx - \omega t), \quad v_y = 0, \quad v_z = Ake^{kz}\cos(kx - \omega t). \tag{57}$$

The components of the fluctuating gravity field in this wave have the form

$$\delta\mathrm{g}_x = \frac{4\pi\rho}{\omega}v_z, \quad \delta\mathrm{g}_y = 0, \quad \delta\mathrm{g}_z = -\frac{4\pi\rho}{\omega}v_x,$$

where v_x and v_z are given by expressions (57). Substituting (57) in (55), we arrive at the well-known dispersion relation of a classical free surface gravity wave in an incompressible liquid of infinite depth [14, 15],

$$\omega = \sqrt{\mathrm{g}k}, \quad \mathrm{V}_G = \frac{\partial\omega}{\partial k} = \frac{1}{2}\sqrt{\frac{\mathrm{g}}{k}}, \tag{58}$$

where V_G is the group velocity of the gravity wave. In this wave, the velocity vector $\mathbf{v}$, at any fixed value of the depth coordinate z in the xz plane, undergoes a uniform rotation in this plane preserving its magnitude. This is clearly seen by representing the flow velocity as

$$\mathbf{v} = [\nabla\times(\mathbf{e}_y f)] = [\nabla f\times\mathbf{e}_y], \quad f = Ae^{kz}\sin(kx - \omega t), \tag{59}$$

where $\mathbf{e}_y$ is the unit vector in the positive direction of the y axis around which, at a fixed value of z, the velocity vector $\mathbf{v}$ executes uniform rotation. However, it should be clearly realized that this rotation has nothing to do with vorticity of the fluid flow, since the vector field of vorticity

$$\mathbf{\Omega} = \nabla\times\mathbf{v} = 0.$$

On the other hand, the requirements

$$\nabla\cdot\mathbf{v} = 0, \quad \nabla\times\mathbf{v} = 0 \tag{60}$$

imply that $\mathbf{v}$ can be represented as the gradient of a scalar function ϕ

$$\mathbf{v} = \nabla\phi, \quad \phi = Ae^{kz}\cos(kx - \omega t). \tag{61}$$

It can be verified that the velocity field in the surface gravity wave $\mathbf{v}$, Eq. (57), obeys the equation

$$\nabla^2\mathbf{v} = 0$$

as well. It is this fact that has been used as a guide in the above adopted classification of the gravity modes in a spherical circumstellar shell as spheroidal and toroidal modes in which the fields of displacements are described by two general solutions of the vector Laplace equation. Thus, the proposed equations of hydrogravity provide a proper account of the canonical gravity waves by accentuating the shear character of this mode. Essentially, this means that gravitational stress endows an incompressible fluid with the mechanical properties typical of viscoelastic materials capable of transmitting shear waves.

4. DISCUSSION

An understanding of physical mechanisms governing the large-scale motions of galactic interstellar medium brought about by seismic vibrations of stars is important in two areas of current astrophysics: asteroseismology and interstellar gas dynamics. In this work, we have investigated the wave motions of galactic interstellar medium promoted by circumstellar gravitational fields of pulsating stars. In doing this, we have set up self-consistent equations of hydrogravity having in appearance some features in common with those lying at the base of the hydromagnetic theory. By examining potential capabilities of such an approach, heavily relayed on the concept of Newtonian gravitational stresses, we have shown that the proposed theory regains the dispersion equation for the canonical gravity waves traveling near the surface of an incompressible inviscid liquid of infinite depth, the wave process being well-known in the theoretical oceanology and physics of planetary atmospheres. Newly highlighted here is the shear character of oscillating flows in this wave, owing its origin to fluctuations of Newtonian gravitational stresses.

Based on this and working from the homogeneous model of a spherical stellar cocoon (a star surrounded by an extended spherical shell of gas–dust medium), we apply the proposed theory of hydrogravity to analysis of the small-amplitude gravity modes generated in the interstellar medium by a neutron star executing spheroidal and torsional vibrations in quiescent, presumably post-starquake, regime. In presented calculations (carried out by two constructively different operational tools), the approximation of incompressible viscous fluid has been adopted. This implies that disturbances outgoing from a pulsating neutron star lead to weak perturbations accompanied by coupled fluctuations of the velocity and gravitational field, whereas the equilibrium density and hydrostatic pressure in the ambient gas–dust shell remain unaffected. Clearly, such an approximation is unwarranted for violent starquakes generating the shock and compressional waves. The extension of the proposed theory to the case of these latter waves requires special investigation, which is out of our present discussion. The practical usefulness of the considered, admittedly idealized, model is that it allows one to attain conclusive inferences regarding the dependence of period and lifetime of considered modes upon characteristic parameters of both a pulsating star and surrounding gas–dust interstellar matter and the multipole degree of oscillations as well. From the physical side, the finding of particular interest is that the frequency of these weakly attenuated modes of hydrogravity in an unbounded dusty envelope is proportional to the frequency of the g-mode in the neutron star bulk. The corresponding period falls in the interval from 0.1 to 20 milliseconds. This inference is in agreement with the Boriakoff–Van Horn conjecture [2, 3] that micropulses of millisecond duration clearly discernible in the windows of the main pulse train owe their existence to pulsations of neutron stars. Together with this, it seems fairly plausible that unpredictable irregularities and perturbations in the ISM mediating the considered waves of hydrogravity should substantially affect the coherency of these micropulses, as extensively discussed in [28].

While the developed theory is presented in the context of neutron star pulsations, it is hoped that the theoretical predictions inferred here can find useful applications to another area of astroseismology. In this connection, it is important that considered mechanisms of gravitational coupling between small-amplitude vibrations of a star and a stellar envelope presume that collective oscillations of charged particles (forming circumstellar plasma) in the quasistatic wave of hydrogravity are accompanied by electromagnetic radiation whose frequency coincides with that for the gravity mode in the star bulk. It is expected, therefore, that such a radiation can be observed in the vicinity of any star surrounded by interstellar plasma and executing small-amplitude nonradial vibrations driven by self-gravity. In this case, the considered quasistatic waves of hydrogravity can exist in the solar envelope, provided the Sun undergoes global nonradial gravity-driven vibrations of small amplitude with the frequencies proportional to the basic frequency of g-mode ω_G. Such an attitude sheds some new light on the known problem of helioseismology [29] regarding 160-minute variability discovered long ago in solar observations [30, 31], which has been interpreted as a manifestation of the gravity-driven vibrations of the Sun (see also [32]). However, in the subsequent years, the authenticity of solar origin of this signal has been the subject of controversy (e.g. [33]). In recent work [34] advocating the solar origin of this signal, it is argued that this variability cannot be ascribed to some terrestrial cause or to an artifact of the data reduction procedure. Notwithstanding the fact that further measurements (preferably with the use of satellite-based telescopes) are needed to attain more definite statements regarding the very source and physical nature of this intriguing signal, we conclude that predictions of the theory developed in this work are in line with the hypothesis about helioseismic origin of this phenomenon.

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APPENDIX

In this Appendix, we show that the results obtained above can be derived from a different mathematical footing. The basic idea of this method is to use the gravitational stress tensor G_{ik} as the dynamical variable of motions together with the density ρ and the velocity field V_i, whose evolution is governed by coupled equations of the form

$$\frac{d\rho}{dt} + \rho\frac{\partial V_k}{\partial x_k} = 0, \qquad (62)$$

$$\rho\frac{dV_i}{dt} + \frac{\partial P}{\partial x_i} - \frac{\partial G_{ik}}{\partial x_k} = \frac{\partial \Pi_{ik}}{\partial x_k}, \qquad (63)$$

$$\frac{dG_{ik}}{dt} + G_{ij}\frac{\partial V_k}{\partial x_j} + G_{kj}\frac{\partial V_i}{\partial x_j} + G_{ik}\frac{\partial V_j}{\partial x_j} = 0. \qquad (64)$$

Such an approach has been used in [8, 9] to compute the gravity modes in the neutron star bulk.

As in the previous section, we focus on disturbances of the gas–dust cocoon triggered by seismic vibrations of a neutron star that are not accompanied by fluctuations in density but solely in the velocity, pressure, and gravitational stresses,

$$\begin{aligned} V_k(\mathbf{r},t) &= V_k^0(0) + v_k(\mathbf{r},t),\\ P(\mathbf{r},t) &= P_0(r) + p(\mathbf{r},t),\\ G_{ik}(\mathbf{r},t) &= G_{ik}^0(r) + \mathrm{g}_{ik}(\mathbf{r},t). \end{aligned} \qquad (65)$$

We note that in this model, the static gravitational stresses in the stationary cloud surrounding the neutron star are determined by the hydrostatic pressure (Eq. (20)) as

$$\begin{aligned} G_{ik}^0(r) &= -P_0(r)\delta_{ik},\\ P_0(r) &= \left[\frac{2\pi}{3}G\,\rho^2(R_s^2 - r^2) + \right.\\ &\left. + \frac{4\pi}{3}G\,\rho(\rho_s - \rho)R_s^3\left(\frac{1}{r} - \frac{1}{R_s}\right)\right]. \end{aligned} \qquad (66)$$

Inserting (65) in (62)–(64), we arrive at linearized equations of gravity-driven fluctuations,

$$\rho\frac{\partial v_i}{\partial t} = -\frac{\partial p}{\partial x_k} + \frac{\partial \mathrm{g}_{ik}}{\partial x_k} + \frac{\partial \pi_{ik}}{\partial x_k}, \qquad (67)$$

$$\frac{\partial \mathrm{g}_{ik}}{\partial t} = 2P_0(r)\,v_{ik} + v_j\nabla_j P_0(r)\delta_{ik}, \qquad (68)$$

$$\pi_{ik} = 2\nu\rho\, v_{ik}, \quad v_{ik} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right). \qquad (69)$$

The energy balance equation is

$$\frac{\partial}{\partial t}\int \frac{\rho\, v^2}{2}\, d\mathcal{V} = -\int [\mathrm{g}_{ik} + \pi_{ik}]\, v_{ik}\, d\mathcal{V}. \qquad (70)$$

This equation is obtained by scalar multiplication of (67) with v_i and integration over the cloud volume, provided that surface stresses are negligible. The next step is to use a separable $\mathbf{r}$ and t representation for both the kinematic characteristics of motion like the field of material displacements and the rate-of-strain tensor

$$v_i(\mathbf{r},t) = a_i(\mathbf{r})\,\dot\alpha(t), \quad v_{ik}(\mathbf{r},t) = a_{ik}\,(\mathbf{r})\,\dot\alpha(t), \qquad (71)$$

and for the strength characteristics of motion such as the stress tensors of gravity and viscosity

$$\begin{aligned} \mathrm{g}_{ik}(\mathbf{r},t) &= [2P_0(r)a_{ik}(\mathbf{r}) + a_j(\mathbf{r})\nabla_j P_0(r)\delta_{ik}]\alpha(t),\\ \pi_{ik} &= 2\,\eta\, a_{ik}\,\dot\alpha(t). \end{aligned} \qquad (72)$$

Substituting (71) and (72) in (70), we obtain the equation of damped harmonic oscillations

$$M\ddot\alpha(t) + D\dot\alpha(t) + K\alpha(t) = 0 \qquad (73)$$

in which the parameters of inertia M, stiffness K, and viscous friction D are given by the integrals

$$\begin{aligned} M &= \int \rho\, a_i(\mathbf{r})\, a_i(\mathbf{r})\, d\mathcal{V},\\ D &= \frac{1}{2}\int \rho\nu[\nabla_i a_k + \nabla_k a_i]\,[\nabla_i a_k + \nabla_k a_i] d\mathcal{V}, \end{aligned} \qquad (74)$$

$$K = \frac{1}{2}\int P_0\,[\nabla_i\, a_k + \nabla_k\, a_i]\,[\nabla_i\, a_k + \nabla_k\, a_i] d\mathcal{V}. \qquad (75)$$

These equations show that the frequency and lifetime of quasistatic modes of hydrogravity can be computed using the above specified fields of spheroidal and toroidal instantaneous displacements. Also, it is noteworthy that the last expression for K is similar to the equation for the rigidity coefficient of a viscoelastic material whose oscillatory response is controlled by Hooke's restoring force. This again leads us to conclude that the gravitational stresses imparts to the gas–dust circumstellar material a portion of shear mechanical rigidity typical of viscoelastic soft matter. Deserving particular emphasis is the fact that the parameter of rigidity (75) computed for both the spheroidal hydrogravity mode (with field of displacement (36)) and toroidal hydrogravity mode (with fields of displacement (37)) has the analytically identical form with that given by Eqs. (40) and (46). Therefore, all physically significant results inferred in the body of this paper can be recovered within the approach outlined in this appendix.

REFERENCES

1. D. Pines, J. Shaham, and M. A. Ruderman, in *Physics of Dense Matter*, IAU Symposium, ed. by C. J. Hansen, Reidel, Doerdrecht (1974), Vol. 53, p. 189.
2. V. Boriakoff, Astronom. J. **208**, L4 (1976).
3. H. M. Van Horn, Astrophys. J. **236**, 899 (1980).
4. G. S. Bisnovatyi-Kogan, V. S. Imshennik, D. K. Nadyozhin, and V. M. Chechetkin, Astrophys. Space. Sci. **35**, 23 (1975).
5. O. Blaes, R. Blandford, P. Goldreich, and P. Madau, Astrophys. J. **343**, 839 (1989).
6. P. M. McDermott, H. M. Van Horn, and C. J. Hansen, Astrophys. J. **325**, 725 (1988).
7. T. E. Strohmayer, Astrophys. J. **372**, 573 (1991).
8. S. I. Bastrukov, V. V. Papoyan, D. V. Podgainy, and F. Weber, in *19th Texas Symposium on Relativistic Astrophysics and Cosmology*; Nucl. Phys. B (Suppl. Ser.) **80** 242 (2000).
9. S. I. Bastrukov, I. V. Molodtsova, D. V. Podgainy, V. V. Papoyan, and F. Weber, Phys. Part. Nucl. **30**, 436 (1999).
10. S. I. Bastrukov, D. V. Podgainy, J. Yang, and F. Weber, Zh. Eksp. Teor. Fiz. **122**, 915 (2002).
11. S. Bastrukov, V. Papoyan, D. Podgainy, and J. Yang, Astrophysics **45**, 232 (2002).
12. H. Lamb, *Hydrodynamics*, Cambridge University Press, Cambridge (1932).
13. J. Lighthill, *Waves in Fluids*, Cambridge University Press, Cambridge (1978).
14. L. D. Landau and E. M. Lifshits, *Fluid Mechanics*, Pergamon Press, New York (1959).
15. K. R. Lang, *Astrophysical Formulas*, Springer-Verlag, Berlin (1980).
16. S. I. Bastrukov, Int. J. Mod. Phys. D **5**, 45 (1996).
17. V. A. Fock, *The Theory of Space, Time, and Gravity*, Pergamon, New York (1959).
18. S. Chandrasekhar, Astrophys. J. **158**, 45 (1969).
19. C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*, Freeman, San Francisco (1970).
20. F. H. Shu, *Physics of Astrophysics*, University Science Books, Mill Valley (1992).
21. L. D. Landau, E. M. Lifshits, and L. P. Pitaevskii, *Electrodynamics of Continuous Media*, Butterworth-Heinemann, Oxford (1995).
22. L. Mestel, *Stellar Magnetism*, Clarendon Press, Oxford (1999).
23. G. S. Bisnovatyi-Kogan, *Stellar Physics*, Springer, Berlin (2002).
24. S. I. Bastrukov, Mod. Phys. Lett. A **8**, 711 (1993).
25. S. I. Bastrukov, Phys. Rev. E **53**, 1917 (1996).
26. S. I. Bastrukov, J. Yang, D. V. Podgainy, and F. Weber, J. Phys. G **29**, 683 (2003).
27. A. N. Timokhin, G. S. Bisnovatyi-Kogan, and H. Spriut, Mon. Not. R. Astron. Soc. **316**, 734 (2000).
28. T. E. Strohmayer, J. M. Cordes, and H. M. Van Horn, Astrophys. J. **389**, 685 (1992).
29. J. Christensen-Dalsgaard, Rev. Mod. Phys. **74**, 1073 (2002).
30. A. B. Severny, V. A. Kotov, and T. T. Tsap, Nature **259**, 87 (1976).
31. J. R. Brookes, G. R. Isaak, and H. B. van der Raay, Nature **259**, 92 (1976).
32. A. B. Severny, V. A. Kotov, and T. T. Tsap, in *IAU Symposium, Advances in Helio- and Asteroseismology* (1988), Vol. 123, p. 33.
33. Y. P. Elsworth, S. M. Jefferis, C. P. McLeod et al., Astrophys. J. **338**, 557 (1989).
34. V. A. Kotov, V. I. Khaneychuk, and T. T. Tsap, KFNT **16**, 49 (2000).