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SYMMETRY-BREAKING SOLUTIONS OF THE GINZBURG–LANDAU EQUATION

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We consider the question of the existence of nonradial solutions of the Ginzburg–Landau equation. We present results indicating that such solutions exist. We seek such solutions as saddle points of the renormalized Ginzburg–Landau free-energy functional. There are two main points in our analysis: searching for solutions that have certain point symmetries and characterizing saddle-point solutions in terms of critical points of certain intervortex energy function. The latter critical points correspond to forceless vortex configurations.

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1. INTRODUCTION

The Ginzburg–Landau equation describes, among other things, macroscopic stationary states of superfluids, Bose–Einstein condensation, and solitary waves in plasmas. In recent years, it has become a subject of active mathematical research (see monographs [1, 2] and [3] and reviews [4–7] for some of the recent references). This equation is simple to write,

$$-\Delta\psi + (|\psi|^2 - 1)\psi = 0, \qquad (1.1)$$

where (in the case of the entire plane $\mathbb{R}^2$) ψ: $\mathbb{R}^2 \to \mathbb{C}$, with the boundary condition

$$|\psi| \;\to\; 1 \quad \text{as } |x| \to \infty, \qquad (1.2)$$

but not easy to analyze. In fact, so far only radially symmetric solutions, i.e., solutions of the form $\psi_n(x) = f_n(r)e^{in\theta}$, where r and θ are polar coordinates for $x \in \mathbb{R}^2$, are known for (1.1)–(1.2) (see [8–17]). Solutions ψ_n are called the n-vortices. We note that $n = \deg\psi_n$, where $\deg\psi$, the degree (or vorticity) of ψ (satisfying (1.2)) is the total index (winding number) at ∞ of ψ considered as a vector field on $\mathbb{R}^2$, i.e.,

$$\deg\psi \;:= \frac{1}{2\pi}\int\limits_{|x|=R} d(\arg\psi)$$

for R sufficiently large.

The existence and properties of the vortex solutions were established only recently. The known facts are as follows.

(i) Existence and uniqueness (modulo symmetry transformations and in a class of radially symmetric functions) [10–13].

(ii) Stability for $|n| \leq 1$ and instability for $|n| > 1$ ([13], earlier results on stability for the disc are due to [15–17]).

(iii) Uniqueness of $\psi_{\pm 1}$ (again, modulo symmetry transformation) in a class of functions ψ with $\deg\psi = \pm 1$ and $\int \left(|\psi|^2 - 1\right)^2 < \infty$ [16].

Therefore, the next question is: are there nonradially symmetric solutions?

In this paper, we present results indicating that such solutions exist. There are two key ingredients in our analysis. First, we characterize nonradially symmetric solutions as critical points of the intervortex energy function described below (see also [18]). Second, we seek solutions having certain point symmetries. The latter fact reduces the number of free parameters describing such solutions to one (the size of the corresponding polygon of vortices).

Solutions breaking the rotational symmetry were found to exist in the case of the Ginzburg–Landau

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equation in the ball $B_R = \{x \in \mathbb{R}^2 | |x| \le R\}$ with the boundary condition $\psi|_{\partial B_R} = e^{in\theta}$ and $|n| \ge 2$ (see [1, 2], Thm IX.1]). However, in the case of the ball, there is an external mechanism leading to the symmetry breaking: the boundary condition. It repels vortices, forcing their confinement. On the other hand, the energy is lowered by breaking up multiple vortices into (+1)- (or (−1)-) vortices and merging vortices of opposite signs. Thus, for R not very small, the lowest energy is reached by a configuration of $|n|$ vortices of vorticities ± 1 depending on the sign of n which, obviously, is not rotationally symmetric.

This paper is organized as follows. In Secs. 2 and 3, we review some material in [13]: the variational formulation of the problem and some specific properties of vortex solutions. In Sec. 4, we define the intervortex energy and discuss its properties. In particular, we discuss the correlation term in (the upper bound on) the expansion of the intervortex energy for large intervortex separations and the definition of G-symmetric vortex energies, where G is a subgroup of the symmetry group of (1.1)

In Sec. 5, we consider point symmetries (C_{Nv}), present one of our main results, Theorem 5.1, on the existence of critical points for C_{Nv}-symmetric intervortex energies, and derive some general relations for those energies. In Sec. 6, we prove Theorem 5.1 and discuss some other cases.

Finally, we have five appendices where all the hard analytic and numerical work is concentrated. In these appendices, we compute various asymptotic expansions beyond the leading order. We feel that these appendices are of interest on their own because they address rather subtle computational issues.

2. RENORMALIZED GINZBURG–LANDAU ENERGY

It is a straightforward observation that Eq. (1.1) is the equation for critical points of the functional

$$\mathcal{E}(\psi) = \frac{1}{2}\int \left(|\nabla\psi|^2 + \frac{1}{2}(|\psi|^2-1)^2\right). \tag{2.1}$$

Indeed, if we define the variational derivative $\partial_\psi \mathcal{E}(\psi)$ of $\mathcal{E}$ by

$$\mathrm{Re}\int \xi \partial_\psi \mathcal{E}(\psi) = \frac{\partial}{\partial\lambda}\mathcal{E}(\psi_\lambda)\Big|_{\lambda=0} \tag{2.2}$$

for any path ψ_λ such that $\psi_0 = \psi$ and $\frac{\partial}{\partial\lambda}\psi_\lambda\big|_{\lambda=0} = \xi$, then the l.h.s. of Eq. (1.1) is equal to $\overline{\partial_\psi \mathcal{E}(\psi)} = \partial_{\bar\psi}\mathcal{E}(\psi)$ for $\mathcal{E}(\psi)$ given by (2.1).

Equation (2.1) is the celebrated Ginzburg–Landau (free) energy. However, there is a problem with it in our context. It is shown in [13] that if ψ is an arbitrary C^1-vector field on $\mathbb{R}^2$ such that $|\psi| \to 1$ as $|x| \to \infty$ uniformly in $\hat{x} = x/|x|$ and $\deg\psi \ne 0$, then $\mathcal{E}(\psi) = \infty$.

We renormalize the Ginzburg–Landau energy functional as follows (see [13]). Let $\chi(x)$ be a smooth positive function on $\mathbb{R}^2$ vanishing at the origin and converging to one at infinity. We define

$$\mathcal{E}_{\rm ren}(\psi) = = \frac{1}{2}\int \left(|\nabla\psi|^2 - \frac{(\deg\psi)^2}{r^2}\chi + F(|\psi|^2)\right) d^2x, \tag{2.3}$$

where

$$F(u) = \frac{1}{2}(u-1)^2. \tag{2.4}$$

Properties of the renormalized energy functional $\mathcal{E}_{\rm ren}(\psi)$ are investigated in [13].

In this paper, we take

$$\chi(x) = \begin{cases} 1 & \text{for } |x| \ge R + R^{-1}, \\ 0 & \text{for } |x| \le R \end{cases} \tag{2.5}$$

for R very large compared to all length scales appearing below.

3. VORTICES

It is shown in [10–13] that for any n, Eq. (1.1) has a solution, unique modulo symmetry transformations, of the form

$$\psi_n(x) = f_n(r)e^{in\theta}, \tag{3.1}$$

where f_n, with $1 > f_n \ge 0$, monotonically increase from $f_n(0) = 0$ to 1 as r increases to ∞. For $n = 0$, $f_n(r) = 1$. For $|n| > 0$, $f_n(r)$ does not admit an explicit expression. These are the n-vortices mentioned in the introduction. Of course, each solution ψ_n generates a one-parameter (for $n = 0$) or a three-parameter (for $|n| > 0$) family of solutions of (1.1). The latter are obtained by applying symmetry transformations to ψ_n.

The function $f_n(r)$ in (3.1) satisfies the ordinary differential equation

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f_n}{\partial r}\right) + \frac{n^2}{r^2}f_n - (1-f_n^2)f_n = 0. \tag{3.2}$$

The (self) energy of the n-vortex is given by $E_{n,R} := \mathcal{E}_{\rm ren}(\psi_n)$. To compute $E_{n,R}$, we use that if ψ is a solution of (1.1), then, due to the formula $\int |\nabla\psi|^2 = -\int \bar{\psi}\Delta\psi$ of integration by parts, we have

$$\mathcal{E}_{\rm ren}(\psi) = \\ = \frac{1}{2}\int\left(1-|\psi|^2-\frac{1}{2}(1-|\psi|^2)^2-\frac{(\deg\psi)^2}{r^2}\chi\right). \quad (3.3)$$

Using this formula for $\psi = \psi_n$ and using the asymptotic expression (which can be easily derived from (3.2), see [19, 20] for the general case)

$$f_n(r) = 1 - \frac{n^2}{2r^2} + O\left(\frac{1}{r^4}\right) \quad (3.4)$$

for $r \gg 1$, we obtain

$$E_{n,R} = \pi n^2 \ln\left(\frac{R}{|n|}\right) + c(|n|) + O\left(\frac{1}{R^2}\right). \quad (3.5)$$

The constant $c(n)$ can be computed numerically (which is not quite trivial, see Appendix 1), which yields

$$\begin{aligned} c(1) = 0.376\,\pi, \quad c(2) = 0.535\,\pi, \\ c(3) = 0.577\,\pi, \quad c(5) = 0.615\,\pi. \end{aligned} \quad (3.6)$$

The asymptotic form of $c(n)$ for $n \gg 1$ is found analytically in Appendix 2.

4. INTERVORTEX ENERGY

In this section, we introduce and discuss a key concept of the intervortex energy (see also [4, 18]). We begin with some definitions.

By a vortex configuration $\underline{c}$, we understand a pair $(\underline{a}, \underline{n})$, where $\underline{a} = (a_1, \ldots, a_K)$, $a_j \in \mathbb{R}^2$, and $\underline{n} = (n_1, \ldots, n_K)$, $n_j \in \mathbb{Z}$, for some $K \geq 1$ (positions of the vortex centers and their vorticities). We consider once-differentiable functions ψ: $\mathbb{R}^2 \to \mathbb{C}$ satisfying $|\psi| \to 1$ as $|x| \to \infty$. We say that the vortex configuration of ψ is $\underline{c} = (\underline{a}, \underline{n})$, $\operatorname{conf}\psi = \underline{c}$, if ψ has zeros (only) at $a_1, \ldots, a_K$ with the respective local indices $n_1, \ldots, n_K$, i.e.,

$$\int\limits_{\gamma_j} d(\arg\psi) = 2\pi n_j \quad (4.1)$$

for any contour γ_j containing a_j, but not the other zeros of ψ, and for $j = 1, \ldots, K$. (Strictly speaking, we have to specify the phase factor, or rotation angle, for each vortex; but these play no role in our considerations and are not displayed or mentioned in what follows.) We now define

$$E_R(\underline{c}) = \inf\left\{\mathcal{E}_{\rm ren}(\psi) | \operatorname{conf}\psi = \underline{c}\right\}. \quad (4.2)$$

We expect that $E_R(\underline{c}) > -\infty$. An argument supporting this statement is presented in [18]. Of course, for bounded domains, this inequality is trivial. We call $E_R(\underline{c})$ the energy of the vortex configuration $\underline{c}$. It plays a central rôle in our analysis. We also note that $E(\underline{c})$ serves as a Hamiltonian for the vortex dynamics in the adiabatic approximation (see [21]).

In what follows, we keep the vortex indices $\underline{n}$ fixed and write $E_R(\underline{a})$ for $E_R(\underline{c})$. It is clear intuitively that a minimizer in (4.2) exists if and only if $\nabla E_R(\underline{a}) = 0$ (the force acting on the vortex centers is zero). However, to establish this fact is not so easy.

Theorem 4.1. If there is a minimizer for variational problem (4.2), then this minimizer satisfies Ginzburg–Landau equation (1.1).

Proof. Let ψ be a minimizer for (4.2). Because we have

$$0 = \frac{\partial}{\partial\lambda}\mathcal{E}_{\rm ren}(\psi + \lambda\xi)\Big|_{\lambda=0} = \\ = \operatorname{Re}\int \bar{\xi}\left(-\Delta\psi + (|\psi|^2 - 1)\psi\right)$$

for any differentiable function ξ: $\mathbb{R}^2 \to \mathbb{C}$ vanishing together with its gradient sufficiently fast at ∞ and vanishing at the points $a_1, \ldots, a_m$, we conclude that ψ satisfies (1.1) for $x \neq a_1, \ldots, a_m$. On the other hand, because $\psi \in H_1^{\rm loc}(\mathbb{R}^2)$, we have that $-\Delta\psi + (|\psi|^2 - 1)\psi \in H_{-1}^{\rm loc}(\mathbb{R}^2)$. Hence, $-\Delta\psi + (|\psi|^2 - 1)\psi = 0$ on $\mathbb{R}^2$.

Arguments and results in [18] (see, in particular, Theorem 3.2) justify making the following conjecture.

Conjecture 4.2. $\nabla E_R(\underline{a}_0) = 0$ for some $\underline{a}_0$ (with $\underline{n}$ fixed) if and only if there is a minimizer for problem (4.2) at the configuration $\underline{a}_0$ and consequently, due to Theorem 4.1, if and only if Ginzburg–Landau equation (1.1) has a solution with the configuration $\underline{a}_0$.

The goal of this paper is to find forceless vortex configurations, i.e., configurations $\underline{c}$ such that

$$\nabla E_R(\underline{a}) = 0. \quad (4.3)$$

For this, we study the intervortex energy $E_R(\underline{a})$ for very small and very large intervortex separations.

Let

$$d_{\underline{a}} = \min_{i \neq j} |a_i - a_j| \quad \text{for} \quad \underline{a} = (a_1, \ldots, a_K).$$

For $d_{\underline{a}}$ large, we prove in Sec. 7 the upper bound

$$E_R(\underline{a}) \leq E_R^{(0)} - A(\underline{a}) + O\big(d_a^{-8/3}\big) + O(R^{-2}), \quad (4.4)$$

where

$$E_R^{(0)} = \sum_{i=1}^{K} E_{n_i,R} + H\left(\frac{\underline{a}}{R}\right)$$

and $A(\underline{c})$ is a homogeneous function of degree -2, provided $\underline{a}$ satisfies $\nabla H(\underline{a}) = 0$. We recall that $E_{n,R} = \mathcal{E}_{\mathrm{ren}}(\psi_n)$ is the self-energy of the n-vortex (see (3.5)) and $H(\underline{a})$ is the energy of the vortex pair interactions,

$$H(\underline{a}) = -\pi \sum_{i\neq j} n_i n_j \ln |a_{ij}|, \qquad (4.5)$$

with $a_{ij} = a_i - a_j$.

The correlation term $A(\underline{a})$ is of importance for us here. We have an explicit expression for it, see Eqs. (A.3.4)–(A.3.5), and compute it explicitly in the cases of interest. We conjecture that $A(\underline{a}) > 0$ always.

We observe that the upper bound (4.4) with the remainder $O\left(d_{\underline{a}}^{-1}\right)$ instead of $-A(\underline{a}) + O\left(d_{\underline{a}}^{-8/3}\right)$ is obtained by choosing the Hartree-type function

$$\psi^{(0)}(x) = \prod_{i=1}^{K} \psi_{n_j}(x - a_i)$$

describing «independent» vortices. For asymptotically forceless configurations, i.e., the ones with $\nabla H(\underline{a}) = 0$, this estimate can be somewhat improved, but in order to move even to the remainder estimate $O\left(d_{\underline{a}}^{-2} \ln d_{\underline{a}}\right)$ in the latter case, one has to refine upon this function and include the leading correlations.

Remark 4.3. As $d_{\underline{a}} \to \infty$, the important asymptotic expression

$$E_R(\underline{a}) = \sum_{i=1}^{K} E_{n_i,R} + H\left(\frac{\underline{a}}{R}\right) + \mathrm{Rem} \qquad (4.6)$$

was proved in [18] with $\mathrm{Rem} = O(d_{\underline{a}}^{-2} \ln d_{\underline{a}})$ in general and $= O(d_{\underline{a}}^{-2})$ if $\nabla H(\underline{a}) = 0$.

As mentioned in the introduction, our second idea is to consider solutions of (1.1) that are invariant under point group transformations. Consequently, we introduce intervortex energy functions invariant under such groups. We consider a subgroup G of the total symmetry group

$$G_{\mathrm{sym}} = O(2) \times T(2) \times U(1)$$

(where $T(n)$ is the group of translations of $\mathbb{R}^n$) of Ginzburg–Landau equation (1). For a G-invariant vortex configuration $\underline{c} = (\underline{a}, \underline{n})$ (i.e., invariant under the spatial part of G), we define the G-invariant vortex interaction energy $E_{R,G}(\underline{a})$ as

$$E_{R,G}(\underline{a}) = \inf\{\mathcal{E}_{\mathrm{ren}}(\psi) | \mathrm{conf}\psi = \underline{c},\ \psi \text{ is } G\text{-invariant}\}$$

(as before, we fix $\underline{n}$ and omit it from the relation).

Theorem 4.1 and Conjecture 4.2 extend obviously to the G-symmetric situation. In particular, we have the following conjecture:

If $\underline{a}_0$ is a critical point of $E_{R,G}(\underline{a})$ (i.e., $\nabla E_{R,G}(\underline{a}_0) = 0$), then Eq. (1.1) has a G-invariant solution.

Our goal in what follows is to find critical points of the G-invariant intervortex energy $E_{R,G}(\underline{a})$ for appropriate groups G, namely, point groups C_{Nv} (see the next section).

5. POINT SYMMETRIES

We seek solutions of Eq. (1.1) having symmetry groups C_{Nv}. These groups consist of rotations around the origin by angles given by integer multiples of $2\pi/N$ and reflection(s) in one (and therefore N) line(s) passing through the origin. Such solutions are determined by fixing vortex configurations that have the desired symmetry group. We consider vortex configurations consisting of N m-vortices uniformly spaced on a circle of radius a and a single $(-k)$-vortex at the center of the circle, which is placed at the origin. Several such configurations and their symmetry lines are shown in Fig. 1. Such configurations have the symmetry group C_{Nv}. The symmetry group C_{Nv} determines such a configuration uniquely up to the vortex values m and k and the size a.

As noted at the end of the previous section, we rely on the argument that C_{Nv}-symmetric solutions are in one-to-one correspondence with critical points of the C_{Nv}-symmetric intervortex energy

$$E_R(\underline{c}) \equiv E_{R,C_{Nv}}(\underline{c})$$

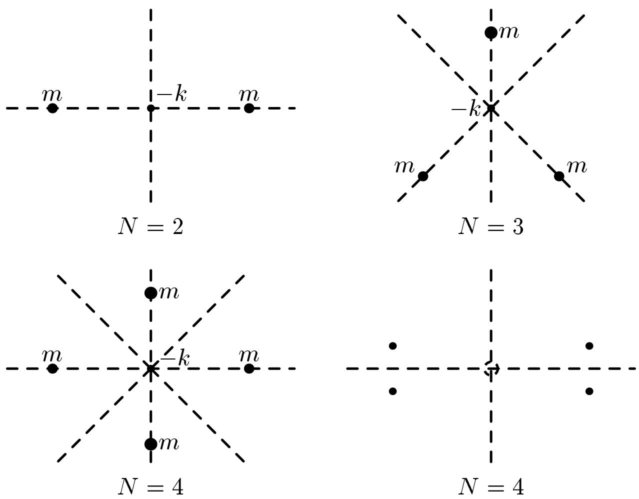

Fig. 1. Symmetric configurations and their reflection lines

(here and in what follows, we consider only C_{Nv}-symmetric intervortex energies and often omit the subscript C_{Nv}). Our goal is to find critical points of $E_R(\underline{c})$. One of the central results in this paper is the following theorem.

Theorem 5.1. There exist critical points of $E_{R,C_{Nv}}(\underline{c})$ among the configurations $\underline{c}$ described above for the parameter values

$$(N, m, k) = (2, 2, 1) \text{ and } (4, 2, 3)$$

(see Fig. 1, a critical value of the parameter a is not specified, but its existence is established).

This theorem is proven in Sec. 6. In the rest of this section, we establish general properties of the energy $E_{R,C_{Nv}}(\underline{c})$ and find a necessary condition on the parameters N, m, and k.

We observe that if $\underline{c}$ is a configuration described above, then

$$\begin{gathered}\nabla_{a_j} E_R(\underline{a}) = \hat{a}_j \partial_{|a_j|} E_R(\underline{a}) \\ \text{and} \quad \nabla_{a_j} H(\underline{a}) = \hat{a}_j \partial_{|a_j|} H(\underline{a}) \quad \forall\, j,\end{gathered} \tag{5.1}$$

where $\hat{a} = a/|a|$ (again, we do not display the parameters $\underline{n}$). In this case, it therefore suffices to investigate the energy $E_R(\underline{a})$ as a function of one variable, the scale parameter a.

We note that if $m \geq 2$, then there is a continuum of configurations, labeled by a parameter $\alpha > 0$, with the same symmetry group C_{Nv} as a given configuration, which have the given configuration as the limit as $\alpha \to 0$. For instance, for $m = 2$, each m-vortex can be split into a pair of 1-vortices with all pairs lying either on the circle or on the lines joining their parent m-vortices to the origin at equal distance α to those m-vortices, see Fig. 2.

By symmetry, the energy of the resulting configurations has a critical point at $\alpha = 0$. A simple analysis of the break-up of a 2-vortex shows that this critical point is a local maximum. Indeed, e.g., for $m = 2$, it was shown in [13] that the linearization of Eq. (1.1) (the Hessian of the energy functional) around the 2-vortex solutions $\psi_2 = f_2(r)e^{2i\theta}$ has exactly one negative mode (an eigenfunction corresponding to a negative eigenvalue) of the form $\xi = e^{4i\varphi}\xi_4(r) + \xi_0(r)$, where $\xi_k(r)$ are some real functions. Then the function $\psi_2 + \lambda\xi$ for $|\lambda|$ sufficiently small lowers the energy of ψ_2. On the other hand, this function has two simple zeros (i.e., of vorticities $+1$) in a vicinity of $x = 0$. Indeed, in the complex notation $z = x_1 + ix_2 \leftrightarrow x = (x_1, x_2)$, $\psi_2(z) = bz^2 + O(z^3)$ and $\xi(z) = c + O(z)$ for some positive numbers b and c in a neighbourhood of $z = 0$. Hence, $\psi_2(z) + \lambda\xi(z) = bz^2 + \lambda c + O(z^3) + O(\lambda z)$, which therefore has two simple zeros $z_\pm = \pm\sqrt{\dfrac{\lambda c}{b}} + O(\lambda^{3/4})$ in a neighbourhood of $z = 0$. This shows in particular that splitting of a 2-vortex lowers the energy.

Proposition 5.2. Let a configuration $\underline{c}_0$, as described above, be asymptotically forceless, i.e., $\nabla H(\underline{a}_0) = 0$. Then

$$k = \frac{1}{2}(N-1)m. \tag{5.2}$$

Proof. By (4.1), the equation $\nabla H(\underline{a}_0) = 0$ for the configuration described is equivalent to the equation

$$\frac{\partial}{\partial a} H(\underline{a}_0) = 0. \tag{5.3}$$

Because

$$H(\underline{a}) = H\left(\frac{\underline{a}}{a}\right) - \pi \sum_{i \neq j} n_i n_j \ln a, \tag{5.4}$$

the latter equation implies that $\sum\limits_{i \neq j} n_i n_j = 0$, which is equivalent to (5.2) due to the relation

$$\sum_{i \neq j} n_i n_j = -2Nmk + N(N-1)m^2. \tag{5.5}$$

We note that Eq. (5.3) implies that if $\nabla H(\underline{a}_0) = 0$, then $\nabla H(\underline{a}) = 0$ for all $\underline{a}$ of the form $\underline{a} = s\underline{a}_0$, $s > 0$. The latter fact implies another proof of (5.2). Indeed, $H(\underline{a}/R)$ behaves as const $\cdot \ln R$ + const for large R. Hence, for an asymptotically force-free configuration (i.e., the one with $\nabla H(\underline{a}) = 0$), the constant in front of $\ln R$ is independent of the scale parameter a. This asymptotic scale invariance implies that the leading term

$$\pi(Nm - k)^2 \ln R$$

for the configuration with $a = 0$ (i.e., when all the vortices collapse to the center of the circle) is equal to the leading term

$$\pi(Nm^2 + k^2) \ln R$$

for the configuration with a very large a, and therefore the vortices in such a configuration can be treated as virtually independent (see (4.4)). Hence,

$$(Nm - k)^2 = Nm^2 + k^2,$$

which implies (5.2).

We observe that Eq. (5.2) is equivalent to the relation

$$H\left(\frac{\underline{a}}{R}\right) = H\left(\frac{\underline{a}}{a}\right) = H(\underline{a}), \quad \text{independent of } a. \tag{5.6}$$

Indeed, this follows from Eqs. (5.4) and (5.5).

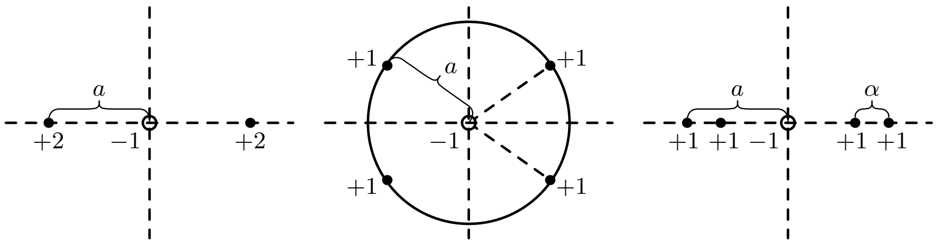

Fig. 2.

Relation (5.2) between k and m is assumed in what follows.

For the configuration above, we now introduce the energy differences

$$\Delta E(\underline{a}) := E_R(\underline{a}) - \pi(Nm-k)^2 \ln R, \tag{5.7}$$

where we recall that $Nm-k$ is the total vorticity of the configuration in question and $E_{n,R}$ is the energy of a single vortex of vorticity n, i.e., $E_{n,R} = \mathcal{E}_{\rm ren}(\psi_n)$. We let ΔE_n denote the energy difference for this vortex,

$$E_{n,R} = \pi n^2 \ln R + \Delta E_n. \tag{5.8}$$

Clearly,

$$E_R(\underline{0}) = E_{Nm-k,R} \quad \text{and} \quad \Delta E(\underline{0}) = \Delta E_{Nm-k}. \tag{5.9}$$

This together with (3.5) implies that (modulo $O(R^{-2})$)

$$\Delta E(\underline{0}) = -\pi(Nm-k)^2 \ln(Nm-k) + c(Nm-k). \tag{5.10}$$

On the other hand, for very large intervortex distances, Eqs. (5.7), (5.6), (4.6), and (3.5) imply that (modulo $O(R^{-2}) + o(a^{-2})$)

$$\Delta E(\underline{a}) \le -\pi(Nm^2 \ln m + k^2 \ln k) + Nc(m) + c(k) + H(\underline{a}) - Ca^{-2}, \tag{5.11}$$

where $C = A(\underline{a}/a)$. We compute $H(\underline{a})$ for the given configuration. Because the distances between the vortices on the circle are $2a\sin\frac{\pi}{N}$, $2a\sin\frac{2\pi}{N}, \ldots, 2a\sin\frac{(N-1)\pi}{N}$, we find

$$H(\underline{a}) = -\pi m^2 N \sum_{k=1}^{N-1} \ln\left(2\sin\frac{k\pi}{N}\right). \tag{5.12}$$

This equation together with Eq. (5.11) yields that for large intervortex distances,

$$\Delta E(\underline{a}) \le -\pi(Nm^2 \ln m + k^2 \ln k) + Nc(m) + c(k) - \pi m^2 N \sum_{k=1}^{N-1} \ln\left(2\sin\frac{k\pi}{N}\right) - Ca^{-2} \tag{5.13}$$

modulo $O(R^{-2}) + o(a^{-2})$.

In the next section, we establish the existence of points $\underline{a}_0$ such that $\nabla E(\underline{a}_0) = 0$ for given configurations by comparing $\Delta E(\underline{0})$ and $\Delta E(\underline{a})$ for large intervortex distances a.

6. THE SIMPLEST CASES. PROOF OF THEOREM 5.1

In this section, we consider some special, in fact the simplest, cases of the vortex configurations introduced in Sec. 5. We recall that every such configuration consists of a vortex of vorticity $-k$ placed at the origin and N vortices, each of vorticity m, distributed equidistantly on the circle of radius a with the center at the origin. Such a configuration is fixed by the symmetry group C_{Nv}, and hence the only remaining free parameter is the radius of the circle a. With a slight abuse of notation, we write $\Delta E(a) = \Delta E(\underline{a})$.

Proof of Theorem 5.1. The correlation coefficient C in Eq. (5.13) is computed for the specified configurations in Appendix 3:

$$C = 8\pi, \quad 20\pi \quad \text{for} \quad (N,m,k) = (2,2,1),\ (4,2,3). \tag{6.1}$$

(We expect that for general (N,m,k), $k = \frac{1}{2}(N-1)m$, C is of the form $\frac{\pi}{4}\cdot$(integer).) Thus,

$$\Delta E(a) \quad \text{monotonically increases to} \quad \Delta E(\infty) \quad \text{as} \quad a \to \infty. \tag{6.2}$$

Moreover, due to (3.6), we have

$$\Delta E(\infty) < \Delta E(0) \tag{6.3}$$

for the configurations $(N,m,k) = (2,2,1),\ (4,2,3)$ (explicit computations are given below). Hence, $\Delta E(a)$ has at least one minimum for these configurations as claimed.

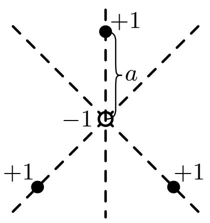

Fig. 3.

Computation of (6.3).

(a) The case $N = 2$, $m = 2$, and $k = 1$ (we recall that $E_R(a) \equiv E_R(\underline{a})$, etc). We have

$$\Delta E(0) \equiv \Delta E_3(0) = c(3) - 9\pi \ln 3 = -9.31\pi. \quad (6.4)$$

On the other hand, Eq. (5.11) implies that for a very large,

$$\Delta E(a) \le c(1) + \big(2c(2) - 8\pi \ln 2\big) - 8\pi \ln 2 - Ca^{-2} + O\left(\frac{\ln a}{a^4}\right) = -9.64\pi - Ca^{-2} + O\left(\frac{\ln a}{a^4}\right). \quad (6.5)$$

(b) The case $N = 4$, $m = 2$, and $k = 3$ (see Fig. 1). In this case,

$$\Delta E(0) = \Delta E_5(0) = c(5) - 25\pi \ln 5 = -39.62\pi. \quad (6.6)$$

On the other hand, Eq. (5.11) implies that for large a, we have the asymptotic behavior

$$\Delta E(a) \le \big(4c(2) - 16\pi \ln 2\big) + \big(c(3) - 9\pi \ln 3\big) - 32\pi \ln 2 - Ca^{-2} + O\left(\frac{\ln a}{a^4}\right) = -40.44\pi - Ca^{-2} + O\left(\frac{\ln a}{a^4}\right). \quad (6.7)$$

Thus (6.3) is shown.

Remarks.

a. We examine the case where $m = 1$, i.e., the vortices on the circle are simple. In this case, $k = (N-1)/2$. Therefore, in the simplest case where $N = 3$ and $k = 1$, we take the $(m = 1)$-vortices equally spaced (Fig. 3).
Equations (4.9), (4.12), and (3.6) imply that in this case, $\Delta E(0) < \Delta E(\infty)$ (in fact, $\Delta E(0) = \Delta E_2(0) = -2.238\pi$ and $\Delta E(\infty) = -1.792\pi$). Numerical computations show (see Appendices 3 and 4) that $\Delta E'(\infty) > 0$ and $\Delta E'(0) > 0$ (in fact, for $a \gg 1$, $\Delta E(a) = 4c(1) - 3\pi \ln 3 - Ca^{-2} = -1.792\pi - Ca^{-2}$ with $C > 0$). In this case, we cannot therefore conclude that a critical point of $E_R(a)$ exists. But a more careful numerical analysis indicates that there probably exist two extremal points of $E_R(a)$, a minimum and a maximum, for $1/\sqrt{2} \le a \le 2$. Similar configurations for large (and odd) N are analyzed in Appendix 5.

b. The case where $N = 2$, $m = 2$, and $k = 1$ is the limiting case of $N = 4$, $m = 1$, and $k = 1$ (see Fig. 2). All three configurations have the same symmetry group C_{2v} generated by rotation by π and reflections in the vertical and horizontal axes passing through the vortex -1. After the symmetry group is fixed, the second and third configurations have two free parameters: the scale parameter a and the angle/distance α between two of its neighboring 1-vortices (see Fig. 3). As $\alpha \to 0$, the second and third configurations are continuously transformed into the first one.

7. UPPER BOUND ON THE INTERVORTEX ENERGY

In this section, we prove inequality (4.4) for the energy $E_R(\underline{a})$ of vortex configurations.

Theorem 7.1. We have the estimate

$$E_R(\underline{a}) \le E_R^{(0)} + \mathrm{Rem} + O(\max |a_j|^2/R^2), \quad (7.1)$$

where $E_R^{(0)} = \sum_{k=1}^{k} E_{n_i,R} + H\left(\frac{\underline{a}}{R}\right)$ and

$$\mathrm{Rem} = \begin{cases} O\big(d_{\underline{a}}^{-2}\big) & \text{if } \nabla H(\underline{a}) = 0, \\ O\big(d_{\underline{a}}^{-2} \ln d_{\underline{a}}\big) & \text{otherwise.} \end{cases} \quad (7.2)$$

Moreover, if $\nabla H(\underline{a}) = 0$, then estimate (7.2) can be improved as

$$\mathrm{Rem} = -A(\underline{a}) + O\big(d_{\underline{a}}^{-8/3}\big) + O\left(\frac{1}{R^2}\right), \quad (7.3)$$

where $A(\underline{a})$, the correlation term, is a homogeneous degree-(-2) function, explicitly given by the conditionally convergent integral

$$A(\underline{a}) = \frac{1}{4}\int \left[|\nabla\varphi_0|^4 - \sum_j |\nabla\varphi_j|^4\right] \quad (7.4)$$

(where $\nabla H(\underline{a}) = 0$ is assumed) with

$$\varphi_0 = \sum_j \varphi_j, \quad \varphi_j(x) = n_j \theta(x - a_j), \qquad \theta(x) = \text{the polar angle of } x \in \mathbb{R}^2. \quad (7.5)$$

Before proceeding to the proof of these estimates, we show that the integral in the r.h.s. of (7.4) is conditionally convergent in the forceless case $\nabla H(\underline{a}) = 0$.

Because the integrand has singularities at the points $a_1,\ldots,a_K$, it suffices to show that the integrals over the discs $D(a_k,\varepsilon)$ centered at a_k and of a radius $\varepsilon>0$ converge. We consider the integral over the disc $D(a_k,\varepsilon)$. Let

$$\varphi_{(k)}(x)=\sum_{j\neq k}\varphi_j(x). \tag{7.6}$$

Because the function $\varphi_{(k)}(x)$ is harmonic in $D(a_k,\varepsilon)$, it has an expansion around the point a_k of the form

$$\varphi_{(k)}(x)=\sum_{m=0}^{\infty}c_m r_k^m\cos m(\theta_k-\theta^{(m)}), \tag{7.7}$$

where r_k and θ_k are the polar coordinates of $x_k=x-a_k$ and c_m and $\theta^{(m)}$ are some constants.

In the forceless case,

$$\nabla\varphi_{(k)}(a_k)=-\frac{1}{2\pi n_k}J\nabla_{a_k}H(\underline{a})=0, \tag{7.8}$$

and therefore

$$\nabla\varphi_{(k)}(x)=c_k(x_k\cos2\theta_k-x_k^{\perp}\sin2\theta_k)+ +O\left(\frac{r_k^2}{d_{\underline{a}}^3}\right), \tag{7.9}$$

where $c_k=O(1/d_{\underline{a}}^2)$ is a constant, $r_k=|x_k|$, and $x^{\perp}=(-x_2,x_1)$. Now, writing

$$\int\limits_{D(a_k,\varepsilon)}\left(|\nabla\varphi|^4-|\nabla\varphi_k|^4\right)= =\int\limits_{D(a_k,\varepsilon)}\left(2|\nabla\varphi_k|^2\alpha_k+\alpha_k^2\right), \tag{7.10}$$

where

$$\alpha_k:=2\nabla\varphi_k\cdot\nabla\varphi_{(k)}+|\nabla\varphi_{(k)}|^2 \tag{7.11}$$

and using (7.9), we see that the singular part of the integral above is

$$4\int\limits_{D(a_k,\varepsilon)}|\nabla\varphi_k|^2\nabla\varphi_k\cdot\nabla\varphi_{(k)}= =4\int\limits_{D(a_k,\varepsilon)}\frac{n_k^2}{r_k^2}\big(-c_k\sin2\theta_k+O(r_k)\big)= =\int\limits_{D(a_k,\varepsilon)}O\left(\frac{1}{r_k}\right)<\infty. \tag{7.12}$$

Therefore, the integral in the r.h.s. of (7.4) is conditionally convergent, in the sense that it is well-defined as a limit of similar integrals with small discs around the points $a_1,\ldots,a_K$ excised, as the radii of those discs tend to 0.

Proof of Theorem 7.1. We prove the upper bound (7.1) using the variational inequality

$$E_R(\underline{a})\ \le\ \mathcal{E}_R(\psi), \tag{7.13}$$

valid for any function ψ having the given vortex configuration $\underline{a}$, and by showing that for an appropriate ψ, $\mathcal{E}_R(\psi)$ is of the form of the r.h.s. of (7.6). Namely, we show that

$$\mathcal{E}_{\mathrm{ren}}(\psi)=E_R^{(0)}+\mathrm{Rem}, \tag{7.14}$$

where Rem is given by either (7.2) or (7.3), as appropriate. Then (7.1) follows from (7.13) and (7.14).

We begin with proving estimate (7.1) with remainder (7.2). Let $\psi_i(x)=\psi^{(n_i)}(x_i)$, where $x_i=x-a_i$, and let $f_i\equiv|\psi_i|$. We consider the class of functions ψ of the form $\psi=fe^{i\varphi_0}$ with a function f such that

$$f=f_i+O\left(\frac{1}{rd_{\underline{a}}^n}\right)\quad\text{if}\ \ r_j\ll d_{\underline{a}},\quad\forall i, \tag{7.15}$$

where $n=2$ if $\nabla H(\underline{a})$ and $n=1$ otherwise and $r_i=|x-a_i|$, and

$$f=1+O\left(\frac{1}{d^2(x,\underline{a})}\right)\quad\text{if}\ d(x,\underline{a})\gg1, \tag{7.16}$$

where

$$d(x,\underline{a})=\min_j|x-a_j|,$$

with the corresponding estimates of their first derivatives.

We construct a function satisfying (7.15) and (7.16). Let $D(z,\rho)$ denote the disc of radius ρ centered at a point z. Let $\{\chi_j\}_1^K$ be a smooth partition of unity, i.e., $\sum\limits_{l=1}^{K}\chi_j=1$, having the properties

$$B\left(a_j,\frac{1}{3}d_{\underline{a}}\right)\subset\mathrm{supp}\chi_j\quad\forall j$$

and

$$\nabla^n\chi_j=O\big(d_{\underline{a}}^{-n}\big),\quad n=0,1,2.$$

Then the function $f=\sum f_j\chi_j$ satisfies (7.15) and (7.16). Indeed, (7.13) is obvious, while (7.14) follows from the relation

$$f_j=1+O(r_j^{-1}). \tag{7.17}$$

We prove the following lemma.

Lemma 7.1. Let ψ satisfy (7.15)–(7.16). Then

$$\mathcal{E}_R(\psi)=E_R^{(0)}+\mathrm{Rem}+O\left(\frac{1}{R^2}\right), \tag{7.18}$$

where $E_R^{(0)}$ is given in Theorem 7.1 and Rem is given by (7.2).

Proof. Let $D_j = D(a_j, r_0)$, the disc with the center at a_j and of the radius $r_0 = d_{\underline{a}}/3$. We decompose the energy functional as

$$\mathcal{E}_R(\psi) = \sum_j \int_{D_j} e(\psi) + \int_{D_R\setminus\cup D_j} e(\psi), \qquad (7.19)$$

where $e(\psi)$ is the energy density,

$$e(\psi) = \frac{1}{2}|\nabla\psi|^2 + \frac{1}{4}(|\psi|^2-1)^2 . \qquad (7.20)$$

Let $e_1(\varphi) = \frac{1}{2}|\nabla\varphi|^2$ and $\langle f(\psi)\rangle = f(\psi) - \sum_k f(\psi_k)$. Equation (4.6) implies

$$\int_{D_R\setminus\cup D_k} e(\psi) = \int_{D_R\setminus\cup D_k} e_1(\varphi_0) + \int_{D_R\setminus\cup D_k} O\big(d(x,\underline{a})^{-4}\big). \qquad (7.21)$$

Next, estimates (7.17) and

$$\nabla|\psi_i| = O(r_j^{-3}) \qquad (7.22)$$

give

$$\int_{D_R\setminus\cup D_k} e_1(\varphi_i) = \int_{D_R\setminus\cup D_k} e(\psi_i) + O(r_0^{-2}). \qquad (7.23)$$

Together with Eq. (7.10), this yields

$$\int_{D_R\setminus\cup D_k} \langle e(\psi)\rangle = \frac{1}{2}\sum_{i\neq j} \int_{D_R\setminus\cup D_k} \nabla\varphi_i\nabla\varphi_j + O(r_0^{-2}). \qquad (7.24)$$

Next, in the region D_i, we have $\psi = e^{i\varphi_0} f_i$, where $f_i \equiv |\psi_i|$. Expansion (7.9) implies that

$$\int_{D_i} \nabla\varphi_i\nabla\varphi_{(i)} = 0. \qquad (7.25)$$

Using this relation, we obtain

$$\int_{D_i} e(\psi) = \int_{D_i} e(\psi_i) + \int_{D_i} e_1(\varphi_{(i)}) + R,$$

where $R = \int_{D_i} (f_i^2-1)\alpha_i$. Expanding

$$\nabla\varphi_{(i)} = \nabla\varphi_{(i)}(a_i) + O\left(\frac{r_i}{d_{\underline{a}}^2}\right) \qquad (7.26)$$

and using that $|\nabla\varphi_{(i)}(x)|^2 = O\big(d(x,\underline{a})^{-2}\big)$, $\nabla\varphi_i(x) = O(r_i^{-1})$, and $\int_{D_i}(1-f_i^2)\nabla\varphi_i = 0$, we obtain

$$R = O\left(\frac{\ln r_0}{d_{\underline{a}}^2}\right).$$

In the forceless case, we can improve this estimate using relation (7.9) again to show that, as in (7.12),

$$\int_{D_i} (f_i^2-1)\nabla\varphi_i\nabla\varphi_{(i)} = \int_{D_i}(f_i^2-1)\left(-c_i\sin 2\theta_i + O\left(\frac{r_i}{d_{\underline{a}}^3}\right)\right) = \int_{D_i}(f_i^2-1)O\left(\frac{r_i}{d_{\underline{a}}^3}\right) = O\left(\frac{r_0}{d_{\underline{a}}^3}\right).$$

This gives

$$R = O\left(\frac{r_0}{d_{\underline{a}}^3}\right) \quad \text{if} \quad \nabla\varphi_i(a_i) = 0.$$

Finally, we observe that due to (7.15),

$$\frac{1}{2}\int_{D_k} |\nabla\varphi_{(k)}|^2 = \sum_{j\neq k}\int_{D_k} e_1(\psi_j) + I_{D_k} = \sum_{j\neq k}\int e(\psi_j) + I_{D_k} + O(r_0^{-2}),$$

where

$$I_D := \frac{1}{2}\sum_{i\neq j}\int_D \nabla\varphi_i\cdot\nabla\varphi_j.$$

Collecting the estimates above, we arrive at

$$\int_{D_k}\langle e(\psi)\rangle = I_{D_k} + O\left(\frac{\ln r_0}{d_{\underline{a}}^2}\right) + O\left(\frac{1}{r_0^2}\right), \qquad (7.27)$$

which together with (7.9) and (7.16) yields

$$\mathcal{E}_R(\psi) = E + \mathrm{Rem}, \qquad (7.28)$$

where Rem is given in (7.2) and

$$E = \int\left(g - \frac{(\deg\psi)^2}{n^2}\chi\right)$$

with

$$g = \sum_j e(\psi_j) + \frac{1}{2}\sum_{i\neq j}\nabla\varphi_i\nabla\varphi_j.$$

Now, by definition of the cut-off function χ ($\chi \geq 0$, $\chi = 1$ for $|x| \geq R$), we have

$$E \leq \int\limits_{B(0,R)} g + \int\limits_{B(0,R)^c} \left(g - \frac{n}{2r^2}\right), \qquad (7.29)$$

where $n = \deg \psi$. We first compute the first integral in the r.h.s.

By definition of $E_{n,R}$ and because $a_i \ll R$, we have

$$\int\limits_{D_R} e(\psi_i) = \int\limits_{D_R + a_i} e(\psi^{(n_i)}) = E_{n_i,R} + O\left(\frac{1}{R^2}\right). \qquad (7.30)$$

We now show that

$$I_{D_R} \equiv \frac{1}{2}\sum_{i \neq j} \int\limits_{D_R} \nabla\varphi_i \nabla\varphi_j = -\sum_{i\neq j} \pi n_i n_j \ln\left(\frac{|a_{ij}|}{R}\right). \qquad (7.31)$$

We compute

$$\int\limits_{D_R} \nabla\varphi_i \nabla\varphi_j = n_i n_j \int\limits_0^{2\pi}\int\limits_0^R \frac{r - a\cos\theta}{r^2 + a^2 - 2ar\cos\theta} dr\, d\theta, \qquad (7.32)$$

where $a = |a_{ij}|$. Furthermore, changing the integration variable as $\theta \to z = e^{i\theta}$ and computing the residue, we find

$$\int\limits_0^{2\pi} \frac{r - a\cos\theta}{r^2 + a^2 - 2ar\cos\theta} d\theta = \frac{\pi}{r} - \frac{r^2 - a^2}{2iar^2} \oint\limits_{|z|=1} \frac{dz}{\left(z - \frac{r}{a}\right)\left(z - \frac{a}{r}\right)} = \frac{\pi}{r} + \frac{\pi}{r}\frac{r^2 - a^2}{|r^2 - a^2|} = \frac{2\pi}{r}\begin{cases} 1 & \text{if } r > a, \\ 0 & \text{if } r < a. \end{cases}$$

The last two equations yield (7.24). We also observe that up to a multiplicative constant, expression (7.24) can be found from the symmetry considerations: the invariance of the integral in the l.h.s. under translations ($a_i \to a_i + h$ and $a_j \to a_j + h$ $\forall h \in \mathbb{R}^2$) and rotations ($a_i \to g a_i$ and $a_j \to g a_j$ $\forall g \in O(2)$) implies that it depends only on $|a_{ij}|$. Its scaling properties under the dilations ($a_i \to \lambda a_i$ and $a_j \to \lambda a_j$ $\forall \lambda \in \mathbb{R}$) imply that it is a multiple of $\ln(|a_{ij}|/R)$.

Equations (7.30) and (7.31) imply that

$$\int\limits_{B(0,R)} g = \sum E_{n_i,R} + H\left(\frac{\underline{a}}{R}\right) + O\left(\frac{1}{R^2}\right). \qquad (7.33)$$

Next, we estimate the second integral in the r.h.s. of (7.29). By Eqs. (7.17) and (7.22), we have

$$g = \frac{1}{2}|\nabla\varphi_0|^2 + O\big(d(x,\underline{a})^{-4}\big).$$

Furthermore, expanding the terms $\nabla\theta(x - a_j)$ in $\nabla\varphi_0(x) = \sum n_j \nabla\theta(x - a_j)$ around the point x, we obtain

$$\nabla\varphi_0(x) = n\nabla\theta(x) - \theta''(x)\sum n_j a_j + O\left(\frac{\sum n_j a_j^2}{d(x,\underline{a})^3}\right), \qquad (7.34)$$

where $\theta''(x)$ is the Hessian of $\theta(x)$. Choosing the origin such that $\sum n_j a_j = 0$ eliminates the second term on the r.h.s.. (Otherwise we could use that by an explicit computation,

$$\theta''(x)\nabla\theta(x) = -\frac{x}{r^4},$$

the integral of which over the exterior of the ball $B(0,R)$ vanishes.) Hence,

$$\int\limits_{B(0,R)^c} \left(g - \frac{n^2}{2r^2}\right) = \int\limits_{B(0,R)^c} O\left(\frac{\sum n_j a_j^2}{d(x,\underline{a})^4}\right) = O\left(\frac{\sum n_j a_j^2}{R^2}\right). \qquad (7.35)$$

Estimates (7.28), (7.29), (7.33), and (7.35) imply (7.7) with Rem given in (7.2).

Remark 7.3. The statement of Lemma 7.2 remains true for a wider class of functions defined by replacing (7.7) by the condition

$$f = f_i + O\left(\frac{1}{r d_{\underline{a}}^n}\right) \text{ and } \int\limits_0^{2\pi} \mathrm{Re}(e^{-i\varphi_0}\psi - f_i) d\theta = O\left(\frac{1}{d_{\underline{a}}^{n+1}}\right) \quad \text{if } |x - a_i| \ll d_{\underline{a}}, \qquad (7.36)$$

with the corresponding estimates of their first derivatives, where $n = 2$ if $\nabla H(\underline{a}) = 0$ and $n = 1$ otherwise.

To prove this, we write ψ in the region D_i as $\psi = e^{i\varphi_0}(f_i + \xi)$, where $f_i \equiv |\psi_i|$. Using relation (7.25) and

$$\int\limits_{D_j} f_j \nabla\varphi_j \nabla\,\mathrm{Im}\,\xi = n_j \int\limits_{D_j} f_j \frac{\partial}{\partial\theta}\mathrm{Im}\,\xi = 0, \qquad (7.37)$$

we obtain

$$\int_{D_i} e(\psi) = \int_{D_i} e(\psi_i) + \int_{D_i} e_1(\varphi_{(i)}) + R + R', \qquad (7.38)$$

where R is given above and

$$R' = \int_{D_i} \Big\{ (|\nabla\varphi_0|^2 + f_i^2 - 1) f_i \mathrm{Re}\,\xi + f_i^2 (\mathrm{Re}\,\xi)^2 + \\ + \frac{1}{2}|\nabla\varphi_0|^2|\xi|^2 + \frac{1}{2}|\nabla\xi|^2 + 2\nabla f_i \nabla \mathrm{Re}\,\xi + f_i \nabla\varphi_{(i)} \nabla \mathrm{Im}\xi + \\ + \mathrm{Im}(\xi\nabla\varphi_0 \cdot \nabla\xi) + \frac{1}{2}(f_i^2 - 1 + 2f_i \mathrm{Re}\,\xi)|\xi|^2 + \\ + \frac{1}{4}|\xi|^4 \Big\}. \qquad (7.39)$$

Using that

$$\xi = O\left(\frac{1}{rd_{\underline{a}}}\right)$$

and

$$\int_0^{2\pi} \mathrm{Re}\,\xi \, d\theta = O\left(\frac{1}{d_{\underline{a}}^2}\right) \quad \text{in} \quad D_j$$

due to (7.36) and that $|\nabla\varphi_i|^2 + f_i^2 - 1 = O(r_i^{-4})$, we find

$$R' = O\left(\frac{\ln r_0}{d_{\underline{a}}^2}\right). \qquad (7.40)$$

We now proceed to proving estimate (7.4) with Rem given by (7.3). First, we describe the class of test functions for which we prove this estimate: $\psi = e^{i\varphi_0} f$ with

$$f = \begin{cases} f_j - \dfrac{1}{2} f_j^{-1}\alpha_j\eta_j \\ \qquad\qquad \text{in } D\big(a_j, \frac{1}{3}d_{\underline{a}}\big) \quad \forall j & (7.41) \\ 1 - \dfrac{1}{2}|\nabla\varphi_0|^2 + O\big(d(x,\underline{a})^{-4}\big) \\ \qquad\qquad \text{in } \left(\bigcup_j D\big(a_j, \frac{1}{4}d_{\underline{a}}\big)\right)^c, & (7.42) \end{cases}$$

where we used definition (7.11) and where η_j are smooth cut-off functions depending only on $r_j = |x_j|$ (i.e., radially symmetric in the x_j variables) satisfying

$$B\left(a_j, \frac{1}{2}d_{\underline{a}}\right) \setminus B\big(a_j, 2d_{\underline{a}}^{\gamma}\big) \subset \mathrm{supp}\eta_j \subset \\ \subset B\left(a_j, \frac{1}{2}d_{\underline{a}}\right) \setminus B(a_j, d_{\underline{a}}^{\gamma}) \qquad (7.43)$$

and

$$\nabla^n \eta_j = O\big(d_{\underline{a}}^{-\gamma n}\big), \quad n = 0, 1, 2, \qquad (7.44)$$

for $\gamma = \frac{1}{3}$ (not optimal). (The f_j^{-1}'s in (7.41) play no important role and are chosen purely with a view of simplifying some expressions below.)

The function

$$f = \sum f_j \chi_j - \sum \frac{1}{2} f_j^{-1}\alpha_j\eta_j \qquad (7.45)$$

satisfies Eqs. (7.41) and (7.42). To prove this, we use the expansion

$$f_j = 1 - \frac{1}{2}|\nabla\varphi_j|^2 + O\big(r_j^{-4}\big) \qquad (7.46)$$

and the estimate

$$\alpha_j = O\big(d_{\underline{a}}^{-2}\big) \text{ in } D\big(a_j, d_{\underline{a}}\big), \qquad (7.47)$$

which is shown by expanding the function $\nabla\varphi_{(j)}(x)$ around a_j and using that

$$\nabla\varphi_{(j)}(a_j) = -\frac{1}{2\pi n_j}\nabla_{a_j} H(\underline{a}) = 0$$

and

$$\nabla\varphi_j(x) = O(r_j^{-1}).$$

Our next task is to prove the following lemma.

Lemma 7.4. Let $\underline{a}$ be forceless in the sense that $\nabla H(\underline{a}) = 0$. Then estimate (7.7) with (7.3) holds for any function ψ satisfying (7.21)–(7.22).

Proof. The proof follows the lines of the proof of Lemma 7.2, but with some subtle modifications considered below.

First of all, instead of $e_1(\psi) = \frac{1}{2}|\nabla\varphi|^2$ used in the proof of Lemma 7.2, we use the density

$$e_2(\varphi) = \frac{1}{2}|\nabla\varphi|^2 - \frac{1}{4}|\nabla\varphi|^4, \qquad (7.48)$$

which is a better approximation to the density $e(\psi)$. We also use (7.27) instead of (7.17). In particular, we have

$$e(\psi_j) = e_2(\varphi_j) + O(r_j^{-6}). \qquad (7.49)$$

We set $f_j := 1 - f_j^2 - |\nabla\varphi_j|^2$. For any k and for $u_k = e^{i\varphi_0}(f_k + \xi)$, where ξ is a real function, we have the identity

$$\langle e(u_k)\rangle = \frac{1}{2}\sum_{i\neq j} \nabla\varphi_i\nabla\varphi_j - A(\varphi) + B_k(\xi) + R_k, \quad (7.50)$$

where

$$B_k(\xi) := -\frac{1}{2}g_k(\alpha_k + 2f_k\xi) + \frac{1}{4}\alpha_k^2 + \alpha_k f_k \xi + f_k^2\xi_k^2 \quad (7.51)$$

and

$$R = \sum_{j\neq k}\big(e_2(\varphi_j) - e(\psi_j)\big) - \frac{1}{2}(g_k - \alpha_k)\xi^2 + f_k\xi_k + \frac{1}{4}\xi_k^4 + \\ + \frac{1}{2}\big(2\nabla f_k \nabla\xi + |\nabla\xi|^2\big). \quad (7.52)$$

We now take $\xi = -\frac{1}{2} f_k^{-1} \alpha_k \eta_k$. Then

$$e(\psi) = e(u_k) \quad \text{on } D\left(a_k, \frac{1}{3} d_{\underline{a}}\right). \tag{7.53}$$

Due to (7.28) and the corresponding estimate for the derivatives of α_j and due to (7.25), (7.27), and (7.29), we have

$$R_k = O\big(d_{\underline{a}}^{-4\gamma-2}\big). \tag{7.54}$$

We note that the form of (7.21) is chosen such that

$$B_k(\xi) = 0 \quad \text{on } B\left(a_k, \frac{1}{2} d_{\underline{a}}\right) \backslash B\big(a_k, d_{\underline{a}}^{\gamma}\big) \subset \{\eta_k = 1\}.$$

Next, we estimate $B_k(\xi)$ on the entire disc $D\big(a_k, \frac{1}{3} d_{\underline{a}}\big)$. Expanding the function $\nabla \varphi_{(k)}(x)$ around the point a_k and using that

$$\nabla \varphi_{(k)}(a_k) = -\frac{1}{2\pi n_k} J \nabla_{a_k} H(\underline{a}) = 0,$$

we find

$$\alpha_k(x) = 2 \nabla \varphi_k(x) \varphi''_{(k)}(a_k) x_k + O\big(r_k d_{\underline{a}}^{-3}\big), \tag{7.55}$$

where $x_k = -a_k$ and φ'' is the Hessian (the matrix of second derivatives) of a function φ. Using this expression in estimating $B_k(\xi)$, we find

$$B_k(\xi) = -g_k \nabla \varphi_k(x) \varphi''_{(k)}(a_k) x_k \bar{\eta}_k + \\ + O\big(r^{-3} d_{\underline{a}}^{-3} + d_{\underline{a}}^{-4}\big) \bar{\eta}_k \quad \text{on } D\left(a_k, \frac{1}{3} d_{\underline{a}}\right), \tag{7.56}$$

where $\bar{\eta}_k = 1 - \eta_k$. The first term in the r.h.s. of this expression is singular at $x_k = x - a_k = 0$, but the integral of it is conditionally convergent and equals 0. Indeed, because the function $\varphi_{(k)}(x)$ is harmonic in $D\big(a_k, \frac{1}{3} d_{\underline{a}}\big)$, we have that (cf. (7.9))

$$\varphi''_{(k)}(a_k) x_k = c\big(x_k \cos 2\theta_k - x_k^{\perp} \sin 2\theta_k\big), \tag{7.57}$$

where $c = O\big(d_{\underline{a}}^{-2}\big)$, $x^{\perp} = (-x_2, x_1)$, and θ_k is the polar angle of x_k (see Eq. (7.9)). Because g_k and $\bar{\eta}_k$ depend only on r_k (we write $(g_k \bar{\eta}_k)(r_k)$ for $g_k(x) \bar{\eta}_k(x)$), we have

$$\int (g_k \bar{\eta}_k)(r_k) \nabla \varphi_k(x) \varphi''_{(k)}(a_k) x_k = \\ = -c \int (g_k \bar{\eta}_k)(r_k) \sin 2\theta_k = 0 \tag{7.58}$$

(strictly speaking, we must first excise a small disc around $x_k = 0$ and then take the radius of this disc to zero).

Equations (7.32), (7.33), (7.35), and (7.37) imply that

$$\int\limits_{D\left(a_k, \frac{1}{3} d_{\underline{a}}\right)} \langle e(\psi) \rangle = \\ = \int\limits_{D\left(a_k, \frac{1}{3} d_{\underline{a}}\right)} \left(\frac{1}{2} \sum_{i \neq j} \nabla \varphi_i \nabla \varphi_j - A(\varphi) \right) + \\ + \quad O\big(d_{\underline{a}}^{-3} + d_{\underline{a}}^{-2-4\gamma} + d_{\underline{a}}^{-4+2\gamma}\big). \tag{7.59}$$

Finally, we derive the estimate

$$\langle e(\psi) \rangle = \frac{1}{2} \sum_{i \neq j} \nabla \varphi_i \nabla \varphi_j - A(\varphi) + O\big(d(x, \underline{a})^{-6}\big) \tag{7.60}$$

on $\left(\bigcup_k D\big(a_k, \frac{1}{4} d_{\underline{a}}\big) \right)^c$. Indeed, Eq. (7.42) implies that

$$e(\psi) = e_2(\varphi_0) + O\big(d(x, \underline{a})^{-6}\big), \tag{7.61}$$

which together with (7.49) implies (7.60).

Now, Eqs. (7.59) and (7.60) with $\gamma = 1/3$ imply

$$\mathcal{E}_R(\psi) = E - A(\underline{a}) + O\big(d_{\underline{a}}^{8/3}\big), \tag{7.62}$$

where the term E is defined after Eq. (7.28) and $A(\underline{a}) = \int A(\varphi)$. Equations (7.29), (7.33), (7.35), and (7.61) imply (7.14) with Rem given by (7.3).

Lemmas 7.2 and 7.4 and inequality (7.13) imply Theorem 7.1.

8. DISCUSSION

In this paper, we investigated the Ginzburg–Landau equation (1.1) appearing in condensed matter physics and nonlinear optics. Specifically, we presented careful arguments supporting the existence of non-radial-symmetric solutions corresponding to vortex configurations c with $N+1$ vortices fixed by the symmetry group C_{Nv}. In these configurations, N m-vortices lie on the circle of radius a and one $(-k)$-vortex is placed at the center of the circle, and the only remaining free parameter is the overall size of the configuration — the radius of the circle a.

Our argument is based on reducing the problem of the existence of solutions corresponding to a given vortex configuration to the existence of critical points of the effective energy of the vortex configurations introduced in this paper. For C_{Nv} configurations, this effective energy is a function of a single variable, a. To prove the existence of critical points of this energy, we investigated it analytically and numerically at large and small values of the parameter a. We found that there are critical points at the vortex configurations ($N = 2$, $m = 2$, $k = 1$) and ($N = 4$, $m = 2$, $k = 3$) and, consequently, we expect the existence of (static) solutions corresponding to these configurations. For the vortex configuration ($N = 3$, $m = 1$, $k = 1$), our numerical analysis indicates that it is very likely that such a critical point exists. Our numerical computations suggest that the critical a's are of the order $O(1)$. Finding their true values requires rather elaborate numerical analysis, which would be desirable to develop but which is presently absent. In addition, we have shown (see Appendix 5) that for the vortex configurations (N, 1, $(N-1)/2$) with $N \gg 1$ odd and a sufficiently large, the energy is greater than the effective energy of a single N-vortex.

All the solutions considered are saddle points of the renormalized Ginzburg–Landau energy functional. Perturbations breaking the C_{Nv} symmetry group can lower the energy of the corresponding solution vortex configuration. However, we expect that under small symmetry breaking perturbations, such solutions lead to long-living metastable states that can be observed experimentally. Moreover, even weak pinning centers can stabilize such solutions. Thus, to experimentally observe the static configurations found in this paper, one would need to create weak pinning potentials satisfying the suggested point symmetry, adjust the radius a at which these potentials are located, and then slowly reduce the strength of these potentials to zero.

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APPENDIX 1

Computation of $c(n)$

In this appendix, we compute the constants $c(n)$ in expression (3.5) for the self-energy $E_{n,R}$ of the n-vortex (see Eq. (3.6)). For this, we derive a convenient formula for $E_{n,R}$. Multiplying Eq. (3.2) by $r^2 f_n'(r)$, where $f'(r) = \partial f(r)/\partial r$, integrating the result over r, observing that the first two integrands are total derivatives, and integrating the last term by parts, we obtain the quantization relation (see [22])

$$\int_0^\infty (1 - f_n^2)^2 r dr = n^2.$$

This equation together with Eq. (3.3) yields an expression for $E_{n,R}$,

$$E_{n,R} = -\frac{\pi}{2} n^2 + \pi \int_0^\infty \left(1 - f_n^2 - \frac{n^2}{r^2}\chi\right) r\, dr.$$

However, we prefer to use a different representation of $E_{n,R}$, which is obtained from above if we write $1 - f_n^2 = (1 - f_n^2) f_n^2 + (1 - f_n^2)^2$ and use the quantization formula above again:

$$E_{n,R} = \frac{\pi}{2} n^2 + \pi \int_0^\infty \left[(1 - f_n^2) f_n^2 - \frac{n^2}{r^2}\chi\right] r\, dr. \quad \text{(A.1.1)}$$

To avoid numerical evaluation of the integral in (A1.1) over an infinite range, we use the expansion of $f_n(r)$ in $1/r$ for large r. However, $f_n(r)$ is not analytic at $r = \infty$; it has an essential singularity at this point. Hence, the resulting series is asymptotic. We truncate this series at the order $O\left(\frac{1}{r^6}\right)$. To compensate for this truncation, we add to the resulting polynomial in $1/r$ a multiple of the decaying solution $e^{-\sqrt{2}r}/\sqrt{r}$ of the linearization of Eq. (3.2) around 1. We should linearize Eq. (3.2) around the resulting polynomial, but the powers of $1/r^2$ lead to similar powers multiplying $e^{-\sqrt{2}r}/\sqrt{r}$, and it therefore suffices to linearize around 1. The result is

$$f_n(r) = \left\{1 - \frac{n^2}{2r^2} - \frac{n^2(1 + n^2/8)}{r^4} - \frac{1}{r^6}\left(\frac{n^4}{2} + \right.\right.$$
$$\left.\left. + \frac{n^2 + 16}{2}\left(n^2 + \frac{n^4}{8}\right)\right) - \ldots\right\} -$$
$$- c\frac{e^{-\sqrt{2}r}}{\sqrt{r}}(1 + \ldots), \quad \text{(A.1.2)}$$

where c is a constant to be determined by a matching procedure. Inserting this expression in Eq. (A1.1), we obtain

$$E_{n,R} - \pi n^2 \ln R = \frac{\pi n^2}{2} + \pi \int_0^{r_0} f_n^2 (1 - f_n^2) r\, dr -$$
$$- \pi n^2 \left(\ln r_0 + \frac{n^2 - 2}{2r_0^2} + \frac{n^2 - 16}{4r_0^4}\right) +$$
$$+ O(r_0^{-6}) \quad \text{(A.1.3)}$$

for any $r_0 > 0$. We choose $6 \le r_0 \le 10$. This relation together with Eq. (3.5) implies that

$$\frac{1}{\pi}c(n) = \frac{n^2}{2} + \int_0^{r_0} f_n^2(1-f_n^2) r\,dr - n^2\left(\ln\frac{r_0}{|n|} + \frac{n^2-2}{2r_0^2} + \frac{n^2-16}{4r_0^4}\right) + O(r_0^{-6}). \quad (A.1.4)$$

For numerical solution of Eq. (3.2), we take the interval $(0.3, r_0)$. Because Eq. (3.2) linearized around the function 1 has the solutions

$$\frac{1}{\sqrt{r}}e^{\pm\sqrt{2}r}, \quad (A.1.5)$$

we should apply the numerical iteration procedure starting from the upper limit, r_0. Then the dangerous, exponentially growing solution would not affect our procedure.

In the range $0 < r \le 0.3$, we use the fact that as Eq. (3.2) shows, the function $f_n(r)$ is analytic in a disc $|r| < O(1)$, and can therefore be presented by a convergent series

$$f_n(r) = \alpha r^n\left\{1 - \frac{r^2}{4(n+1)} + \frac{r^4}{8(n+2)} \times \left(\frac{1}{4(n+1)} + \alpha^2\delta_{n,1}\right) + \frac{r^6}{12(n+3)} \times \left[\alpha^2\left(\delta_{n,2} - \frac{3}{4(n+1)}\delta_{n,1}\right) - \frac{1}{8(n+2)} \times \left(\frac{1}{4(n+1)} + \alpha^2\delta_{n,1}\right)\right] + \ldots\right\} \quad (A.1.6)$$

for some number $\alpha > 0$. Here, $\delta_{n,k}$ is the Kronnecker symbol, $\delta_{n,k} = 1$ for $n = k$ and $= 0$ for $n \ne k$. (We expect that the pole closest to the origin lies on the imaginary axis.)

To finish the computation of $c(n)$, we must find the value of the parameters α and c. This is done by matching solution (A1.2) for small r with solution (A1.6), for large r. Specifically, using Eq. (A1.2), we compute $f_n(r_0)$ and $f_n'(r_0)$ for various values of the parameter c. Using these values as initial conditions, we integrate Eq. (3.2) backward to $r = 0.3$, which yields $f_{\text{right}}(0.3)$ and $f'_{\text{right}}(0.3)$. On the other hand, using Eq. (A1.6), we compute $f_{\text{left}}(0.3)$ and $f'_{\text{left}}(0.3)$ for various values of the parameter α. We then match $f_{\text{right}}(0.3)$ and $f'_{\text{right}}(0.3)$ with $f_{\text{left}}(0.3)$ and $f'_{\text{left}}(0.3)$ by minimizing $\left[\left(f_{\text{right}}(0.3) - f_{\text{left}}(0.3)\right)^2 + \left(f'_{\text{right}}(0.3) - f'_{\text{right}}(0.3)\right)^2\right]^{1/2}$. This yields the values of the parameters c and α. After this, we compute $c(n)$ using formulas (A.1.4) and (A.1.6).

APPENDIX 2

Large-n asymptotic form of the vortex (self) energy

In this appendix, we find the large-n asymptotic form of the constant $c(n)$ in expression (3.6) for the (self) energy of the n-vortex. For this, we use the large-n asymptotic expression for the function $f_n(r)$ defined in (3.2),

$$f_n(r) = \begin{cases} \sqrt{1-n^2/r^2} & \text{if } r-n \gg (n/2)^{1/3}, \\ (2/n)^{1/3}\,g(z) & \text{if } |r-n| \ll n, \end{cases} \quad (A.2.1)$$

where the variable z is defined by

$$r = n + \left(\frac{n}{2}\right)^{1/3} z \quad (A.2.2)$$

and the function $g(z)$ is a solution of the equation

$$g'' + zg - g^3 = 0. \quad (A.2.3)$$

The function $g(z)$ has the asymptotic form

$$\begin{aligned} g(z) &= z^{1/2} \quad \text{if } z \gg 1, \\ g(z) &= \text{const}\,\phi(z) \quad \text{if } z \ll -1, \end{aligned} \quad (A.2.4)$$

where $\phi(z)$ is the Eiry function. In particular, we have

$$g(z) = \frac{0.39}{(-z)^{1/4}}e^{-2(-z)^{3/2}/3} \quad \text{for } z \ll -1. \quad (A.2.5)$$

Inserting expression (A.2.1)–(A.2.2) in Eq. (A.1.1) and using (A.2.4) and (A.2.5), we find that

$$c(n) = \alpha n^{2/3}\pi + c + O(n^{-2/3}), \quad (A.2.6)$$

where c is some constant and

$$\alpha = 2^{1/3}\int_{-\infty}^{\infty}\left(g^2(z) - z\theta(z)\right)dz, \quad (A.2.7)$$

with $\theta(z) = 1$ for $z \ge 0$ and $= 0$ for $z < 0$. Multiplying Eq. (A.2.3) by $g'(z)$ and integrating the result, we find that $\alpha = 0$, and therefore

$$c(n) = c + O(n^{-2/3}) \quad (A.2.8)$$

as $n \to \infty$. A rough numerical computation yields the following value for the constant c:

$$c \approx 0.7\pi. \quad (A.2.9)$$

APPENDIX 3

Computation of correlation coefficients

In this appendix, we compute the correlation function

$$A = A(\underline{a}) = \frac{1}{4}\int\left[|\nabla\varphi_0|^4 - \sum_j |\nabla\varphi_j|^4\right] \qquad \text{(A.3.1)}$$

with

$$\varphi_0 = \sum_j \varphi_j \text{ and } \varphi_j(x) = n_j\theta(x-a_j), \qquad \text{(A.3.2)}$$

$$\theta(x) = \text{the polar angle of } x \in \mathbb{R}^2 \qquad \text{(A.3.3)}$$

(see Eq. (4.4)) for configurations of $K = N+1$ vortices with N vortices of vorticity m lying on the circle of radius a and one vortex of vorticity $-k$ at the center of this circle, such that $\nabla H(\underline{a}) = 0$.

We write $\underline{a} = a\cdot\underline{b}$ where $\underline{b}$ is a fixed configuration with N vortices on the unit circle and one at the center. Changing the integration variable in (A.3.4) as $x = ay$, we find

$$A(\underline{a}) = Ca^{-2}, \qquad \text{(A.3.4)}$$

where C depends on $\underline{b}$ only. Our task is now to find the sign of C for the configurations of interest. We write $A = A(\underline{a})$.

1. $N = 2$, $m = 2$, and $k = 1$. In this case, there are two double vortices on the circle and one single vortex of the opposite vorticity at the center (see Fig. 1). Below, we use the dimensionless variable

$$\rho = \frac{|x|}{a}. \qquad \text{(A.3.5)}$$

For the configuration under consideration, we have

$$A = \frac{1}{4a^2}\int_0^\infty \rho d\rho \times \int_0^{2\pi} d\theta\left\{\frac{48}{\alpha} - \frac{16\cos(2\theta)}{\alpha\rho^2} + \frac{64\cos^2(2\theta)}{\alpha^2} - \frac{64}{\alpha^2}\big(1+2\rho^2+2\rho^2\cos(2\theta)\big)\right\}, \qquad \text{(A.3.6)}$$

where

$$\alpha = \rho^4 + 1 + 2\rho^2\cos(2\theta). \qquad \text{(A.3.7)}$$

(In general, for a_j, $j = 1,\ldots,N$, distributed equidistantly on the circle of radius a, $\alpha = \prod_{j=1}^{N}(x-a_j)^2/a^{2N}$.) First, we take the integral over θ. For this, we change the integration variable as $\theta \to z = \exp(2i\theta)$, i.e., we write the inner integral in (A.3.8) as an integral over the unit circle. A simple calculation gives

$$\int_0^{2\pi}\frac{d\theta}{\alpha^2} = \frac{2\pi(1+\rho^4)}{|1-\rho^4|^3}, \qquad \int_0^{2\pi}\frac{d\theta}{\alpha^2}\cos(2\theta) = -\frac{4\pi\rho^2}{|1-\rho^4|^3}, \qquad \text{(A.3.8)}$$

$$\int_0^{2\pi}\frac{d\theta}{\alpha} = \frac{2\pi}{|1-\rho^4|}, \qquad \int_0^{2\pi}\frac{d\theta}{\alpha}\cos(2\theta) = -\frac{2\pi}{|1-\rho^4|}\min\left\{\rho^2, \frac{1}{\rho^2}\right\}, \qquad \text{(A.3.9)}$$

$$\int_0^{2\pi}\frac{d\theta}{\alpha^2}\cos^2(2\theta) = \frac{\pi}{|1-\rho^4|^3} \times \begin{cases} 1+4\rho^4-\rho^8 & \text{for } \rho < 1, \\ (\rho^8+4\rho^4-1)/\rho^4 & \text{for } \rho > 1. \end{cases} \qquad \text{(A.3.10)}$$

Inserting expressions (A.3.7)–(A.3.10) in Eq. (A.3.7), we obtain

$$A = \frac{4\pi}{a^2}\left\{2\int_0^1 dx\frac{1-x}{(1+x)^3} + \int_1^\infty \frac{dx}{(1+x)^3}\left(3x+1+\frac{3}{x}+\frac{1}{x^2}\right)\right\}.$$

This gives

$$A = \frac{8\pi}{a^2}. \qquad \text{(A.3.11)}$$

Hence, in the configuration under consideration, the energy $E_R(\underline{a})$ is given by

$$\frac{1}{\pi}E_R(\underline{a}) - 9\ln R = -9.64 - \frac{8}{a^2} + O\left(\frac{\ln a}{a^4}\right). \qquad \text{(A.3.12)}$$

2. $N = 3$, $m = 1$, and $k = 1$. Similarly to Eq. (A.3.8), we obtain

$$A = \frac{1}{4a^2}\int_0^\infty d\rho\rho\int_0^{2\pi} d\theta\left\{\frac{6}{\alpha}(1+2\rho^2) - \frac{12\sin(3\theta)}{\rho\alpha} - \frac{9(1+\rho^2)}{\alpha^2}(1+\rho^2+2\rho^4) + \frac{36\rho^2\sin^2(3\theta)}{\alpha^2} - \frac{36\rho^5\sin(3\theta)}{\alpha^2}\right\}, \qquad \text{(A.3.13)}$$

where $\alpha = \rho^6 + 1 + 2r^3 \sin(3\theta)$. The integrals in Eq. (A.3.13) can be taken explicitly. To do this, we set $z = \exp(3i\theta)$, and then

$$\int_0^{2\pi} \frac{d\theta}{\alpha} = \frac{2\pi}{|1-\rho^6|}, \qquad \int_0^{2\pi} \frac{d\theta}{\alpha} \sin(3\theta) = -\frac{2\pi}{|1-\rho^6|} \min\left(\rho^3, \frac{1}{\rho^3}\right), \tag{A.3.14}$$

$$\int_0^{2\pi} \frac{d\theta}{\alpha^2} = \frac{2\pi(1+\rho^6)}{|1-\rho^6|^3}, \qquad \int_0^{2\pi} \frac{d\theta}{\alpha^2} \sin(3\theta) = -\frac{4\pi\rho^3}{|1-\rho^6|^3} \tag{A.3.15}$$

and

$$\int_0^{2\pi} \frac{d\theta}{\alpha^2} \sin^2(3\theta) = \frac{\pi}{|1-\rho^6|^3} \times \left\{ \begin{array}{ll} 1 + 4\rho^6 - \rho^{12} & \text{for } \rho < 1, \\ (\rho^{12} + 4\rho^6 - 1)/\rho^6 & \text{for } \rho > 1. \end{array} \right. \tag{A.3.16}$$

Inserting expressions (A.3.14)–(A.3.16) in Eq. (A.3.13), we obtain

$$A = \frac{3\pi}{4a^2} \left\{ \int_0^1 dx \frac{5x + 9x^2 - 1 - 2x^3 - 2x^4}{(1+x+x^2)^3} + \int_1^\infty dx \left(\frac{4}{1+x+x^2} - \frac{9}{(1+x+x^2)^2} + \frac{10x+18}{(1+x+x^2)^3} + \frac{6x+2}{x^2(1+x+x^2)^3} \right) \right\}. \tag{A.3.17}$$

A simple calculation of integrals in Eq. (A.3.16) gives explicit answers for A:

$$A = \frac{2\pi}{a^2}. \tag{A.3.18}$$

Hence, the energy for such configurations is given by

$$\frac{1}{\pi} E_R(a) - 4 \ln R = -1.792 - \frac{2}{a^2}. \tag{A.3.19}$$

3. $N = 4$, $m = 2$, and $k = 3$. In this case, there are four double vortices in the corners of a rectangle and a (-3)-vortex in the center. For this configuration, we have

$$A = \frac{16}{a^2} \int_0^{2\pi} d\theta \int_0^\infty \frac{d\rho\,\rho}{\alpha} \left\{ \frac{4\rho^{12}}{\alpha} + \frac{36\rho^4 \cos^2(4\theta)}{\alpha} + 4.5\rho^4 + 13.5\cos(4\theta) + \frac{24\rho^8}{\alpha}\cos(4\theta) - \frac{1}{\alpha}\left[(\rho^2+1)^6 - 2\rho^2(\rho^2+1)^2(\rho^4+1) + 4\rho^6\right] - 2\rho^4 \cos(4\theta)\left(3(\rho^2+1)^2 - 2\rho^2\right)/\alpha \right\}, \tag{A.3.20}$$

where

$$\alpha = \rho^8 + 1 - 2\rho^4 \cos(4\theta).$$

The change of variables $2\theta \to \tilde{\theta} + \pi/2$, $\rho^8 \to \tilde{\rho}^4$ reduces the integrals over θ in Eq. (A.3.20) to those in Eqs. (A.3.8)–(A.3.10). As a result, we obtain

$$A = \frac{16\pi}{a^2} \left\{ \int_0^1 dx \left[\frac{1-3x}{1+x+x^2+x^3} + \frac{2(5x^5 + 23x^4 + 18x^3 + 6x^2 - 3x - 1)}{(1+x+x^2+x^3)^3} + \int_1^\infty dx \left(\frac{7.5}{1+x^2} - \frac{1.5}{x^2(1+x^2)} - \frac{4(1+x+x^2)}{x^2(1+x+x^2+x^3)} - \frac{2}{(1+x+x^2+x^3)^3} \left(x^5 + 11x^4 - 2x^3 - 22x^2 - 31x - 21 - \frac{12}{x} - \frac{4}{x^2} \right) \right) \right] \right\}. \tag{A.3.21}$$

Direct calculation of the integrals in Eq. (A.3.11) gives

$$A = \frac{80\pi}{a^2}, \tag{A.3.22}$$

and therefore the energy of the configuration in question is

$$\frac{1}{\pi} E_R(a) - 25 \ln R = -40.44 - \frac{80}{a^2}. \tag{A.3.23}$$

We note that for all the configurations under consideration, the correlation term A is given by

$$A = \frac{\pi}{4a^2} M$$

where M is an integer, i.e., the quantity given by the integral in A is quantized. Moreover, the «quantization» takes place separately for the integrals over regions $r < 1$ and $r > 1$. We conjecture that this property is general and holds for any forceless configuration.

APPENDIX 4

Inequality $E'_R(0) > 0$

In this appendix, we show that $E_R(a) - E_R(0) > 0$ for the configuration consisting of N 1-vortices equidistributed on the circle of radius a and one $(-\frac{N-1}{2})$-vortex at the center and for a sufficiently small. We assume that N is odd but otherwise arbitrary.

For $a = 0$, the configuration in question collapses to a single $\frac{N+1}{2}$-vortex, $\psi_{\frac{N+1}{2}}$, sitting at the origin. Let L be the Hessian of $\mathcal{E}_{\rm ren}(\psi)$ at $\psi = \psi_{\frac{N+1}{2}}$. It was shown in [13] that the subspaces

$$\left\{u_1(r)e^{im\theta} + u_2(r)e^{i(2\frac{N+1}{2}-m)\theta} | u_k \in \in L^2(rdr),\ k = 1, 2\right\}, \quad (A.4.1)$$

$m = \frac{N+1}{2}, \frac{N+1}{2} + 1, \ldots$, which are orthogonal to each other and span the entire Hilbert space $L^2(\mathbb{R}^2)$, are invariant under the action of the operator L. Moreover, it was shown that in the sectors with $m \geq 3\frac{N-1}{2} - 1$, L is nonnegative and 0 is not its eigenvalue (actually, the statement in [13] is formulated for $m \geq 3\frac{N-1}{2}$, but the proof works also for $m = 3\frac{N-1}{2} - 1$), while in the sectors

$$\frac{N+1}{2} + 2 \leq m \leq 2\frac{N+1}{2},$$

the operator L has negative eigenvalues. We now observe that the sectors with $\frac{N+1}{2} \leq m \leq 3\frac{N-1}{2} - 2$ do not have the C_{Nv} symmetry and, consequently, are forbidden in our case. Therefore, on the subspace invariant under the action of the group C_{Nv}, $L \geq 0$ and 0 is now its eigenvalue. The latter implies that

$$E_R(a) - E_R(0) > 0 \quad (A.4.2)$$

for any odd N and for sufficiently small a.

APPENDIX 5

Large-N asymptotic forms

In this appendix, we find asymptotic behavior of the energy of the circular asymptotically forceless configurations, i.e., the ones with $\nabla H(\underline{a}) = 0$, for large values of N. More precisely, the configurations we consider consist of N 1-vortices equally spaced on the circle of radius a and with the center at the origin and one $(-k)$-vortex at the center. We recall that the condition $\nabla H(\underline{a}) = 0$ is equivalent to the relation $k = -(N-1)/2$. We assume in addition that N is odd and $a \gg N$.

According to Eq. (5.10) and because

$$\sin\frac{\pi k}{N} = \sin\frac{\pi(N-k)}{N},$$

the energy of the above configuration is

$$E_R(\underline{a}) = \pi\left(\frac{N+1}{2}\right)^2 \ln R - \pi\left(\frac{N-1}{2}\right)^2 \times \ln\left(\frac{N-1}{2}\right) + Nc(1) - 2\pi N \sum_{k=1}^{\frac{N-1}{2}} \ln\left(2\sin\frac{\pi k}{N}\right), \quad (A.5.1)$$

where we use the notation $E_R(a) = E_R(\underline{a})$. For $a = 0$ (the «initial state»), the energy is given by Eq. (3.5),

$$E_R(0) = \pi\left(\frac{N+1}{2}\right)^2 \ln R - \pi\left(\frac{N+1}{2}\right)^2 \ln\left(\frac{N+1}{2}\right). \quad (A.5.2)$$

To calculate the sum in Eq. (A.5.1), we use the Euler expansion

$$\sum_{k=M}^{L} f(k) = \int\limits_{M-\frac{1}{2}}^{L+\frac{1}{2}} f(x)\,dx - \frac{1}{24}\left(f'\left(L+\frac{1}{2}\right) - f'\left(M-\frac{1}{2}\right)\right) \quad (A.5.3)$$

and

$$\int\limits_0^{\pi/2} \ln(2\sin x)\,dx = 0, \quad (A.5.4)$$

$$\sum_{k=1}^{M} \ln k = \ln\Gamma(M+1),$$

where $\Gamma(x)$ is Euler gamma-function,

$$\sum_{k=1}^{\frac{N-1}{2}} \ln\left(2\sin\frac{\pi k}{N}\right) = \sum_{k=1}^{M} \ln\left(2\sin\left(\frac{\pi k}{N}\right)\right) - \frac{N}{\pi}\int\limits_0^{\frac{\pi M}{N}} dz\,\ln(2\sin Z) + \sum_{k=M+1}^{\frac{N-1}{2}} \ln\left(2\sin\left(\frac{\pi k}{N}\right)\right) - \frac{N}{\pi}\int\limits_{\frac{\pi M}{N}}^{\pi/2} dz\,\ln(2\sin Z) = \sum_{k=1}^{M} \ln\left(\frac{2\pi k}{N}\right) - M\left(\ln\left(\frac{2\pi M}{N}\right) - 1\right) - \frac{1}{2}\ln\left(\frac{2\pi M}{N}\right) + \frac{1}{24M},$$

where $1 \ll M \ll N$. For $N \gg 1$, this yields

$$\sum_{k=1}^{\frac{N-1}{2}} \ln\left(2\sin\frac{\pi k}{N}\right) = \frac{1}{2}\ln N \qquad \text{(A.5.5)}$$

modulo terms $O(1)$ in N. As a result, we have the energy difference

$$E_R(a) - E_R(0) = N\left[c(1) + \left(\frac{1}{2} - \ln 2\right)\pi\right] = \\ = 0.183\pi N. \qquad \text{(A.5.6)}$$

Thus, for ($N \gg 1$)-vortices placed equidistantly on a circle of radius $a \gg N$, the energy is greater than the effective energy of a single N-vortex.

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REFERENCES

1. F. Bethuel, H. Brezis, and F. Hélein, Calc. Variat. and PDE **1**, 123 (1993).
2. F. Bethuel, H. Brezis, and F. Hélein, *Ginzburg–Landau Vortices,* Birkhäuser, Basel (1994).
3. F. Pacard and T. Rivière, *Linear and Nonlinear Aspects of Vortices*: *The Ginzburg–Landau Model*, Progress in Nonlinear Diffirential Equations and Their Applications **39**, Publ.: Birkhäuser, Boston (2000).
4. Yu. N. Ovchinnikov and I. M. Sigal, Physica A **261**, 143 (1998).
5. T. Rivière, Manuscripta Math. **108**, 217 (2002).
6. F. Dalfovo, S. Giogini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).
7. A. L. Fetter and A. A. Svidzinsky, E-print archives cond-mat/0102003 (2001).
8. V. L. Ginzburg and L. Pitaevskii, Sov. Phys. JETP **7**, 585 (1958).
9. P. Hagan, SIAM J. Applied Math. **42**, 762 (1982).
10. M. Hervé and R. Hervé, Ann. Inst. Henri Poincaré, Analyse non linéaire **11**, 427 (1994).
11. Y. Chen, C. Elliot, and T. Qui, Proc. Roy. Soc. Edinburgh **124A**, 1068 (1994).
12. P. Fife and L. A. Peletier, Quart. Appl. Math. **54**, 85 (1996).
13. Yu. N. Ovchinnikov and I. M. Sigal, CRM Proceedings and Research Notes **12**, 199 (1997).
14. S. Chanillo and M. Kiesling, Compt. Rendu. Acad. Sci., Paris, Série I, **321**, 1023 (1995).
15. P. Mironescu, J. Funct. Anal. **130**, 334 (1995).
16. P. Mironescu, Compt. Rendu. Acad. Sci., Paris, Série I, **323**, 593 (1996).
17. E. Lieb and M. Loss, Math. Res. Lett. **1**, 701 (1994).
18. Yu. N. Ovchinnikov and I. M. Sigal, Euro. J. Applied Math. **13**, 153 (2002).
19. I. Shafrir, C. R. Acad. Sci. Paris **318**, Série I, 327 (1994).
20. Yu. N. Ovchinnikov and I. M. Sigal, Rev. Math. Phys. **12**, 287 (2000).
21. Yu. N. Ovchinnikov and I. M. Sigal, Nonlinearity **11**, 1277 (1998).
22. H. Brezis, F. Merle, and T. Rivière, Arch. Rational Mech. Anal. **126**, 35 (1994).