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INTERACTION CORRECTIONS TO THERMAL TRANSPORT COEFFICIENTS IN DISORDERED METALS: THE QUANTUM KINETIC EQUATION APPROACH

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We consider the singular electron–electron interaction corrections to the transport coefficients in disordered metals to test the validity of the Wiedemann – Franz law. We develop a local, quantum kinetic equation approach in which the charge and energy conservation laws are explicitly satisfied. To obtain the local description, we introduce bosonic distribution functions for the neutral low-energy collective modes (electron–hole pairs). The resulting system of kinetic equations enables us to distinguish between the different physical processes involved in the charge and energy transport: the elastic electron scattering affects both, while the inelastic processes influence only the latter. Moreover, the neutral bosons, although incapable of transporting charge, contribute significantly to the energy transport. In our approach, we calculate on equal footing the electric and thermal conductivities and the specific heat in any dimension. We find that the Wiedemann – Franz law is always violated by the interaction corrections; the violation is larger for one- and two-dimensional systems in the diffusive regime $T\tau \ll \hbar$ and is due to the energy transported by neutral bosons. For two-dimensional systems in the quasi-ballistic regime $T\tau \gg \hbar$, the inelastic scattering of the electron on the bosons also contributes to the violation.

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1. INTRODUCTION

It is well-known that measurement of the thermal transport coefficient may provide additional information about the scattering processes in disordered metals. In particular, the Wiedemann – Franz [1] law holds as long as elastic scattering dominates in the system[1)],

$$L = \frac{\kappa}{\sigma T} = \frac{\pi^2}{3e^2}, \qquad (1.1)$$

where κ and σ are the respective thermal and electric conductivities in the system, T is the temperature in energy units ($k_B = 1$), and e is the electron charge. On the other hand, for the deep inelastic forward scattering, the Wiedemann – Franz law is violated [2], and hence the Lorentz number L is smaller than the universal value, $L < \pi^2/3e^2$.

Historically, the transport (in particular, thermal transport) coefficients were first calculated using the Boltzmann equation (BE) [3]. The advantage of this approach is that it allows a clear separation of the scales in the problem: a particle moves freely most of the time and rarely scatters on other particles or impurities. The BE is applicable at the time scale much larger than the time that it takes for the scattering to happen, and hence all the scattering events are encoded into the local collision integral. All the quantum mechanical part of the calculation is then reduced to solving the scattering problems for the relevant physical processes. This gives the precise form of the collision integral but does not affect the general structure of the BE. The great advantage of the BE is that its structure illuminates the relevant conservation laws.

In the late 50s, an alternative approach became popular — the so-called Kubo formulas [4]. In this approach, the transport equation is not derived but rather the connection of the transport coefficient to the equilibrium correlation function of certain current operators is used. (The Kubo approach to the thermal

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[1)] It was shown by G. V. Chester and A. Thellung (Proc. Phys. Soc. (London) **77**, 1005 (1961)) that Eq. (1.1) remains valid for arbitrary scattering strength as long as the scattering rates and the density of states are smooth ($\mathcal{C}_2$) functions of energy near the Fermi level.

transport was claimed to be put on rigorous footing by Luttinger [5] based on the assumption that there exists some spatial scale in the system such that the gradient expansion is possible for perturbations smooth at that scale). Being exact, the Kubo formulas are formally applicable even in the regime where the transport equation cannot be justified (the evolution cannot be separated into free motion and rare collisions).

However, in practice, the possibility of explicit calculations within the Kubo formula is somewhat limited. The most spectacular results of the Kubo-formula calculations — such as the Maki – Thompson [6, 7], Aslamazov – Larkin [8], and weak localization [9] corrections to the electrical conductivity — require a small parameter, which is the same parameter that determines the applicability of the Boltzmann equation. This means that all these effects can also be described in terms of quantum corrections to the collision integral (for weak localization, this was done in Ref. [13]). The most relevant effect for this paper, the Altshuler – Aronov [10] interaction correction to the electrical conductivity in two dimensions [11, 14]

$$\delta\sigma_{AA} = -\frac{e^2}{2\pi^2\hbar}\ln\left(\frac{\hbar}{T\tau}\right)\times \\ \times\left[1+3\left(1-\frac{1}{F_0^\sigma}\ln(1+F_0^\sigma)\right)\right], \quad (1.2)$$

originates from the elastic scattering of electrons on a self-consistent potential (Friedel oscillation) [15, 16] and can be once again obtained from the correction to the collision integral [17].

The success of the Kubo formulas in the description of the quantum and interaction effects in thermal transport is by far more modest and controversial. Particularly, despite a 20-year history, there is no consensus on the answer to a natural question: how does the logarithmic correction to conductivity (1.2) translate into a correction to Wiedemann – Franz law (1.1)?

The first attempt to answer this question was made by Castellani at al. [19] by analyzing Ward identities for a disordered Fermi liquid; they found that the Wiedemann – Franz law should hold for interacting disordered electrons. Their claim was later disputed by Livanov et al. [20]: in a «quantum kinetic equation» approach[2)], a logarithmic divergence for the thermal conductivity in two dimensions was found to have even the sign opposite to the Wiedemann – Franz law. More recently, Niven and Smith [22] applied the Kubo formula and again found a logarithmically divergent contribution (for the Coulomb but not a short-range interaction) in addition to the one that follows from the Wiedemann – Franz law.

The reason for this confusion in the literature is twofold. Technically, the identification of the correct form of the current operator is complicated by the presence of the electron–electron interaction (the energy current operator in the form defined by Luttinger [5] is cumbersome for use due to the presence of the additional disorder and interaction potentials in it, whereas the superficially more elegant expression in the Matsubara frequency representation does not in fact correspond to any conservation law for the interacting system and violates gauge invariance, see Appendix B). Physically, the use of the diagram calculation within the Kubo formula prevents one from clearly identifying the relevant scattering processes, because each diagram taken separately describes some mixture of such processes and does not have a physical meaning individually.

This situation calls for the development of the kinetic equation description, which takes the interaction correction of the Altshuler – Aronov type into account for both the electric and thermal transport. The advantage of this approach is that it allows keeping track of the conservation laws explicitly and thus excludes any ambiguity in the definition of the currents. This paper is devoted to the development and application of this method.

We use the units with $\hbar = 1$ throughout the paper and restore the Planck constant in the final results only. This paper is organized as follows: in Sec. 2, we discuss some general features of the kinetic equation approach using a simple «toy model». In Sec. 3, we present our final expression for the kinetic equation describing interacting electrons in disordered metals. Section 4 summarizes the results for the thermal conductivity and the specific heat obtained by solving the kinetic equation. The derivation of the kinetic equation is presented in Sec. 5, and the calculation of the transport coefficients and the specific heat is given in Sec. 6. Some mathematical details are relegated to the Appendices.

2. STRUCTURE OF THE KINETIC EQUATION: CURRENTS AND SPECIFIC HEAT

The purpose of this section is to show how the structure of the kinetic equation permits the proper identification of the relevant currents. We first recall how

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[2)] The quantum kinetic equation with the necessary conservation laws was not actually derived in Ref. [20] and we are therefore unable to compare their approach with ours.

to calculate the specific heat from the kinetic equation once the conservation laws are obtained (this enables a direct check against the much simpler thermodynamic calculation). We then discuss the locality requirement for a proper kinetic equation. This requirement determines the number of the necessary degrees of freedom (i.e., independent distribution functions) that must be introduced into the kinetic description.

2.1. Kinetic equation and conservation laws

As a specific example, we here consider electron-like and hole-like excitations coupled to neutral bosons in the presence of an external electric field $\mathbf{E}$. (As we see later, the system of interacting electrons can be effectively described at low temperatures by such a coupled system for the scattering at small momentum transfer in the particle–hole channel.) The kinetic equations for electrons and bosons have the form

$$\left[\frac{\partial}{\partial t}+v_F\mathbf{n}\cdot\nabla+ev_F\mathbf{n}\cdot\mathbf{E}\frac{\partial}{\partial\varepsilon}\right]f=\mathrm{St}_e\{f,N\}, \quad (2.1a)$$

$$\left[\frac{\partial}{\partial t}+v(\omega)\mathbf{n}\cdot\nabla\right]N=\mathrm{St}_b\{f,N\}, \quad (2.1.b)$$

where $f=f(\varepsilon,\mathbf{n};t,\mathbf{r})$ is the distribution function for the electrons with charge e, v_F is the Fermi velocity, and $\mathbf{n}$ is the direction of the momentum. The energy ε is counted from the Fermi level such that $f(\varepsilon>0)$ describes electron-like excitations and $1-f(-\varepsilon)$, $\varepsilon>0$, corresponds to hole-like excitations. Concentrating only on the corrections that are singular in T, we neglect the dependence of the electron velocity on the energy (the electron–hole asymmetry) because it does not introduce anything but a small correction regular in powers of T^2.

The bosonic function $N=N(\omega,\mathbf{n};t,\mathbf{r})$ is the distribution function for the bosons with the velocity $\mathbf{v}(\omega)$. All the interaction effects are included into the collision integrals St_e and St_b; for example, an electron-like excitation can decay into a less energetic electron and a neutral boson, or an electron and a hole can annihilate into a neutral bosons, etc. By locality, the collision integrals depend on the same variables as the distribution functions, i.e., $\mathrm{St}_e=\mathrm{St}_e(\varepsilon,\mathbf{n};t,\mathbf{r})$ and $\mathrm{St}_b=\mathrm{St}_b(\omega,\mathbf{n};t,\mathbf{r})$.

In thermodynamic equilibrium with $\mathbf{E}=0$, the Fermi function for fermions and the Planck function for the neutral bosons,

$$f_F(\varepsilon)=\frac{1}{\exp(\varepsilon/T)+1},\qquad N_P(\omega)=\frac{1}{\exp(\omega/T)-1}, \quad (2.2)$$

solve the kinetic equation. The temperature T is here a constant determined by the initial conditions for the kinetic equation.

Being an effective description for the slow dynamics of the original quantum mechanical system, the kinetic equation must respect the conservation laws of the original system: (i) the total charge conservation and (ii) the total energy conservation. These two conditions are enforced by the requirements

$$\int d\varepsilon\ \nu\langle\mathrm{St}_e\{f,N\}\rangle_{\mathbf{n}}=0 \quad (2.3a)$$

and

$$\int d\varepsilon\varepsilon\ \nu\langle\mathrm{St}_e\{f,N\}\rangle_{\mathbf{n}}+ \\ +\int d\omega\ \omega\ b(\omega)\langle\mathrm{St}_b\{f,N\}\rangle_{\mathbf{n}}=0 \quad (2.3b)$$

for the collision integrals; here, ν is the density of states (DOS) of the electrons (we neglect its energy dependence) and $b(\omega)$ is the density of states of the bosons. We also introduce the short notation for the angular integral

$$\langle\ldots\rangle_{\mathbf{n}}\equiv\int\frac{d\mathbf{n}}{\Omega_d}\cdots\ , \quad (2.4)$$

where Ω_d is the total solid angle in d dimensions.

Let the electron density ρ be given by

$$\rho(t,\mathbf{r})=e\nu\int d\varepsilon\langle f(\varepsilon,\mathbf{n};t,\mathbf{r})\rangle_{\mathbf{n}}. \quad (2.5)$$

Integrating Eq. (2.1a) over the energy and the direction of the momentum and using Eq. (2.3a), we arrive at the continuity equation

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\mathbf{j}=0, \quad (2.6)$$

with the electron current density defined as

$$\mathbf{j}(t,\mathbf{r})=e\nu v_F\int d\varepsilon\langle\mathbf{n}f(\varepsilon,\mathbf{n};t,\mathbf{r})\rangle_{\mathbf{n}}. \quad (2.7)$$

(Strictly speaking, Eq. (2.6) fixes only the longitudinal component of the current, i.e., an arbitrary curl may be added to Eq. (2.7). We do not consider the effect of the magnetic field here and therefore disregard such magnetization currents.)

We now turn to the analysis of the energy conservation. We multiply Eq. (2.1a) by $\nu\varepsilon$ and integrate over ε and $\mathbf{n}$. Next, we multiply Eq. (2.1b) by $\omega b(\omega)$ and integrate over ω and $\mathbf{n}$. Adding the two results together and using Eqs. (2.3b) and (2.7), we obtain

$$\frac{\partial u_{tot}}{\partial t}+\nabla\cdot\mathbf{j}^{\varepsilon}_{tot}=\mathbf{j}\cdot\mathbf{E}, \quad (2.8)$$

where

$$u_{tot} = u_e(t,\mathbf{r}) + u_b(t,\mathbf{r}), \tag{2.9a}$$

$$u_e(t,\mathbf{r}) = \nu \int d\varepsilon\ \varepsilon\ \langle f(\varepsilon,\mathbf{n};t,\mathbf{r})\rangle_{\mathbf{n}}, \tag{2.9b}$$

$$u_b(t,\mathbf{r}) = \int d\omega\ \omega\ b(\omega) \langle N(\omega,\mathbf{n};t,\mathbf{r})\rangle_{\mathbf{n}}, \tag{2.9c}$$

and

$$\mathbf{j}^{\varepsilon}_{tot} = \mathbf{j}^{\varepsilon}_{e} + \mathbf{j}^{\varepsilon}_{b}, \tag{2.10a}$$

$$\mathbf{j}^{\varepsilon}_{e}(t,\mathbf{r}) = v_F \nu \int d\varepsilon\ \varepsilon\ \langle \mathbf{n} f(\varepsilon,\mathbf{n};t,\mathbf{r})\rangle_{\mathbf{n}}, \tag{2.10b}$$

$$\mathbf{j}^{\varepsilon}_{b}(t,\mathbf{r}) = \int d\omega\, \omega\, b(\omega) v(\omega) \langle \mathbf{n} N(\omega,\mathbf{n};t,\mathbf{r})\rangle_{\mathbf{n}}. \tag{2.10c}$$

The right-hand side of Eq. (2.8) is nothing but the Joule heat. For a homogeneous system, the gradient term in the left-hand side vanishes, and by virtue of the energy conservation, expression (2.9) must be identified with the total energy density of the system. On the other hand, for $\mathbf{E} = 0$, Eq. (2.8) has the form of the continuity equation for the energy density; therefore, Eqs. (2.10) must be identified with the total energy current density. This statement is not entirely trivial. One could imagine that for an interacting system, the DOS entering the respective expressions (2.5) and (2.9) for the charge and the energy density are renormalized differently. Energy conservation equation (2.8) eliminates such a possibility.

The conservation of energy, Eq. (2.8), is valid for any rate of the energy flow into and out of the system. On the other hand, the collision integrals in Eqs. (2.1) define a certain time scale τ_{in}: the dynamics slow at the scale of τ_{in} can be characterized by distribution functions (2.2) with a time-dependent temperature $T(t)$ (corrections to such an adiabatic description are of the order of $\tau_{in}\partial_t \ln T$). Substituting this form of the distribution function in Eq. (2.9) and then using the result in Eq. (2.8), we find, for a homogeneous system,

$$c_V(T)\frac{\partial T}{\partial t} = \mathbf{j}\cdot\mathbf{E}, \tag{2.11}$$

where

$$c_V = \frac{\partial}{\partial T}\left[\nu \int d\varepsilon\ \varepsilon f_F(\varepsilon) + \int d\omega\ \omega\ b(\omega) N_P(\omega)\right] \tag{2.12}$$

is nothing but the specific heat of the system. The latter quantity may be calculated independently by applying the standard diagram technique for equilibrium systems. The agreement of such a calculation with the structure of the kinetic equation result in (2.12) is the most important check of the consistency of our description of the thermal transport.

2.2. Locality of the kinetic equation and the number of the degrees of freedom

The form of the collision integrals local in space and time is clearly a simplified description. Actually, the collision integral may be nonlocal at the time scale of the order of $\hbar/T$ and at the spatial scale of the order of $\hbar v_F/T$. We say that such a description is *local* and the description where the nonlocality is involved at larger spatial and time scales is *nonlocal*.

The number of distribution functions to be introduced into the description is governed by the locality of the kinetic equation. We use the model in Eqs. (2.1) to illustrate the point. We had a local description in terms of the fermionic and bosonic distribution functions. But we can try to eliminate the bosonic distribution function and obtain a description in terms of the electronic degrees of freedom only.

Assuming that the deviation of the distribution function from its equilibrium value is small, we can linearize the bosonic collision integral to the form

$$\mathrm{St}_b\{f,N\} = -\hat{I}\left(N - \tilde{N}\{f\}\right), \tag{2.13}$$

where $\hat{I}$ is some positive definite integral operator and $\tilde{N}\{f\}$ is the functional of the fermionic distribution function $f(\varepsilon)$ such that for $f(\varepsilon) = f_F(\varepsilon)$, $\tilde{N}(\omega) = N_P(\omega)$. Using Eq. (2.13), we can formally solve Eq. (2.1b) as

$$N = \frac{1}{\partial/\partial t + v(\omega)\mathbf{n}\cdot\nabla + \hat{I}}\hat{I}\tilde{N}\{f\}. \tag{2.14}$$

Substituting Eq. (2.14) in Eq. (2.1a), we apparently obtain the kinetic equation in terms of the electron distribution function only,

$$\left[\frac{\partial}{\partial t} + v_F\mathbf{n}\cdot\nabla + ev_F\mathbf{n}\cdot\mathbf{E}\frac{\partial}{\partial\varepsilon}\right] f = \mathrm{St}^{?}_{e}\{f\},$$
$$\mathrm{St}^{?}_{e}\{f\} \equiv \tag{2.15}$$
$$\equiv \mathrm{St}_e\left\{f, \frac{1}{\partial/\partial t + v(\omega)\mathbf{n}\cdot\nabla + \hat{I}}\hat{I}\tilde{N}\{f\}\right\}.$$

If we are interested in the linear response to a weak and smooth external perturbation, the description in terms of this single kinetic equation is completely equivalent to the original coupled system (2.1). However, there are clear drawbacks: the presence of the integral operator $\hat{I}$ in the collision integral makes it nonlocal on the scale determined by the kinetic equation itself rather than by the temperature. Moreover, although it is still easy to identify the continuity equation for the electron charge using Eq. (2.3a), there is no longer a relation similar to Eq. (2.3b). This is why the analysis of the energy conservation law becomes cumbersome: calculation of the specific heat and energy current requires the time expansion of the collision integral, which in turns seems to require the knowledge of the concrete form of the inelastic collision integral.

The example we have just considered is somewhat trivial because the system was separated into fermionic and bosonic modes from the very beginning. The problem that we consider in this paper is how to include the collective modes of the interacting electron system into the kinetic equation. Indeed, in this case, any calculation gives the result in terms of the electronic distribution function only, and it is not clear a priori how to introduce the occupation numbers for the collective modes into the description.

As we show in what follows, it may be possible to reverse our previous argument. We consider a system of interacting electrons and find that the interactions are described by a *nonlocal* collision integral. We therefore introduce bosonic degrees of freedom that allow rewriting the nonlocal kinetic equation in terms of coupled, local kinetic equations. This then makes it possible to identify the energy density and energy current density as sums of the fermionic and bosonic contributions. A specific example is briefly discussed in the next subsection.

2.3. Degrees of freedom for the kinetics of a disordered Fermi liquid

We now focus on the disordered, interacting Fermi liquid. For simplicity, we consider the interaction in the singlet channel only. Our goal is to show that the thermodynamic result for the interaction correction to the specific heat has indeed the kinetic equation structure (2.12). As a result, we are then able to determine the necessary number of the bosonic degrees of freedom for the local kinetic equation. For the paper to be self-contained, we briefly review the thermodynamic approach, referring the reader to the literature for further details[3].

The thermodynamic calculation of the specific heat c_V is based on the relation between c_V and the thermodynamic potential Ω:

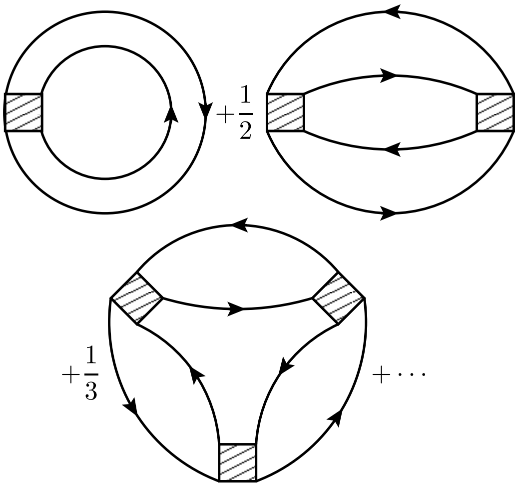

Fig. 1. Leading singular contribution to the thermodynamic potential for the clean system. The shaded box corresponds to F/ν, defined through the two-particle vertex Γ^ω, see Ref. [26]; the solid lines are coherent parts of the electron Green's functions. For the disordered system, the polarization bubbles should be dressed by impurity scattering [12]

$$c_V = -T\frac{\partial^2\Omega}{\partial T^2}. \tag{2.16}$$

The thermodynamic potential can be written as the sum of the thermodynamic potential Ω_0 for noninteracting quasiparticles and a correction $\delta\Omega$ associated with soft modes in the system. Keeping such a correction is legitimate because it turns out to be a more singular function of the temperature than the T^3 correction due to the electron–hole asymmetry.

The correction $\delta\Omega$ is given by the sum of the so-called ring diagrams, see Fig. 1. The Matsubara representation for this diagram is

$$\delta\Omega = \frac{T}{2}\sum_{\omega_n}\int\frac{d^dq}{(2\pi)^d}\ln\left(1+\frac{F}{\nu}\Pi(i|\omega_n|,\mathbf{q})\right), \tag{2.17}$$

where F is the coupling constant, $\omega_n = 2\pi Tn$ are the bosonic Matsubara frequencies, and Π is the polarization operator. The explicit expression for this operator is not important for the present discussion and is given later, see Eq. (5.43a).

A straightforward calculation, relegated to Appendix A, enables us to rewrite Eq. (2.17) as

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[3] See, e.g., Refs. [12, 26] and references therein.

$$\delta\Omega = -\int \frac{d\omega}{2\pi}\left(\frac{1}{2}\operatorname{cth}\frac{\omega}{2T}\right) \times \\ \times \int \frac{d^dq}{(2\pi)^d}\operatorname{Im}\operatorname{Tr}\left[\ln\hat{\mathcal{L}}^\rho - \ln\hat{\mathcal{L}}^g\right]. \quad (2.18)$$

The explicit expressions for the bosonic propagators $\mathcal{L}^\rho$ and $\mathcal{L}^g$ are not relevant (they can be found from Eq. (3.11), with the trace understood as the sum or integration over all variables other than ω, $\mathbf{q}$); we just mention here that $\mathcal{L}^\rho = \mathcal{L}^g$ in the absence of interaction ($F = 0$). Substituting Eq. (2.18) in Eq. (2.16) and integrating over ω by parts, we find

$$\delta c_V = \frac{\partial}{\partial T}\int_0^\infty d\omega\, \omega N_P(\omega)\left[b^\rho(\omega) - b^g(\omega)\right], \quad (2.19a)$$

where the densities of states are defined as

$$b^\rho(\omega) = \frac{1}{\pi}\operatorname{Im}\int \frac{d^dq}{(2\pi)^d}\partial_\omega \operatorname{Tr}\ln\mathcal{L}^\rho, \quad (2.19b)$$

$$b^g(\omega) = \frac{1}{\pi}\operatorname{Im}\int \frac{d^dq}{(2\pi)^d}\partial_\omega \operatorname{Tr}\ln\mathcal{L}^g. \quad (2.19c)$$

The function $b^\rho(\omega)$ has the physical meaning of the density of states (DOS) of the bosonic degrees of freedom in the system (soft electron–hole pairs). The function $b^g(\omega)$ has the meaning of the density of states of fictitious bosons (we call them «ghosts») that describe soft electron–hole pairs in the absence of interaction. The physical meaning of the minus sign in front of $b^g(\omega)$ is that with the formation of collective modes, some degrees of freedom are removed from the description of the noninteracting system; the ghost bosons in the last term in Eq. (2.19a) take this reduction into account.

Comparison of Eq. (2.19a) with Eqs. (2.12) and (2.9) suggests the following expression for the contribution of the collective modes to the energy density in the nonequilibrium case:

$$u_b = \int_0^\infty d\omega\, \omega\left[N^\rho(\omega)b^\rho(\omega) - N^g(\omega)b^g(\omega)\right]; \quad (2.20)$$

here, $N^\rho = N^g = N_P$ in the equilibrium and have to be found from some kinetic equation otherwise (this definition requires that Eq. (2.8) holds for an arbitrary distribution function). A similar expression can be obtained for the contribution due to the interaction in the triplet channel by introducing an additional propagator $\mathcal{L}^\sigma$ and the distribution function N^σ. This means that the proper local kinetic equation must include four distribution functions: one for the fermions, $f(\varepsilon)$, and three for the bosons, $N^{\rho,\sigma,g}(\omega)$. We derive such a description in the subsequent sections.

3. FINAL FORM OF THE KINETIC EQUATION AND SCATTERING PROCESSES

In this section, we summarize the final form of the quantum kinetic equation, the conservation laws, and the corresponding currents. The explicit expressions of the collision integrals are given in subsection 3.2. The detailed derivation of these results is presented in Sec. 5.

In accord with the previous section, the kinetics of the system is described by the electron distribution function $f(\varepsilon, \mathbf{n}; t, \mathbf{r})$, the «distribution functions» $\hat{N}^\rho$ and $\hat{N}^\sigma$ of the bosonic singlet and triplet excitations, and the «distribution function» $\hat{N}^g$ of the ghost excitation.

The electron distribution function $f(\varepsilon, \mathbf{n}; t, \mathbf{r})$ is diagonal in the space of momentum directions. On the contrary, bosonic excitations are characterized by the density matrices $N^\alpha(\omega, \mathbf{q}; \mathbf{n}_i, \mathbf{n}_j; t, \mathbf{r})$ ($\alpha = \rho, \sigma, g$) that may not be diagonal in the space of momentum direction $\mathbf{n}$. Only in the thermal equilibrium, with

$$\begin{aligned} &f_{eq}(\varepsilon, \mathbf{n}; t, \mathbf{r}) = f_F(\varepsilon), \\ &N^\alpha_{eq}(\omega, \mathbf{q}; \mathbf{n}_i, \mathbf{n}_j; t, \mathbf{r}) = \Omega_d\delta(\widehat{\mathbf{n}_i, \mathbf{n}_j})N_P(\omega) \end{aligned} \quad (3.1)$$

and with the Fermi and Planck distribution functions given by Eq. (2.2), the matrices $N^\alpha(\omega, \mathbf{q}; \mathbf{n}_i, \mathbf{n}_j; t, \mathbf{r})$ acquire the diagonal form[4]. But even outside the equilibrium, these matrices have the property

$$N^\alpha(\omega, \mathbf{q}; \mathbf{n}_i, \mathbf{n}_j) = \\ = -\left[N^\alpha(-\omega, -\mathbf{q}; \mathbf{n}_j, \mathbf{n}_i) + \Omega_d(\widehat{\mathbf{n}_i, \mathbf{n}_j})\right] \quad (3.2)$$

(hereafter, the spectator t, $\mathbf{r}$ variables might be suppressed.)

Strictly speaking, $f(\varepsilon, \mathbf{n}; t, \mathbf{r})$ is a 2×2 density matrix in the spin space and N^σ is a 3×3 density matrix in the angular momentum $L = 1$ space; however, this is not important in the calculations below and we write the equations for the diagonal components only. To account for the threefold degeneracy of the triplet mode, we explicitly introduce factors of 3 in the corresponding collision integrals and currents.

For compactness, we use the operator notation for matrices in the space of momentum directions, such that, for example, $\hat{N}$ should be understood as an operator acting on a function $a(\mathbf{n}_i)$ as

$$\left[\hat{N}a\right](\mathbf{n}_i) \equiv \int \frac{d\mathbf{n}_j}{\Omega_d}N(\mathbf{n}_i, \mathbf{n}_j)a(\mathbf{n}_j). \quad (3.3)$$

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[4] As given in Eq. (3.1), the equilibrium distribution functions $N^\alpha_{eq}(\omega)$ are defined only for $\omega > 0$; for $\omega < 0$, they are found using the property in Eq. (3.2).

3.1. Kinetic equations and conserved currents

The kinetic equation for the electrons in the electric field **E** (we do not consider the magnetic field effects) has the canonical form

$$\begin{gathered}\left[\partial_t+\mathbf{v}\cdot\nabla+e\mathbf{v}\cdot\mathbf{E}\frac{\partial}{\partial\varepsilon}\right]f(\varepsilon,\mathbf{n};t,\mathbf{r})=\widehat{\mathrm{St}}_e(\varepsilon,\mathbf{n};t,\mathbf{r}),\\ \hat{\mathrm{St}}_e=\widehat{\mathrm{St}}_\tau f+\widehat{\mathrm{St}}^{e\text{-}\rho}\{f,N^\rho\}+3\widehat{\mathrm{St}}^{e\text{-}\sigma}\{f,N^\sigma\}-\\ -4\widehat{\mathrm{St}}^{e\text{-}g}\{f,N^g\}+\widehat{\mathrm{St}}^{e\text{-}e}\{f\},\end{gathered}\tag{3.4}$$

where the first term in the right-hand side is the «bare» collision integral

$$\mathrm{St}_\tau(\mathbf{n}_i,\mathbf{n}_j)=\frac{1}{\tau(\theta_{ij})}-(\widehat{\mathbf{n}_i,\mathbf{n}_j})\int\frac{d\mathbf{n}_k}{\tau(\theta_{ik})},\tag{3.5}$$

with $\theta_{ij}=\widehat{\mathbf{n}_i\mathbf{n}_j}$, and the other terms, which are to be written shortly, take the interaction effects into account.

The bosonic distributions, for $\alpha=\rho,\sigma,g$, are governed by

$$\begin{gathered}\omega\left[\left\{\frac{1}{1+\hat{F}^\alpha};\partial_t\hat{N}^\alpha\right\}+\left\{\hat{\boldsymbol{s}}^\alpha(\omega,\mathbf{q});\nabla\hat{N}^\alpha\right\}+\right.\\ \left.+\,i\left[\hat{H}^\alpha_{e-h}(\omega,\mathbf{q});\hat{N}^\alpha\right]\right]=\\ =\widehat{\mathrm{St}}^{\alpha\text{-}e}\{N^\alpha,f\}(\omega,\mathbf{q};\mathbf{n}_i,\mathbf{n}_j;t,\mathbf{r}),\end{gathered}\tag{3.6}$$

where the commutator and anticommutator are defined as

$$\left\{\hat{A};\hat{B}\right\}\equiv\frac{1}{2}(\hat{A}\hat{B}+\hat{B}\hat{A}),\quad\left[\hat{A};\hat{B}\right]\equiv\hat{A}\hat{B}-\hat{B}\hat{A}.\tag{3.7}$$

The operators $\hat{H}^\alpha_{e-h}$ acting in the angular (momentum direction) space are defined as

$$\hat{H}^\alpha_{e-h}(\omega,\mathbf{q})=\mathbf{v}\cdot\mathbf{q}-\frac{\omega}{1+\hat{F}^\alpha}\tag{3.8}$$

and the velocity operator is

$$\hat{\boldsymbol{s}}^\alpha(\omega,\mathbf{q})=\frac{\partial\hat{H}^\alpha_{e-h}(\omega,\mathbf{q})}{\partial\mathbf{q}}=\\ =\mathbf{v}+\omega\frac{\partial}{\partial\mathbf{q}}\left(\frac{\hat{F}^\alpha}{1+\hat{F}^\alpha}\right).\tag{3.9}$$

The action of the operators $\hat{F}^\alpha$ in the angular space is the same as in Eq. (3.3); these operators are given by

$$\begin{gathered}\hat{F}^g=0,\\ [\hat{F}^\sigma](\mathbf{n}_i,\mathbf{n}_j)=F^\sigma(\theta_{ij}),\\ [\hat{F}^\rho](\mathbf{n}_i,\mathbf{n}_j)=\nu V(\mathbf{q})+F^\rho(\theta_{ij}),\end{gathered}\tag{3.10}$$

where $F^{\rho,\sigma}(\theta)$ are the Landau Fermi-liquid interaction parameters. The angular-independent term $\nu V(\mathbf{q})$ takes the long-range part of the Coulomb density–density interaction into account.

To characterize the density of states for the bosonic excitations, we introduce the propagators $\hat{\mathcal{L}}^\alpha(\omega,\mathbf{q};\mathbf{n}_i,\mathbf{n}_j)$, $\alpha=\rho,\sigma,g$ as

$$\left[i\hat{H}^\alpha_{e-h}(\omega,\mathbf{q})-\widehat{\mathrm{St}}_\tau\right]\hat{\mathcal{L}}^\alpha=\hat{1}.\tag{3.11}$$

They describe the propagation of an electron–hole pair scattered by the disorder potential. This propagation is affected by the corresponding interactions for $\alpha=\rho,\sigma$, and it reduces to the usual diffusion for the ghosts.

We are now prepared to write the conservation laws that must be satisfied by the collision integrals independently of their explicit form or the particular shape of the distribution functions. The conservation of the number of particles is ensured by the condition

$$\begin{gathered}\int\widehat{\mathrm{St}}^{e\text{-}\alpha}\{f,N^\alpha\}(\varepsilon,\mathbf{n};t,\mathbf{r})\,d\mathbf{n}\,d\varepsilon=0,\\ \alpha=g,\rho,\sigma,\\ \int\widehat{\mathrm{St}}^{e\text{-}e}\{f\}(\varepsilon,\mathbf{n};t,\mathbf{r})\,d\mathbf{n}\,d\varepsilon=0,\end{gathered}\tag{3.12a}$$

and the impurity collision integral (3.5) preserves the number of particles on each energy shell,

$$\int\widehat{\mathrm{St}}_\tau f(\varepsilon,\mathbf{n};t,\mathbf{r})\,d\mathbf{n}=0.\tag{3.12b}$$

The conservation of energy during purely electron–electron collisions is ensured by

$$\int\varepsilon\,\widehat{\mathrm{St}}^{e\text{-}e}\{f\}(\varepsilon,\mathbf{n};t,\mathbf{r})\,d\mathbf{n}\,d\varepsilon=0.\tag{3.12c}$$

Finally, the conservation of energy during the electron–boson collision is guaranteed by the conditions

$$\begin{gathered}\nu\int\varepsilon\,\widehat{\mathrm{St}}^{e\text{-}\alpha}\{f,N^\alpha\}(\varepsilon,\mathbf{n};t,\mathbf{r})\,\frac{d\mathbf{n}\,d\varepsilon}{\Omega_d}+\\ +\int\mathrm{Tr}\left[\hat{\mathcal{L}}^\alpha(\omega)\widehat{\mathrm{St}}^{\alpha\text{-}e}\{f,N^\alpha\}(\omega;t,\mathbf{r})\right]\frac{d\omega}{2\pi}=\\ =-i\int\mathrm{Tr}\left[\left[\hat{H}^\alpha_{e-h}(\omega);\hat{\mathcal{L}}^\alpha(\omega)\right]\hat{N}^\alpha(\omega;t,\mathbf{r})\right]\frac{\omega d\omega}{2\pi};\end{gathered}\tag{3.12d}$$

for $\alpha=g,\rho,\sigma$, where the trace is defined as

$$\mathrm{Tr}\hat{A}\hat{B}=\int\frac{d\mathbf{n}_1d\mathbf{n}_2}{\Omega_d^2}\times\\ \times\int\frac{d^dq}{(2\pi)^d}A(\mathbf{q};\mathbf{n}_1,\mathbf{n}_2)B(\mathbf{q};\mathbf{n}_2,\mathbf{n}_1).\tag{3.13}$$

The existence of conservation laws (3.12) immediately enables us to establish the expressions for the conserved currents in the spirit of Sec. 2.1. By integrating both sides of Eq. (3.4) over ε and $\mathbf{n}$, we find the relations

$$\begin{gathered}\frac{\partial\rho}{\partial t}+\nabla\cdot\mathbf{j}=0,\\ \rho(t,\mathbf{r})=e\nu\int f(\varepsilon,\mathbf{n};t,\mathbf{r})\frac{d\varepsilon\, d\mathbf{n}}{\Omega_d},\\ \mathbf{j}(t,\mathbf{r})=e\nu v_F\int \mathbf{n}f(\varepsilon,\mathbf{n};t,\mathbf{r})\frac{d\varepsilon\, d\mathbf{n}}{\Omega_d},\end{gathered}\tag{3.14}$$

which express the conservation of charge in terms of the usual charge density and electric current density, cf. Eqs. (2.5)–(2.7).

Turning to the energy conservation, we multiply Eq. (3.4) by ε and then integrate over $\mathbf{n}$ and ε. Similarly, we multiply Eq. (3.6) by $\hat{\mathcal{L}}^\alpha$, take the trace (3.13), and integrate over ω. Adding the results together, we find

$$\begin{gathered}\frac{\partial u_{tot}}{\partial t}+\nabla\cdot\mathbf{j}^\varepsilon_{tot}=\mathbf{j}\cdot\mathbf{E},\\ u_{tot}=u_e+u_\rho+3u_\sigma-4u_g,\\ \mathbf{j}^\varepsilon_{tot}=\mathbf{j}^\varepsilon_e+\mathbf{j}^\varepsilon_\rho+3\mathbf{j}^\varepsilon_\sigma-4\mathbf{j}^\varepsilon_g.\end{gathered}\tag{3.15a}$$

The electronic contributions to the energy density and current density are given by

$$\begin{gathered}u_e(t,\mathbf{r})=\nu\int\frac{d\varepsilon\, d\mathbf{n}}{\Omega_d}\varepsilon\, f(\varepsilon,\mathbf{n};t,\mathbf{r}),\\ \mathbf{j}^\varepsilon_e(t,\mathbf{r})=\nu v_F\int\frac{d\varepsilon\, d\mathbf{n}}{\Omega_d}\varepsilon\,\mathbf{n}f(\varepsilon,\mathbf{n};t,\mathbf{r}).\end{gathered}\tag{3.15b}$$

The contributions of the bosonic neutral excitations are

$$\begin{gathered}u_\alpha(t,\mathbf{r})=\\ =\int\mathrm{Tr}\left\{\frac{1}{1+\hat{F}^\alpha}\hat{\mathcal{L}}^\alpha(\omega)\hat{N}^\alpha(\omega;t,\mathbf{r})\right\}\frac{\omega\, d\omega}{2\pi},\\ j^\varepsilon_\alpha(t,\mathbf{r})=\\ =\int\mathrm{Tr}\left\{\hat{\boldsymbol{s}}^\alpha(\omega)\hat{\mathcal{L}}^\alpha(\omega)\hat{N}^\alpha(\omega;t,\mathbf{r})\right\}\frac{\omega\, d\omega}{2\pi}\end{gathered}\tag{3.15c}$$

for $\alpha=g,\rho,\sigma$.

Equations (3.14)–(**??**) constitute our main results: the conserved currents are defined in terms of the distribution functions of the quasiparticles that describe low-energy excitations of the interacting electron gas for interaction in the particle–hole channel. In contrast with previous calculations [20–23], we explicitly show the validity of the continuity equation for energy transport; no such proof has been presented before in the quantum kinetic equation approach[5]. Moreover, we believe that the form of the energy current in those references is not correct, because it is not gauge invariant, see Appendix B for more details. As an additional benefit, our approach enables us to clearly identify the contributions of the collective modes and the scattering processes involved (this last task is accomplished by analyzing the explicit form of the collision integrals, which is also needed to calculate the transport coefficients). The derivation of the collision integrals can be found in Sec. 5; in the next section, we summarize the results and give them a physical interpretation.

3.2. The collision integrals

In this section, we give the explicit form of various collision integrals and verify that they satisfy conservation laws (3.12). To shorten the formulas, we introduce the combinations of the distribution functions

$$\begin{gathered}\Upsilon^\alpha_{ij;kl}(\varepsilon,\omega,\mathbf{q};t,\mathbf{r})\equiv N^\alpha(\omega,\mathbf{q};\mathbf{n}_i,\mathbf{n}_j;t,\mathbf{r})\times\\ \times\Big\{f(\varepsilon,\mathbf{n}_k;t,\mathbf{r})-f(\varepsilon-\omega,\mathbf{n}_k;t,\mathbf{r})\Big\}+\Omega_d\delta(\widehat{\mathbf{n}_i\mathbf{n}_j})\times\\ \times\Big\{f(\varepsilon,\mathbf{n}_l;t,\mathbf{r})\left[1-f(\varepsilon-\omega,\mathbf{n}_k;t,\mathbf{r})\right]\Big\}\end{gathered}\tag{3.16a}$$

and

$$\begin{gathered}\Psi_{ij;kl}(\varepsilon,\varepsilon_1;\omega)\equiv f(\varepsilon-\omega,\mathbf{n}_i)\left[1-f(\varepsilon,\mathbf{n}_j)\right]\times\\ \times f(\varepsilon_1,\mathbf{n}_k)\left[1-f(\varepsilon_1-\omega,\mathbf{n}_l)\right]-\\ -f(\varepsilon,\mathbf{n}_i)\left[1-f(\varepsilon-\omega,\mathbf{n}_j)\right]f(\varepsilon_1-\omega,\mathbf{n}_k)\times\\ \times\left[1-f(\varepsilon_1,\mathbf{n}_l)\right].\end{gathered}\tag{3.16b}$$

It follows from Eq. (3.2) that $\Upsilon^\alpha_{ij;kl}$ has the property

$$\int d\varepsilon\,\Upsilon^\alpha_{ij;kl}(\varepsilon,\omega,\mathbf{q})=\int d\varepsilon\,\Upsilon^\alpha_{ji;lk}(\varepsilon,-\omega,-\mathbf{q}).\tag{3.16c}$$

It is easy to check that $\Upsilon=\Psi=0$ in the thermal equilibrium (Eq. (3.1)). The combination in (3.16b) enters the collision integral in the symmetric form

$$\Psi^s_{ij}\equiv\frac{1}{4}\Big[\Psi_{ij;\,ij}+\Psi_{ji;\,ij}+\Psi_{ij;\,ji}+\Psi_{ji;\,ji}\Big].\tag{3.16d}$$

It is worth noting that the terms involving four distribution functions f are in fact cancelled from Ψ^s; besides, it has the properties

$$\begin{gathered}\int\Psi^s_{ij}(\varepsilon,\varepsilon_1;\omega)\;d\varepsilon\, d\varepsilon_1=0,\\ \int a(\omega)\int\varepsilon\,\Psi^s_{ij}(\varepsilon,\varepsilon_1;\omega)\;d\varepsilon\, d\varepsilon_1\, d\omega=0\end{gathered}\tag{3.16e}$$

for any even function $a(\omega)$. Finally, we introduce the vertex γ for the impurity scattering

$$\gamma^k_{ij}\equiv\frac{1}{\tau(\theta_{ij})}\Omega_d\left[\delta(\widehat{\mathbf{n}_j\mathbf{n}_k})-\delta(\widehat{\mathbf{n}_i\mathbf{n}_k})\right]\tag{3.17}$$

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[5] The current operator used in Ref. [21] does not satisfy the continuity equation for the long-range interaction potential.

and the short notation

$$\mathcal{L}^{\alpha}_{ij} \equiv \mathcal{L}^{\alpha}(\omega, \mathbf{q}; \mathbf{n}_i, \mathbf{n}_j).$$

The boson–electron collision integral is then explicitly expressed as

$$\widehat{\mathrm{St}}^{\alpha\text{-}e}(\omega, \mathbf{q}; \mathbf{n}_1, \mathbf{n}_2; t, \mathbf{r}) = -\int d\varepsilon \int \frac{d\mathbf{n}_3 d\mathbf{n}_4}{\Omega_d^2} \times \\ \times \left\{ \gamma_{14}^3 \Upsilon^{\alpha}_{32;41}(\varepsilon, \omega, \mathbf{q}; t, \mathbf{r}) + \right. \\ \left. + \gamma_{34}^2 \Upsilon^{\alpha}_{13;14}(\varepsilon, \omega, \mathbf{q}; t, \mathbf{r}) \right\}, \quad (3.18)$$

for $\alpha = g, \rho, \sigma$. The formula for the electron–boson collision integral can be conveniently decomposed into local (l) and nonlocal (n) (in the sense of Sec. 2) parts

$$\widehat{\mathrm{St}}^{e\text{-}\alpha} = \widehat{\mathrm{St}}_l^{e\text{-}\alpha} + \widehat{\mathrm{St}}_n^{e\text{-}\alpha}. \quad (3.19\text{a})$$

The local part of the collision integral is

$$\widehat{\mathrm{St}}_l^{e\text{-}\alpha}(\varepsilon, \mathbf{n}_1) = \frac{1}{\nu} \int \frac{d\omega}{2\pi} \frac{1}{\omega} \int \frac{d^d q}{(2\pi)^d} \int \frac{d\mathbf{n}_2 d\mathbf{n}_3 d\mathbf{n}_4}{\Omega_d^3} \times \\ \times \{ \gamma_{12}^3 [\mathcal{L}^{\alpha}_{34} \Upsilon^{\alpha}_{41;21}(\varepsilon, \omega, \mathbf{q}) + \Upsilon^{\alpha}_{34;21}(\varepsilon, \omega, \mathbf{q}) \bar{\mathcal{L}}^{\alpha}_{41}] + \\ + \gamma_{21}^3 [\mathcal{L}^{\alpha}_{34} \Upsilon^{\alpha}_{42;21}(\varepsilon, \omega, \mathbf{q}) + \\ + \Upsilon^{\alpha}_{34;21}(\varepsilon, \omega, \mathbf{q}) \bar{\mathcal{L}}^{\alpha}_{42}] \}, \quad (3.19\text{b})$$

where the bar denotes Hermitian conjugation,

$$\bar{\mathcal{L}}^{\alpha}(\omega, \mathbf{q}; \mathbf{n}_i, \mathbf{n}_j) = \mathcal{L}^{\alpha}(-\omega, -\mathbf{q}; \mathbf{n}_j, \mathbf{n}_i). \quad (3.19\text{c})$$

Using Eq. (3.11) and definitions (3.13) and (3.17), we can verify that the pair of equations (3.18), (3.19b) satisfies the energy conservation law Eq. (3.12d) on its own.

The function in Eq. (3.19b) also satisfies the particle number conservation law (3.12a). To verify this, we change the variables as $(\omega, \mathbf{q}) \to (-\omega, -\mathbf{q})$ in the terms containing $\bar{\mathcal{L}}^{\alpha}$ and then use Eqs. (3.16b) and (3.19c) to rewrite the integral in Eq. (3.19b) in terms of $\mathcal{L}^{\alpha}$ only:

$$\int \widehat{\mathrm{St}}_l^{e\text{-}\alpha}(\varepsilon, \mathbf{n}_1) d\varepsilon\, d\mathbf{n}_1 = \cdots \int \frac{d\mathbf{n}_1 \ldots d\mathbf{n}_4}{\Omega_d^4} \times \\ \times \{ \gamma_{12}^3 [\mathcal{L}^{\alpha}_{34} \Upsilon^{\alpha}_{41;21}(\varepsilon, \omega, \mathbf{q}) - \mathcal{L}^{\alpha}_{14} \Upsilon^{\alpha}_{43;12}(\varepsilon, \omega, \mathbf{q})] + \\ + \gamma_{21}^3 [\mathcal{L}^{\alpha}_{34} \Upsilon^{\alpha}_{42;21}(\varepsilon, \omega, \mathbf{q}) - \mathcal{L}^{\alpha}_{24} \Upsilon^{\alpha}_{43;12}(\varepsilon, \omega, \mathbf{q})] \}.$$

Performing the $\mathbf{n}_3$-integration using the delta functions in Eq. (3.17), we obtain the result that is antisymmetric under the $\mathbf{n}_1 \leftrightarrow \mathbf{n}_2$ permutation. Hence, the above expression vanishes after the $\mathbf{n}_{1,2}$-integrations.

The physical meaning of collision integrals (3.18) and (3.19b) is as follows. In the absence of disorder, the electron–hole pair propagates for an infinitely long time. Due to the impurity potential, the decay of the pair into an electron and a hole moving in different directions as shown in Fig. 2 is allowed. Equations (3.18) and (3.19b) are the probabilities for such a decay. (See also Sec. 5 after Eq. (5.67) for further discussion.)

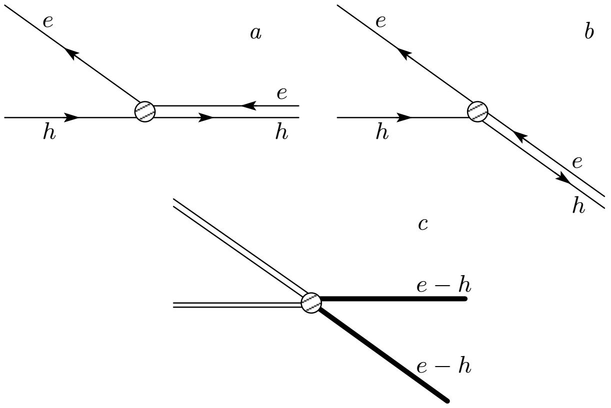

Fig. 2. The scattering amplitudes leading to the creation of the same electron and hole out of different electron–hole pairs (double lines) (a, b) and their interference contribution (c). The impurity is denoted by filled circles

By construction, the nonlocal contribution to the collision integral

$$\widehat{\mathrm{St}}_n^{e\text{-}\alpha}(\varepsilon, \mathbf{n}_1) = \frac{2}{\nu} \int \frac{d\omega}{2\pi} \frac{1}{\omega^2} \int \frac{d^d q}{(2\pi)^d} \int \frac{d\mathbf{n}_2 \ldots d\mathbf{n}_7}{\Omega_d^6} \times \\ \times \gamma_{13}^2 \gamma_{46}^5 [\mathcal{L}^{g}_{14} - \mathcal{L}^{g}_{34}] \operatorname{sh}^2\left(\frac{\omega \partial}{2 \partial \varepsilon}\right) [f(\varepsilon, \mathbf{n}_6) - f(\varepsilon, \mathbf{n}_4)] \times \\ \times \int d\varepsilon_1 \{ [\Upsilon^{\alpha}_{57;64}(\varepsilon_1, \omega, \mathbf{q}) + \Upsilon^{\alpha}_{57;46}(\varepsilon_1, \omega, \mathbf{q})] \bar{\mathcal{L}}^{\alpha}_{72} + \\ + \mathcal{L}^{\alpha}_{57} [\Upsilon^{\alpha}_{72;13}(\varepsilon_1, \omega, \mathbf{q}) + \Upsilon^{\alpha}_{72;31}(\varepsilon_1, \omega, \mathbf{q})] \} \quad (3.19\text{d})$$

satisfies its own conservation law

$$\int \varepsilon^m\, \widehat{\mathrm{St}}_n^{e\text{-}\alpha}(\varepsilon, \mathbf{n}_1)\, d\varepsilon = 0, \quad m = 0, 1, \quad (3.19\text{e})$$

i.e., preserves the energy and the number of electrons moving along a given momentum direction $\mathbf{n}$. Moreover, one can see that the collision integral (3.19d) does not contribute to the linear response at all because $\Upsilon = 0$ and f is independent of the angle in the thermodynamic equilibrium. The nonlocality of this collision integral indicates that the task formulated in Sec. 2 has not been fully accomplished. Technically, this nonlocality can be decoupled by introducing a density matrix that is nondiagonal in the boson–ghost space. We choose not to pursue this line because the term in Eq. (3.19d) does not contribute to any observable quantity of interest and does not affect any conservation laws.

The electron–electron collision integral can be split into elastic, nonlocal, and local parts:

$$\widehat{\mathrm{St}}^{e\text{-}e}(\varepsilon, \mathbf{n}_1) = \widehat{\mathrm{St}}^{e\text{-}e}_{el}(\varepsilon, \mathbf{n}_1) + \\ + \widehat{\mathrm{St}}^{e\text{-}e}_{n}(\varepsilon, \mathbf{n}_1) + \widehat{\mathrm{St}}^{e\text{-}e}_{l}(\varepsilon, \mathbf{n}_1). \quad (3.20a)$$

The elastic term describes the scattering of the electron on the static self-consistent potential created by all the other electrons,

$$\widehat{\mathrm{St}}^{e\text{-}e}_{el} = \frac{2}{\nu}\mathrm{Re}\int \frac{d\omega}{2\pi}\,\frac{1}{\omega}\int \frac{d^dq}{(2\pi)^d}\int \frac{d\mathbf{n}_2\ldots d\mathbf{n}_6}{\Omega_d^5}\gamma_{13}^{2}\gamma_{46}^{5}\times \\ \times \left[\mathcal{L}^{\rho} + 3\mathcal{L}^{\sigma} - 4\mathcal{L}^{g}\right]_{52}\left[f(\varepsilon-\omega, \mathbf{n}_6) - f(\varepsilon-\omega, \mathbf{n}_4)\right]\times \\ \left[\mathcal{L}^{g}_{14} f(\varepsilon, \mathbf{n}_3) + \mathcal{L}^{g}_{34} f(\varepsilon, \mathbf{n}_1)\right]. \quad (3.20b)$$

Its physical origin is discussed in detail in Ref. [17]. Being elastic, it preserves the number of particles, for each energy shell,

$$\int \widehat{\mathrm{St}}^{e\text{-}e}_{el}(\varepsilon, \mathbf{n}_1)\,d\mathbf{n}_1 = 0, \quad (3.20c)$$

as can be seen from the property $\gamma_{13}^2 = -\gamma_{31}^2$ of vertex (3.17).

The nonlocal term

$$\widehat{\mathrm{St}}^{e\text{-}e}_{n} = -\frac{4}{\nu}\mathrm{Re}\int \frac{d\omega}{2\pi}\,\frac{1}{\omega^2}\times \\ \times \int \frac{d^dq}{(2\pi)^d}\int \frac{d\mathbf{n}_2\ldots d\mathbf{n}_6}{\Omega_d^5}\gamma_{13}^{2}\gamma_{46}^{5}\times \\ \times \int d\varepsilon_1 \{\, \left[\mathcal{L}^{g}_{14} - \mathcal{L}^{g}_{34}\right]\left[\bar{\mathcal{L}}^{\rho} + 3\bar{\mathcal{L}}^{\sigma} - 4\bar{\mathcal{L}}^{g}\right]_{52}\times \\ \times \Psi^{s}_{46}(\varepsilon, \varepsilon_1; \omega) \quad (3.20d)$$

describes the inelastic electron–electron collisions during which the bosons and ghosts act as virtual states. (The function Ψ^s was introduced in Eq. (3.16d).) The real part being an even function, we can use Eq. (3.16e) to verify that Eq. (3.20d) satisfies the conservation law

$$\int \varepsilon^m\, \widehat{\mathrm{St}}^{e\text{-}e}_{n,l}(\varepsilon, \mathbf{n}_1)\,d\varepsilon = 0, \quad m = 0, 1. \quad (3.20e)$$

As indicated, the same law is satisfied by the local (and elastic) term:

$$\widehat{\mathrm{St}}^{e\text{-}e}_{l} = \frac{2}{\nu}\int \frac{d\omega}{2\pi}\,\frac{1}{\omega^2}\times \\ \times \int \frac{d^dq}{(2\pi)^d}\int \frac{d\mathbf{n}_2\ldots d\mathbf{n}_6}{\Omega_d^5}\int d\varepsilon_1\gamma_{13}^{2}\gamma_{46}^{5}\times \\ \times \left[\mathcal{L}^{g}_{14} + \mathcal{L}^{g}_{34}\right]\left[\mathcal{L}^{\rho} + 3\mathcal{L}^{\sigma} - 4\mathcal{L}^{g}\right]_{52}\times \\ \times \mathrm{sh}^2\left(\frac{\omega\partial}{2\partial\varepsilon}\right)\left[f(\varepsilon, \mathbf{n}_6) - f(\varepsilon, \mathbf{n}_4)\right]\times \\ \times \left[f(\varepsilon_1, \mathbf{n}_1)[1 - f(\varepsilon_1 - \omega, \mathbf{n}_3)] + (\mathbf{n}_1 \leftrightarrow \mathbf{n}_3)\right]. \quad (3.20f)$$

Therefore, Eq. (3.20e) enables us to conclude that both collision integrals (3.20d) and (3.20f) do not affect the transport coefficients (in the case where they can be considered perturbations in comparison to the bare impurity collision integral).

We note that although it might not be evident, the present form of the kinetic equation permits the proper identification of the inelastic kernel that determines the phase relaxation time; further details can be found in Appendix G.

4. SUMMARY OF THE RESULTS FOR THE THERMAL TRANSPORT AND SPECIFIC HEAT

In this section, we present our final answers for the interaction corrections to the thermal conductivity and the specific heat. They are obtained by solving the kinetic equations and then substituting the solutions in definitions (3.15) of the energy and energy current densities. The explicit calculations are performed in Sec. 6. We consider short-range impurities for which the scattering time τ is independent of the scattering angle, $\tau(\theta) = \tau$. We report our results for quasi-one-dimensional and three-dimensional systems in the diffusive limit $T\tau \ll \hbar$; for two-dimensional systems, we do not put such a restriction on the temperature range. However, common to all dimensionalities is the zeroth harmonic approximation for the Fermi-liquid constants (see Eq. (6.14)).

4.1. Thermal conductivity

In the absence of a magnetic field, the thermal conductivity tensor is diagonal, $\kappa_{\mu\nu} = \kappa\delta_{\mu\nu}$, and we write the expression for the diagonal components simply as

$$\kappa = \kappa_{WF} + \Delta\kappa. \quad (4.1)$$

The first term is given by the Wiedemann – Franz law $\kappa_{WF} = L\sigma T$ with the inclusion of the interaction corrections to the conductivity and the Lorentz number given in Eq. (1.1). The second term causes a violation of the Wiedemann – Franz law. In the diffusive limit and for low dimensionality, the main contribution to $\Delta\kappa$ is due to the long-range nature of the bosonic energy transport, which originates from the long-range part of the interaction in the singlet channel. In the quasiballistic case, large contribution also comes from the inelastic scattering of the electron on the bosons. Smaller corrections arise due to the triplet channel bosonic transport and to the energy dependence of the elastic scattering.

For quasi-one-dimensional and three-dimensional systems in the diffusive limit, we write

$$\Delta\kappa = \delta\kappa^{\rho} + 3\delta\kappa^{\sigma} + \delta\kappa_{el},$$

where the bosonic corrections include the ghost contributions

$$\delta\kappa^{\alpha} = \kappa^{\alpha} - \kappa^{g}, \ \alpha = \rho, \sigma$$

(see Eq. (6.12) for the definition of κ^{α}) and we neglect the inelastic contributions $\delta\kappa_{in}$, which are smaller by the parameter $T\tau/\hbar$.

For quasi-one-dimensional systems, the explicit expressions are

$$\delta\kappa_{el} = \frac{1}{8\sqrt{2\pi}}\zeta\left(\frac{3}{2}\right)\sqrt{\frac{DT}{\hbar}} \times \\ \times\left\{-1+3\left[1-\frac{2}{F_0^{\sigma}}\left(1+F_0^{\sigma}-\sqrt{1+F_0^{\sigma}}\right)\right]\right\}, \quad (4.2a)$$

$$\delta\kappa^{\rho} = \frac{3}{8\sqrt{2\pi}}\zeta\left(\frac{3}{2}\right)\sqrt{\frac{DT}{\hbar}}ak\ln^{1/2}\left(\frac{\hbar D k^2}{T}\right), \quad (4.2b)$$

$$\delta\kappa^{\sigma} = \frac{3}{8\sqrt{2\pi}}\zeta\left(\frac{3}{2}\right)\sqrt{\frac{DT}{\hbar}}\left[\sqrt{1+F_0^{\sigma}}-1\right], \quad (4.2c)$$

where a is a length of the order of the wire width, $k = \sqrt{4\pi e^2\nu}$ is the inverse screening length in the bulk, and $D = \tau v_F^2/d$ is the diffusion constant (in d dimensions).

For three-dimensional systems, the results are

$$\delta\kappa_{el} = \frac{5}{48\sqrt{2\pi^3}}\zeta\left(\frac{5}{2}\right)\sqrt{\frac{T^3}{\hbar^3 D}} \times \\ \times\left\{1+3\left[1-\frac{2}{F_0^{\sigma}}\left(1-\frac{1}{\sqrt{1+F_0^{\sigma}}}\right)\right]\right\}, \quad (4.3a)$$

$$\delta\kappa^{\rho} = \frac{15}{32\sqrt{2\pi^3}}\zeta\left(\frac{5}{2}\right)\sqrt{\frac{T^3}{\hbar^3 D}}, \quad (4.3b)$$

$$\delta\kappa^{\sigma} = \frac{15}{32\sqrt{2\pi^3}}\zeta\left(\frac{5}{2}\right) \times \\ \times\sqrt{\frac{T^3}{\hbar^3 D}}\left[1-\frac{1}{\sqrt{1+F_0^{\sigma}}}\right]. \quad (4.3c)$$

In these expressions, $\zeta(x)$ is the Riemann zeta function, with $\zeta(3/2) \approx 2.612$ and $\zeta(5/2) \approx 1.341$.

For two-dimensional systems, we separate the corrections due to the singlet and the triplet channel interactions:

$$\Delta\kappa = \Delta\kappa_s + 3\Delta\kappa_t. \quad (4.4a)$$

With logarithmic accuracy, the singlet channel contribution is

$$\Delta\kappa_s = \frac{T}{6\hbar}g_1\left(2\pi\frac{T\tau}{\hbar}\right)\ln\left(\frac{\hbar v_F k}{T}\right) - \\ - \frac{T}{24\hbar}g_2\left(\pi\frac{T\tau}{\hbar}\right)\ln\left(1+\frac{\hbar^2}{(T\tau)^2}\right) - \\ - \frac{\pi^2}{15}\frac{T}{\hbar}\left(\frac{T\tau}{\hbar}\right)^2\ln\left(\frac{E_F}{T}\right), \quad (4.4b)$$

where $k = 2\pi e^2\nu$ is the two-dimensional inverse screening length. The cross-over functions g_1 and g_2 are given in Eq. (6.42). Here, we note that $g_1(x), g_2(x) \approx 1$ for $x \ll 1$ and $g_1(x) \approx 3/x$, $g_2(x) \approx 14x^2/15$ for $x \gg 1$.

For the triplet channel, we have

$$\Delta\kappa_t = \begin{cases} -\dfrac{T}{18\hbar}\left[1-\dfrac{1}{F_0^{\sigma}}\ln\left(1+F_0^{\sigma}\right)\right] + \\ +\dfrac{T}{12\hbar}\ln\left(1+F_0^{\sigma}\right), \quad T\tau \ll \hbar, \\ -\dfrac{\pi^2}{15}\dfrac{T}{\hbar}\left(\dfrac{T\tau}{\hbar}\right)^2\ln\left(\dfrac{E_F}{T}\right) \times \\ \times\left(\dfrac{F_0^{\sigma}}{1+F_0^{\sigma}}\right)^2, \quad T\tau \gg \hbar. \end{cases} \quad (4.4c)$$

In the diffusive limit $T\tau \ll \hbar$, our results are consistent with those in Ref. [22], even though the form of the energy current operator used in this reference is, in our opinion, incorrect, see Appendix B.

4.2. Specific heat

The specific heat is given by

$$c_V = \frac{\pi^2}{3}\nu T + \delta c_V, \quad (4.5)$$

where the first term is the usual noninteracting electronic contribution and the second term is the bosonic interaction correction.

For quasi-one-dimensional and three-dimensional systems,

$$\delta c_V = (1 + 3\Lambda_d) \left(\frac{T}{\hbar D}\right)^{d/2} a_d. \tag{4.6}$$

The two terms in the first bracket are respectively the singlet and triplet channel contributions. The singlet channel term is considered in the unitary limit and is therefore independent of any interaction parameter. On the other hand, the Fermi-liquid parameter for the interaction in the triplet channel enters Eq. (4.6) as

$$\Lambda_d = 1 - \frac{1}{(1 + F_0^\sigma)^{d/2}} \tag{4.7}$$

and the numerical factors $a_{1,3}$ are

$$a_3 = \frac{15}{32\pi\sqrt{2\pi}}\zeta\left(\frac{5}{2}\right),$$

$$a_1 = -\frac{3}{8\sqrt{2\pi}}\zeta\left(\frac{3}{2}\right).$$

For two-dimensional systems, with logarithmic accuracy, the result is

$$\delta c_V = -\left(\frac{T}{\hbar D}\right)\left[(1 + 3\Lambda_2)\frac{1}{12}\ln\left(\frac{E_F}{T}\right) + \right. \\ \left. + \left(1 + 3\Lambda_2^2\right)\frac{3}{4\pi}\zeta(3)T\tau\right], \tag{4.8}$$

where $\zeta(3) \approx 1.202$. The first term in the right-hand side extends the logarithmic behavior known in the diffusive limit to higher temperatures (the upper cutoff is of the order of the Fermi energy E_F and not $\hbar/\tau$); the second term becomes relevant in the quasiballistic limit and coincides with the correction calculated in Ref. [29] for the clean Fermi liquid. In the diffusive limit, our results are the same as those obtained in Ref. [12] by an explicit thermodynamic calculation.

5. DERIVATION OF THE KINETIC EQUATION

This section is devoted to the derivation of the local kinetic equation. We first introduce the Eilenberger equation and some basic notation. Next, we perform a (generalized) gauge transformation: this is the crucial step that enables us to obtain the local description. Then we introduce the bosonic degrees of freedom and derive the collision integrals.

5.1. Eilenberger equation

Our starting point for the derivation of the kinetic equation is the Eilneberger equation (Eq. (5.7)). For disordered metals, the derivation of this equation starting from the action for the interacting electron gas in the presence of disorder can be found in Ref. [17]. We briefly summarize it here to introduce some notation and a straightforward generalization to the angle-dependent impurity scattering rate and Fermi-liquid parameter (see Eqs. (5.8) and (5.15)).

The interaction with small momentum and energy transfer in the singlet channel (the triplet channel is to be discussed in Sec. 5) is decoupled using the two Hubbard–Stratonovich fields $\phi_\pm(t, \mathbf{r}, \mathbf{n})$. For the purpose of the one-loop approximation that we use, these fields can be considered Gaussian with the propagators

$$\begin{aligned} \langle\!\langle \phi_+(1)\phi_+(2)\rangle\!\rangle &= -\frac{i}{2}D^K(1,2), \\ \langle\!\langle \phi_+(1)\phi_-(2)\rangle\!\rangle &= -\frac{i}{2}D^R(1,2), \\ \langle\!\langle \phi_-(1)\phi_+(2)\rangle\!\rangle &= -\frac{i}{2}D^A(1,2), \\ \langle\!\langle \phi_-(1)\phi_-(2)\rangle\!\rangle &= 0, \end{aligned} \tag{5.1}$$

where $\langle\!\langle\ldots\rangle\!\rangle$ denotes averaging over the fields $\phi_\pm$. We use the short notation

$$\begin{aligned} &(i) \equiv (t_i, \mathbf{r}_i, \mathbf{n}_i), \quad \int di \equiv \int dt_i d\mathbf{r}_i \int \frac{d\mathbf{n}_i}{\Omega_d}, \\ &(i^*) \equiv (\mathbf{r}_i, \mathbf{n}_i), \quad \int di^* \equiv \int d\mathbf{r}_i \int \frac{d\mathbf{n}_i}{\Omega_d}, \end{aligned} \tag{5.2}$$

where $i = 1, 2, \ldots$ and Ω_d is the total solid angle.

We introduce the disorder-averaged Green's function of the electron in the field $\phi_\pm$ in its matrix form in the Keldysh space,

$$\widehat{G}(1,2|\phi) = \begin{pmatrix} G^R(1,2|\phi) & G^K(1,2|\phi) \\ G^Z(1,2|\phi) & G^A(1,2|\phi) \end{pmatrix}_K, \tag{5.3}$$

such that its average over the fluctuating field $\phi_\pm$ gives the usual expressions for the physical propagators:

$$\begin{aligned} &\langle\!\langle G^R(1,2)\rangle\!\rangle = \\ &= -i\theta(t_1 - t_2)\langle\psi(1)\psi^\dagger(2) + \psi^\dagger(2)\psi(1)\rangle, \\ &\langle\!\langle G^A(1,2)\rangle\!\rangle = \\ &= i\theta(t_2 - t_1)\langle\psi(1)\psi^\dagger(2) + \psi^\dagger(2)\psi(1)\rangle, \\ &\langle\!\langle G^K(1,2)\rangle\!\rangle = -i\langle\psi(1)\psi^\dagger(2) - \psi^\dagger(2)\psi(1)\rangle, \\ &\langle\!\langle G^Z(1,2)\rangle\!\rangle = 0\,. \end{aligned} \tag{5.4}$$

Here, $\theta(t)$ is the Heaviside step function, $\psi^\dagger$ and ψ are the fermionic creation/annihilation operators in the

Heisenberg representation, and quantum mechanical averaging $\langle\ldots\rangle$ is performed with an arbitrary distribution function to be found from the solution of the kinetic equation.

For the disorder-averaged Green's function, the semiclassical approximation is obtained by integrating the Wigner transform of $\widehat{G}(1,2|\phi)$ over the distance from the Fermi surface:

$$\widehat{G}(t_1,t_2,\mathbf{p},\boldsymbol{R}) = \int d^2r\, e^{i\mathbf{P}\cdot\mathbf{r}}\widehat{G}(1,2|\phi), \tag{5.5}$$

where

$$\begin{gathered}\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2, \quad \boldsymbol{R}=\frac{1}{2}(\mathbf{r}_1+\mathbf{r}_2),\\ \mathbf{P}=\mathbf{p}-\frac{1}{2}\left[\mathbf{A}\left(t_1,\boldsymbol{R}\right)+\mathbf{A}\left(t_2,\boldsymbol{R}\right)\right],\\ \hat{g}(t_1,t_2,\mathbf{n},\mathbf{r}) = \\ = \frac{i}{\pi\nu}\int\limits_{-\infty}^{\infty} d\xi\; \widehat{G}\left(t_1,t_2,\mathbf{n}\left[p_F+\frac{\xi}{v_F}\right],\mathbf{r}\right),\end{gathered} \tag{5.6}$$

where $\mathbf{A}$ is the vector potential of an external electromagnetic field, p_F is the Fermi momentum, v_F is the Fermi velocity, and ν is the density of states on the Fermi level. The dynamics of the semiclassical Green's function $\hat{g}$ in the matrix form is governed by the Eilenberger equation [24]

$$\begin{gathered}\left[\tilde{\partial}_t+\mathbf{v}\cdot\tilde{\nabla}+\boldsymbol{\omega}_c\cdot\left(\mathbf{n}\times\frac{\partial}{\partial\mathbf{n}}\right)\right]\hat{g}+i\left[\hat{\phi};\hat{g}\right] = \\ = \frac{\left[\hat{g}\overset{\circ}{,}\widehat{\mathrm{St}}_\tau\hat{g}\right]}{2},\end{gathered} \tag{5.7}$$

where $\mathbf{v}=v_F\mathbf{n}$, the action of the «bare» collision integral on any function $a(\mathbf{n})$ is defined as

$$\begin{gathered}\left[\widehat{\mathrm{St}}_\tau a\right](\mathbf{n}) = \int\frac{d\mathbf{n}_1}{\Omega_d}\mathrm{St}_\tau(\mathbf{n},\mathbf{n}_1)a(\mathbf{n}_1)\,,\\ \widehat{\mathrm{St}}_\tau(\mathbf{n}_1,\mathbf{n}_2) = \frac{1}{\tau(\theta_{12})}-\delta(\widehat{\mathbf{n}_1\mathbf{n}_2})\int\frac{d\mathbf{n}_2}{\tau(\theta_{12})},\end{gathered} \tag{5.8}$$

and $\theta_{12}=\widehat{\mathbf{n}_1\mathbf{n}_2}$ (for the short-range impurity, $\tau(\theta)$ is independent of θ; however, the formulas derived here are valid for an arbitrary impurity scattering). The time convolution of two matrices $\hat{a}(t_1,t_2)$ and $\hat{b}(t_1,t_2)$ is given by

$$\begin{gathered}\hat{a}\circ\hat{b} = \int dt_3\; \hat{a}(t_1,t_3)\hat{b}(t_3,t_2),\\ \left[\hat{a}\overset{\circ}{,}\hat{b}\right] = \hat{a}\circ\hat{b}-\hat{b}\circ\hat{a}.\end{gathered} \tag{5.9}$$

Defining the commutator between a matrix $\hat{c}(t,\mathbf{r},\mathbf{n})$ and $\hat{g}$ as

$$\begin{gathered}[\hat{c};\hat{g}] = \hat{c}(t_1,\mathbf{r},\mathbf{n})\hat{g}(t_1,t_2,\mathbf{n},\mathbf{r}) - \\ - \hat{g}(t_1,t_2,\mathbf{n},\mathbf{r})\hat{c}(t_2,\mathbf{r},\mathbf{n}),\end{gathered} \tag{5.10}$$

we express the covariant derivatives in Eq. (5.7) as

$$\tilde{\partial}_t\hat{g} = \partial_{t_1}\hat{g}+\partial_{t_2}\hat{g}+i\left[\hat{\varphi};\hat{g}\right], \tag{5.11a}$$

$$\tilde{\nabla}\hat{g} = \nabla\hat{g}+i\left[\hat{\mathbf{A}};\hat{g}\right] \tag{5.11b}$$

with $\hat{\mathbf{A}}=\mathbf{A}\hat{\mathbb{1}}_K$ and $\hat{\varphi}=\varphi\hat{\mathbb{1}}_K$. Here, $\hat{\mathbb{1}}_K$ denotes the unit matrix in the Keldysh space and φ is the scalar potential for an external electromagnetic field such that

$$e\mathbf{E} = -\nabla\varphi+\partial_t\mathbf{A}, \quad e\boldsymbol{B} = -c\nabla\times\mathbf{A}.$$

The vector $\boldsymbol{\omega}_c = e\boldsymbol{B}/(mc)$ has the magnitude of the cyclotron frequency and the direction of the magnetic field $\boldsymbol{B}$. Finally, $\hat{\phi}$ is the matrix in the Keldysh space:

$$\hat{\phi} = \begin{pmatrix}\phi_+ & \phi_-\\ \phi_- & \phi_+\end{pmatrix}_K. \tag{5.12}$$

The matrix Green's function $\hat{g}$ is subject to the constraints

$$\hat{g}(\mathbf{n},\mathbf{r})\circ\hat{g}(\mathbf{n},\mathbf{r}) = \delta(t_1-t_2)\hat{\mathbb{1}}_K, \tag{5.13a}$$

$$\mathrm{Tr}\,\hat{g}(t,t,\mathbf{n},\mathbf{r}) = 0. \tag{5.13b}$$

In thermal equilibrium, the relation

$$\begin{gathered}g^K(t_1,t_2) = \left[g^R\circ n - n\circ g^A\right](t_1,t_2),\\ n(t_1,t_2) = \int\frac{d\varepsilon}{2\pi}\exp\left(i\varepsilon(t_2-t_1)\right)n(\varepsilon),\\ n(\varepsilon) = 1-2f_F(\varepsilon) = 2\,\mathrm{th}\,\frac{\varepsilon}{2T}\end{gathered} \tag{5.14}$$

must hold independently of the form of the spectral functions $g^{R,A}$.

In what follows, we assume that there is no magnetic field, $\mathbf{B}=0$ and $\boldsymbol{\omega}_c=0$, but no gauge choice is made: although one could set $\mathbf{A}=0$ by a gauge transformation, both the scalar and vector external potentials are left arbitrary in order to keep track of the gauge invariance of the equations.

The propagators defined by Eq. (5.1) satisfy the matrix Dyson equation

$$\begin{gathered}\hat{D}(1,2) = \hat{D}_0(1,2) + \\ + \int d3\int d4\; \hat{D}_0(1,3)\hat{\Pi}(3,4)\hat{D}(4,2),\\ \hat{D}_0(1,2) = -\left[V(\mathbf{r}_{12})+\frac{F^\rho(\theta_{12})\,\delta(\mathbf{r}_{12})}{\nu}\right]\times \\ \times\,\delta(t_{12})\hat{\mathbb{1}}_K,\end{gathered} \tag{5.15}$$

where $V(\mathbf{r})$ is the long-range part of the interaction (for the Coulomb interaction, $V(\mathbf{r}) = e^2/|\mathbf{r}|$, $\theta_{12} = \widehat{\mathbf{n}_1\mathbf{n}_2}$, $\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2$, and $t_{12} = t_1 - t_2$). The matrix propagator is denoted by $\hat{D}$ and $\hat{\Pi}$ is the matrix polarization operator. They have a structure similar to the Green's function one:

$$\hat{D} = \begin{pmatrix} D^R & D^K \\ 0 & D^A \end{pmatrix}_K, \quad \hat{\Pi} = \begin{pmatrix} \Pi^R & \Pi^K \\ 0 & \Pi^A \end{pmatrix}_K. \tag{5.16}$$

The polarization operators are given by variational derivatives of the solutions of the Eilenberger equation (5.7),

$$\Pi^R(1,2) = \Pi^A(2,1) = \\ = \nu \left[\delta_{12} + \frac{\pi}{2} \frac{\delta g^K(t_1, t_1, \mathbf{n}_1, \mathbf{r}_1)}{\delta \phi_+(t_2, \mathbf{r}_2, \mathbf{n}_2)} \right], \tag{5.17a}$$

$$\Pi^K(1,2) = \frac{\pi\nu}{2} \left[\frac{\delta(g^K + g^Z)(t_1, t_1, \mathbf{n}_1, \mathbf{r}_1)}{\delta \phi_-(t_2, \mathbf{r}_2, \mathbf{n}_2)} \right], \tag{5.17b}$$

where

$$\delta_{12} \equiv \Omega_d \delta(\widehat{\mathbf{n}_1\mathbf{n}_2}) \delta(\mathbf{r}_1 - \mathbf{r}_2) \delta(t_1 - t_2), \tag{5.18}$$

with Ω_d being the total solid angle.

5.2. The gauge transformation

With Eilenberger equation (5.7) at hand, one could proceed as in Ref. [17] in order to derive an equation for the distribution function. But the resulting inelastic part of the collision integral, expressed in terms of the electron distribution function only, is nonlocal and the evaluation, e.g., of the thermal conductivity would require the time and spatial gradient expansion of this term in the spirit of Eq. (2.15). As we already discussed, such a route makes the energy conservation in the kinetic equation obscure. Here, we follow a different approach, inspired by the following considerations [25]: if the fluctuating fields were uniform, they would be eliminated from Eq. (5.7) by a gauge transformation

$$\hat{g} \to \exp\left\{ -i \int^{t_1} \hat{\phi}(t)\, dt \right\} \hat{g} \exp\left\{ i \int^{t_2} \hat{\phi}(t)\, dt \right\}. \tag{5.19}$$

In other words, the position-independent fluctuations of the ϕ fields define a time-dependent position of the energy levels but the occupation numbers for such levels do not change. Therefore, such fluctuations affect neither the electric transport nor the electron contribution to the thermal transport in the system. Moreover, if the path of the electron were a straight line, all smooth fluctuating fields would still be eliminated in the eikonal approximation and, once again, they should not affect the electron contribution to the transport. To eliminate such spurious contributions, we use the gauge transformation described below.

We introduce a new matrix field $\hat{K}(t, \mathbf{n}, \mathbf{r})$,

$$\hat{K} = \begin{pmatrix} K_+ & K_- \\ K_- & K_+ \end{pmatrix}, \tag{5.20}$$

which is a functional of the field $\hat{\phi}$ and is used to perform the «generalized» gauge transformation

$$\hat{g} \to e^{-i\hat{K}(t_1, \mathbf{n}, \mathbf{r})} \hat{g}\, e^{i\hat{K}(t_2, \mathbf{n}, \mathbf{r})}. \tag{5.21}$$

This transformation is unitary and therefore preserves constraints (5.13). As we see in what follows, it leads to the local kinetic equations. Applying the transformation to Eilenberger equation (5.7), we obtain

$$\left[\tilde{\partial}_t + \mathbf{v} \cdot \tilde{\nabla} \right] \hat{g} - i \left[(\partial_t + \mathbf{v} \cdot \nabla) \hat{K} - \hat{\phi}, \hat{g} \right] = \\ = \frac{1}{2} \left[\hat{g} \overset{\circ}{,} \widehat{\mathrm{St}}{}_\tau^\phi \hat{g} \right], \tag{5.22}$$

where

$$\left[\widehat{\mathrm{St}}{}_\tau^\phi \hat{g} \right] (t_1, t_2, \mathbf{n}) \equiv \int \frac{d\mathbf{n}_1}{\Omega_d} \widehat{\mathrm{St}}_\tau(\mathbf{n}, \mathbf{n}_1) \times \\ \times e^{i\hat{K}(t_1, \mathbf{n})} e^{-i\hat{K}(t_1, \mathbf{n}_1)} \hat{g}(t_1, t_2, \mathbf{n}_1) e^{i\hat{K}(t_2, \mathbf{n}_1)} e^{-i\hat{K}(t_2, \mathbf{n})}.$$

The «bare» impurity collision integral and the derivatives are defined respectively in Eq. (5.8) and (5.11). We suppress the argument $\mathbf{r}$, which is the same in all functions.

We seek a perturbative solution of the Eilenberger equation in form (5.22) in the one-loop approximation; for this, it suffices to retain only the terms at most quadratic in the K fields in the collision integral. In the lowest order, $\hat{g}$ has the form

$$\hat{g} = \begin{pmatrix} \delta(t_1 - t_2) & g^K \\ 0 & -\delta(t_1 - t_2) \end{pmatrix}. \tag{5.23}$$

We require that this form be preserved even in the first and second order in K, i.e., the corrections to the spectrum (described by $g^{R,A}$) be indeed eliminated by the gauge transformation.

In the linear order, the retarded, advanced, and «Z» components of Eq. (5.22) vanish if K_- satisfies the equation

$$\Big(\partial_t + \mathbf{v} \cdot \nabla \Big) K_- + \widehat{\mathrm{St}}_\tau K_- = \phi_-. \tag{5.24}$$

The solution of integro-differential equation (5.24) can be written in terms of the diffuson propagator $\mathcal{L}^g(t_1, t_2; \mathbf{n}_1, \mathbf{n}_2; \mathbf{r}_1, \mathbf{r}_2)$, the retarded solution of the classical kinetic equation

$$\left(\partial_{t_1} + \mathbf{v}_1 \cdot \nabla_{\mathbf{r}_1} - \widehat{\mathrm{St}}_\tau\right) \mathcal{L}^g = \delta_{12}, \tag{5.25}$$

where δ_{12} is defined in Eq. (5.18). Using Eq. (5.25), we find

$$\begin{gathered} K_-(1) = -\int d2\, \bar{\mathcal{L}}^g(1,2)\phi_-(2), \\ \bar{\mathcal{L}}^g(1,2) = \mathcal{L}^g(2,1), \end{gathered} \tag{5.26}$$

where we use the short notation in (5.2). In the operator notation, Eq. (5.26) can be rewritten as

$$K_- = -\hat{\bar{\mathcal{L}}}^g \phi_-. \tag{5.26$'$}$$

To simplify further manipulations, we introduce the following function of three angular variables:

$$\gamma \begin{pmatrix} \mathbf{n}_k \\ \mathbf{n}_i ; \mathbf{n}_j \end{pmatrix} \equiv \frac{1}{\tau(\theta_{ij})} \times \\ \times \Omega_d \left[\delta(\widehat{\mathbf{n}_j\mathbf{n}_k}) - \delta(\widehat{\mathbf{n}_i\mathbf{n}_k})\right] \equiv \gamma^k_{ij}. \tag{5.27}$$

This function is related to the impurity collision integral (5.8) by

$$\begin{gathered} \int \frac{d\mathbf{n}_2}{\Omega_d} \gamma \begin{pmatrix} \mathbf{n}_3 \\ \mathbf{n}_1 ; \mathbf{n}_2 \end{pmatrix} = \left[\widehat{\mathrm{St}}_\tau\right](\mathbf{n}_1, \mathbf{n}_3), \\ \int \frac{d\mathbf{n}_1}{\Omega_d} \gamma \begin{pmatrix} \mathbf{n}_3 \\ \mathbf{n}_1 ; \mathbf{n}_2 \end{pmatrix} = -\left[\widehat{\mathrm{St}}_\tau\right](\mathbf{n}_2, \mathbf{n}_3), \\ \int \frac{d\mathbf{n}_3}{\Omega_d} \gamma \begin{pmatrix} \mathbf{n}_3 \\ \mathbf{n}_1 ; \mathbf{n}_2 \end{pmatrix} = 0. \end{gathered} \tag{5.28}$$

With δg^K denoting the first-order correction to g^K, the Keldysh component of the Eilenberger equations in the linear order is

$$\begin{gathered} i\left(\tilde{\partial}_t + \mathbf{v} \cdot \tilde{\nabla} - \widehat{\mathrm{St}}_\tau\right)\delta g^K + \\ + \left[\left(\partial_t + \mathbf{v}\cdot\nabla - \widehat{\mathrm{St}}_\tau\right) K_+ - \phi_+ ; g^K\right] = \hat{Q}K_- + \\ + \int \frac{d\mathbf{n}_2 d\mathbf{n}_3}{\Omega_d^2} \gamma \begin{pmatrix} \mathbf{n}_3 \\ \mathbf{n} ; \mathbf{n}_2 \end{pmatrix} \times \\ \times \left[K_+(\mathbf{n}_3); g^K(\mathbf{n}_2) - g^K(\mathbf{n})\right] \end{gathered} \tag{5.29}$$

with (cf. Eq. (5.11))

$$\begin{gathered} \tilde{\partial}_t \delta g^K = \left(\partial_{t_1} + \partial_{t_2}\right)\delta g^K + i[\varphi; \delta g^K], \\ \tilde{\nabla}\delta g^K = \nabla\delta g^K + i[\mathbf{A}; \delta g^K]. \end{gathered}$$

Here, the operator $\hat{Q}$ is local in space,

$$\hat{Q}K_- = \int dt_3 \int \frac{d\mathbf{n}_1}{\Omega_d} Q(t_1, t_2, \mathbf{n}; t_3, \mathbf{n}_1; \mathbf{r}) K_-(t_3, \mathbf{n}_1, \mathbf{r})$$

with the kernel

$$\begin{gathered} Q(t_1, t_2, \mathbf{n}_1; t_3, \mathbf{n}_2) = \frac{1}{2} \int \frac{d\mathbf{n}_3}{\Omega_d} \gamma \begin{pmatrix} \mathbf{n}_2 \\ \mathbf{n}_1 ; \mathbf{n}_3 \end{pmatrix} \times \\ \times \left[P g^K(t_1, t_3, \mathbf{n}_1) g^K(t_3, t_2, \mathbf{n}_3) + \right. \\ \left. + (\mathbf{n}_1 \leftrightarrow \mathbf{n}_3)\right]. \end{gathered} \tag{5.30}$$

We suppress the spectator argument $\mathbf{r}$, which is the same in each term of the equation; the last term means that terms similar to the ones shown but with the angular arguments of the Green's function switched must be added. The principal value sign P in Eq. (5.30) means that the part of the product of the Green's functions divergent as $t_1 \to t_2$,

$$g^K(t_1, t_2, \mathbf{n}, \mathbf{r})\Big|_{t_1 \to t_2} = -\frac{2i}{\pi(t_1 - t_2)} + \text{regular}, \tag{5.31}$$

must be eliminated,

$$\begin{gathered} P g^K(t_1, t_3) g^K(t_3, t_2) \equiv \\ \equiv g^K(t_1, t_3) g^K(t_3, t_2) - 4\delta(t_1 - t_3)\delta(t_3 - t_2), \end{gathered}$$

or, equivalently,

$$\begin{gathered} P g^K(t_1, t_3) g^K(t_3, t_2) \equiv \\ \equiv \frac{1}{2} \sum_{\sigma = \pm 1} g^K(t_1, t_3 + \sigma i0) g^K(t_3 + \sigma i0, t_2). \end{gathered}$$

It is worth noting that all nonequilibrium effects contribute to the regular part in Eq. (5.31) but not to the singular part; the states deep in the Fermi sea, which are not perturbed, contribute to it.

To solve Eq. (5.59), we define a new field $\tilde{K}_-$ by the relation

$$\tilde{K}_-(t, \mathbf{n}, \mathbf{r}) = (i\partial_t)^{-1} \hat{M} K_-, \tag{5.32}$$

where the operator $\hat{M}$ is shown below to be related to certain products of the Green's functions g^K, see Eq. (5.38). The operator $\hat{M}$ is Hermitian and local in space but not in the momentum direction and time. We again use the operator notation

$$\begin{gathered} \hat{M} K_- \equiv \\ \equiv \int dt_1 \int \frac{d\mathbf{n}_1}{\Omega_d} M(t, \mathbf{n}; t_1, \mathbf{n}_1; \mathbf{r}) K_-(t_1, \mathbf{n}_1, \mathbf{r}). \end{gathered} \tag{5.33}$$

We require K_+ to satisfy the equation

$$\left(\partial_t + \mathbf{v}\cdot\nabla - \widehat{\mathrm{St}}_\tau\right)K_+ = \phi_+ - 2\tilde{K}_-, \qquad (5.34)$$

whose solution is

$$K_+ = \hat{\mathcal{L}}^g\phi_+ - 2\hat{\mathcal{L}}^g\tilde{K}_-. \qquad (5.35)$$

The operator notation here is the same as in Eq. (5.26).

The next task is to choose the «best» form for the operator $\hat{M}$ to maximally simplify the further perturbative expansion. Writing $\delta g^K = \delta g^K_+ + \delta g^K_-$, we obtain the equations

$$i\hat{L}\delta g^K_- = \hat{Q}K_- + 2\left[\tilde{K}_-, g^K\right], \qquad (5.36a)$$

$$i\hat{L}\delta g^K_+ = \int\frac{d\mathbf{n}_2 d\mathbf{n}_3}{\Omega_d^2}\gamma\begin{pmatrix}\mathbf{n}_3\\ \mathbf{n};\mathbf{n}_2\end{pmatrix}\times \\ \times\left[K_+(\mathbf{n}_3), \left(g^K(\mathbf{n}_2) - g^K(\mathbf{n})\right)\right], \qquad (5.36b)$$

where $\hat{L} \equiv \left(\tilde{\partial}_t + \mathbf{v}\cdot\tilde{\nabla} - \widehat{\mathrm{St}}_\tau\right)$.

We note that the right-hand side of Eq. (5.36b) vanishes in equilibrium, because $g^K = \langle g^K\rangle_{\mathbf{n}}$. Therefore, δg^K_+ also vanishes in equilibrium and cannot contribute to the equilibrium properties such as the specific heat. Moreover, even outside the equilibrium, $\delta g^K_+(t_1, t_1, \mathbf{n}, \mathbf{r}) = 0$, because the right-hand side of Eq. (5.36b) vanishes, see the remark after Eq. (5.31). This means that δg^K_+ does not contribute to the electron density or current.

We are now ready to choose the operator $\hat{M}$. We require that $\delta g^K_-(t_1, t_1, \mathbf{n}, \mathbf{r}) = 0$, i.e., δg^K_- also does not contribute to the electron density or current. This means that the right-hand side of Eq. (5.36a) must vanish for $t_1 = t_2$ for any field K_-. Imposing this requirement, we obtain

$$\hat{M}(t_1, t_2, \mathbf{n}, \tilde{\mathbf{n}}, \mathbf{r}) = \frac{\pi}{4}\hat{Q}(t_1, t_1, \mathbf{n}; t_2, \tilde{\mathbf{n}}; \mathbf{r}). \qquad (5.37)$$

Together with Eq. (5.30), it yields

$$M(t_1, \mathbf{n}_1; t_2, \mathbf{n}_2; \mathbf{r}) = \frac{\pi}{8}\int\frac{d\mathbf{n}_3}{\Omega_d}\gamma\begin{pmatrix}\mathbf{n}_2\\ \mathbf{n}_1;\mathbf{n}_3\end{pmatrix}\times \\ \times\left[Pg^K(t_1, t_2, \mathbf{n}_1)g^K(t_2, t_1, \mathbf{n}_3) + \right. \\ \left. + (\mathbf{n}_1 \leftrightarrow \mathbf{n}_3)\right]. \qquad (5.38)$$

Expression (5.27) for the vertex γ enables us to establish the following properties of kernel (5.38):

$$M(t_1, \mathbf{n}_1; t_2, \mathbf{n}_2) = M(t_1, \mathbf{n}_2; t_2, \mathbf{n}_1) = \\ = M(t_2, \mathbf{n}_2; t_1, \mathbf{n}_1); \qquad (5.38')$$

the operator $\hat{M}$ is therefore Hermitian.

It is instructive to find $\hat{M}$ in the thermal equilibrium. With Eq. (5.14), because Eq. (5.24) implies that the retarded and advanced components of $\hat{g}$ are still given by Eq. (5.23), it follows from Eqs. (5.28) and (5.38) that

$$M_{eq}(t_1, \mathbf{n}_1; t_2, \mathbf{n}_2; \mathbf{r}) = \\ = \int\frac{d\omega}{2\pi}e^{i\omega(t_2 - t_1)}\hat{M}_{eq}(\omega; \mathbf{n}_1, \mathbf{n}_2), \\ M_{eq}(\omega; \mathbf{n}_1, \mathbf{n}_2) = -\omega\,\mathrm{cth}\left(\frac{\omega}{2T}\right)\left[\widehat{\mathrm{St}}_\tau\right](\mathbf{n}_1, \mathbf{n}_2). \qquad (5.39)$$

Equation (5.39) is useful in checking the fluctuation–dissipation theorem.

5.3. Polarization operators and propagators

The knowledge of the linear-order corrections to the Green's function permits the calculation of the polarization operators as variational derivatives of the original Green's functions (i.e., before the gauge transformation) in the limit $t_2 \to t_1$, see Eq. (5.17). In the linear order, the corrections to the original Green's functions are given by the relations (cf. Eq. (5.21))

$$\delta g^K \to \delta g^K - i\left[K_+, g^K\right] - 2K_-\delta(t_1 - t_2), \qquad (5.40a)$$

$$\delta g^Z \to 2K_-\delta(t_1 - t_2). \qquad (5.40b)$$

By construction in the previous subsection,

$$\lim_{t_2\to t_1}\delta g^K(t_1, t_2, \mathbf{n}, \mathbf{r}) = 0, \qquad (5.41)$$

and using Eq. (5.31), we have

$$\lim_{t_2\to t_1} -i\left[K_+, g^K\right] = -\frac{2}{\pi}\partial_t K_+(t, \mathbf{n}, \mathbf{r}). \qquad (5.42)$$

Substituting these results in Eq. (5.17) and using Eqs. (5.26), (5.32), and (5.35), we obtain

$$\Pi^R(1,2) = \nu\left[\delta_{12} - \partial_{t_1}\mathcal{L}^g(1,2)\right], \\ \Pi^A(2,1) = \nu\left[\delta_{12} - \partial_{t_2}\bar{\mathcal{L}}^g(1,2)\right]; \qquad (5.43a)$$

we use notation (5.2) throughout this subsection. The result for the Keldysh component is

$$\Pi^K(1,2) = 2i\nu\left[\hat{\mathcal{L}}^g\hat{M}\hat{\bar{\mathcal{L}}}^g\right](1,2). \qquad (5.43b)$$

The actions of the operators $\hat{M}$ and $\hat{D}$ are defined in Eqs. (5.26) and (5.33).

It is easy to check that the fluctuation–dissipation relation between the polarization operators holds in the thermal equilibrium. As follows from Eqs. (5.25) and (5.26),

$$-2\hat{\mathcal{L}}^g \hat{\mathrm{S}}\mathrm{t}_\tau \hat{\bar{\mathcal{L}}}^g = \hat{\mathcal{L}}^g + \hat{\bar{\mathcal{L}}}^g. \qquad (5.44)$$

We perform the time Fourier transformation for all the propagators and the polarization operators in thermodynamic equilibrium,

$$A(1,2) = \int \frac{d\omega}{2\pi} e^{i\omega(t_2-t_1)} A(\omega; 1^*, 2^*). \qquad (5.45)$$

Substituting Eqs. (5.39) and (5.44) in Eq. (5.43b), we obtain that in equilibrium,

$$\Pi^K_{eq}(\omega; 1^*, 2^*) = \\ = \left[\Pi^R(\omega; 1^*, 2^*) - \Pi^A(\omega; 1^*, 2^*)\right] \operatorname{cth}\frac{\omega}{2T}. \qquad (5.46)$$

With the help of Eq. (5.15), we then derive the fluctuation–dissipation relation

$$D^K_{eq}(\omega; 1^*, 2^*) = \\ = \left[D^R(\omega; 1^*, 2^*) - D^A(\omega; 1^*, 2^*)\right] \operatorname{cth}\frac{\omega}{2T}. \qquad (5.47)$$

With the expressions for the polarization operators obtained above, we can solve Dyson equation (5.15) and obtain the explicit expressions for the interaction propagators. In the operator notation,

$$\nu \hat{D}^R = -\frac{1}{1 + \hat{F} - \partial_t \hat{F} \hat{\mathcal{L}}^g} \hat{F}, \qquad (5.48a)$$

$$\nu \hat{D}^A = -\hat{F} \frac{1}{1 + \hat{F} + \partial_t \hat{\bar{\mathcal{L}}}^g \hat{F}}, \qquad (5.48b)$$

$$\hat{D}^K = 2i\nu \hat{D}^R \hat{\mathcal{L}}^g \hat{M} \hat{\bar{\mathcal{L}}}^g \hat{D}^A, \qquad (5.48c)$$

where the action of the operator $\hat{F}$ on any function $a(t, \mathbf{n}, \mathbf{r})$ is defined by

$$\left[\hat{F} a\right](t; \mathbf{n}, \mathbf{r}) \equiv \int \frac{d\mathbf{n}_1}{\Omega_d} \left[F^\rho(\widehat{\mathbf{n}\mathbf{n}}_1) a(t, \mathbf{n}_1, \mathbf{r}) + \right. \\ \left. + \int d\mathbf{r}_1 \nu V(\mathbf{r} - \mathbf{r}_1) a(t, \mathbf{n}_1, \mathbf{r}_1)\right], \qquad (5.49)$$

see also the text after Eq. (5.15).

To find the propagators for the fields $K_\pm$ given in Eqs. (5.26′) and (5.35), defined as

$$\begin{aligned} \langle\!\langle K_+(1) K_+(2) \rangle\!\rangle &= \frac{i}{2} \mathcal{K}^K(1,2), \\ \langle\!\langle K_+(1) K_-(2) \rangle\!\rangle &= \frac{i}{2} \mathcal{K}^R(1,2), \\ \langle\!\langle K_-(1) K_+(2) \rangle\!\rangle &= \frac{i}{2} \mathcal{K}^A(1,2), \\ \langle\!\langle K_-(1) K_-(2) \rangle\!\rangle &= 0, \end{aligned} \qquad (5.50)$$

we use Eqs. (5.1) and (5.48) and obtain the retarded and advanced propagators

$$\hat{\mathcal{K}}^R = \hat{\mathcal{L}}^g \hat{D}^R \hat{\mathcal{L}}^g, \quad \hat{\mathcal{K}}^A = \hat{\bar{\mathcal{L}}}^g \hat{D}^A \hat{\bar{\mathcal{L}}}^g, \qquad (5.51a)$$

whereas the result for the Keldysh propagator is

$$\hat{\mathcal{K}}^K = -\hat{\mathcal{L}}^g \hat{D}^K \hat{\bar{\mathcal{L}}}^g + 2i\left[\hat{\mathcal{L}}^g (\partial_t)^{-1} \hat{M} \hat{\bar{\mathcal{L}}}^g \hat{D}^A \hat{\bar{\mathcal{L}}}^g - \right. \\ \left. - \hat{\mathcal{L}}^g \hat{D}^R \hat{\mathcal{L}}^g \hat{M} (\partial_t)^{-1} \hat{\bar{\mathcal{L}}}^g\right]. \qquad (5.51b)$$

The fluctuation–dissipation relation between the D propagators in Eq. (5.47), the equilibrium form for $\hat{M}$ in Eq. (5.39), and identity (5.44) allow us to verify the fluctuation–dissipation relation for the $\mathcal{K}$ propagators:

$$\mathcal{K}^K_{eq}(\omega; 1^*, 2^*) = \left[\mathcal{K}^R(\omega; 1^*, 2^*) - \mathcal{K}^A(\omega; 1^*, 2^*)\right] \times \\ \times \operatorname{cth}\frac{\omega}{2T}. \qquad (5.52)$$

5.4. Additional bosonic fields

Equation (5.51b) together with Eqs. (5.51a) and (5.38) allows expressing the Keldysh propagator $\mathcal{K}^K$ in terms of the electron distribution function. This relation, however, is nonlocal on the spatial scale much larger than the temperature length

$$L_T \approx \min\left[\frac{\hbar v_F}{T}, v_F\sqrt{\frac{\hbar\tau}{T}}\right], \qquad (5.53)$$

recall the discussion in Sec. 2.2. Indeed, the collision integral and all physical quantities are then given by integrals of the type

$$\mathcal{I}^\alpha = \int d\omega f(\omega) \mathcal{K}^\alpha(\omega), \quad \alpha = R, A, K,$$

where the function $f(\omega)$ depends on its argument at the characteristic scale of T. A retarded function is an analytic function of ω at $\operatorname{Im}\omega > 0$, which implies that for $\alpha = R$, the integral is determined only by singularities of $f(\omega)$, i.e., $\mathcal{I}^R \approx \mathcal{K}^R(\omega = iT)$. This immediately restricts the spatial scales to L_T. The same argument applies to the advanced case, because of the analyticity at $\operatorname{Im}\omega < 0$. But the function $\mathcal{K}^K(\omega)$ is not analytic. Moreover, according to Eq. (5.51b), it contains overlapping singularities of the retarded and advanced propagators. This means that the characteristic frequencies entering $\mathcal{I}^K$ are determined by the poles of the propagator rather than by the width of the function f, i.e., the spatial scale may by far exceed L_T and any expression of the type $\mathcal{I}^K$ is therefore nonlocal.

To overcome this difficulty, the standard parameterization of the Keldysh function $D^K = D^R \circ N - N \circ D^A$

is usually introduced and the kinetic equation for the distribution function N is then derived. All the nonlocality in the problem is then contained in the partial solution of the kinetic equation, to be compared with Eq. (2.14), whereas the kinetic equation itself is *local.*

In what follows, we adopt this program in a slightly modified form. We introduce a new retarded propagator $\mathcal{L}^{\rho}(1,2)$ (to be compared with Eq. (5.25))

$$\left[i\hat{H}_{e-h}(i\partial_{t_1}, -i\nabla_1) - \widehat{\mathrm{St}}_{\tau}\right]\mathcal{L}^{\rho} = \delta_{12}, \qquad \hat{H}_{e-h}(\omega,\mathbf{q}) = \mathbf{v}\cdot\mathbf{q} - \frac{\omega}{1+\hat{F}} \tag{5.54}$$

and its advanced counterpart $\bar{\mathcal{L}}^{\rho}(1,2) = \mathcal{L}^{\rho}(2,1)$. The multiplications in Eq. (5.54) are to be understood in the operator sense and the action of the operator $\hat{F}$ on a function $a(t,\mathbf{n},\mathbf{r})$ is defined by Eq. (5.49).

For $\hat{F} = 0$, $\mathcal{L}^{\rho}$ ($\bar{\mathcal{L}}^{\rho}$) reduces to the usual diffuson $\mathcal{L}^{g}$ ($\bar{\mathcal{L}}^{g}$). Physically, $\mathcal{L}^{\rho}$ describes the spectrum of a propagating electron–hole pair and the operator in the left-hand side of Eq. (5.54) corresponds to the kinetic equation for the collective mode in the Fermi-liquid theory [26]. The operator $\hat{H}_{e-h}(\omega,\mathbf{q})$ can be interpreted as a «Hamiltonian» (see also Appendix C) of the interacting electron–hole pair.

In terms of $\mathcal{L}^{\rho}$ and $\mathcal{L}^{g}$, Eqs. (5.51) acquire the form

$$\nu\hat{\mathcal{K}}^{R} = (\partial_t)^{-1}\left[\hat{\mathcal{L}}^{g} - \hat{\mathcal{L}}^{\rho}\right], \qquad \nu\hat{\mathcal{K}}^{A} = -\left[\hat{\bar{\mathcal{L}}}^{g} - \hat{\bar{\mathcal{L}}}^{\rho}\right](\partial_t)^{-1}, \tag{5.55a}$$

$$\nu\hat{\mathcal{K}}^{K} = -2i\,(\partial_t)^{-1}\left[\hat{\mathcal{L}}^{g}\hat{M}\hat{\bar{\mathcal{L}}}^{g} - \hat{\mathcal{L}}^{\rho}\hat{M}\hat{\bar{\mathcal{L}}}^{\rho}\right](\partial_t)^{-1}. \tag{5.55b}$$

We introduce two bosonic «distribution functions» (the density matrices, to be more precise) $\hat{\mathcal{N}}^{g}$ and $\hat{\mathcal{N}}^{\rho}$ that satisfy the equations

$$(\hat{\mathcal{L}}^{g})^{-1}\hat{\mathcal{N}}^{g} + \hat{\mathcal{N}}^{g}(\hat{\bar{\mathcal{L}}}^{g})^{-1} = 2\hat{M}, \tag{5.56a}$$

$$(\hat{\mathcal{L}}^{\rho})^{-1}\hat{\mathcal{N}}^{\rho} + \hat{\mathcal{N}}^{\rho}(\hat{\bar{\mathcal{L}}}^{\rho})^{-1} = 2\hat{M}. \tag{5.56b}$$

The operator $\hat{M}$ is defined in Eqs. (5.33) and (5.38) and, in a more explicit notation, the action of the operators $\hat{\mathcal{N}}^{\rho,g}$ on any function $a(t,\mathbf{n},\mathbf{r})$ is to be understood as

$$\left[\hat{\mathcal{N}}^{\rho,g}a\right](1) = \int d2\;\mathcal{N}^{\rho,g}(1,2)\,a(2),$$

where short notation (5.2) is used. We note that the above equations imply that the bosonic functions $\mathcal{N}^{\rho,g}$ are symmetric:

$$\mathcal{N}^{\rho,g}(1,2) = \mathcal{N}^{\rho,g}(2,1). \tag{5.57}$$

Equations (5.56) allow us to rewrite Eq. (5.55b) as

$$\nu\hat{\mathcal{K}}^{K} = -i\,(\partial_t)^{-1}\left[\hat{\mathcal{L}}^{g}\hat{\mathcal{N}}^{g} + \hat{\mathcal{N}}^{g}\hat{\bar{\mathcal{L}}}^{g}\right](\partial_t)^{-1} + i\,(\partial_t)^{-1}\left[\hat{\mathcal{L}}^{\rho}\hat{\mathcal{N}}^{\rho} + \hat{\mathcal{N}}^{\rho}\hat{\bar{\mathcal{L}}}^{\rho}\right](\partial_t)^{-1}. \tag{5.58}$$

This expression is local in the sense discussed above and is used in the construction of the conserved energy current. Obtaining the local expression, however, requires the introduction of two additional bosonic distribution functions: $\mathcal{N}^{\rho}$, describing the interacting electron–hole pairs, and the ghost field distribution $\mathcal{N}^{g}$, subtracting the contribution of the electron–hole pairs in the absence of interactions.

Closing this subsection, we rewrite Eq. (5.56) in a form resembling the kinetic equation in Sec. 2. We substitute Eqs. (5.25) and (5.54) in Eq. (5.56) and obtain

$$\left[\partial_t + \mathbf{v}\cdot\nabla;\ \hat{\mathcal{N}}^{g}\right] = \widehat{\mathrm{St}}^{\,b}\left\{\mathcal{N}^{g}, g^{K}\right\}, \tag{5.59a}$$

$$\left[i\hat{H}_{e-h}(i\partial_{t_1}, -i\nabla_1); \hat{\mathcal{N}}^{\rho}\right] = \widehat{\mathrm{St}}^{\,b}\left\{\mathcal{N}^{\rho}, g^{K}\right\}, \tag{5.59b}$$

where the collision integrals are

$$\widehat{\mathrm{St}}^{\,b}\left\{\mathcal{N}^{\alpha}, g^{K}\right\} \equiv 2\left\{\widehat{\mathrm{St}}_{\tau}; \hat{\mathcal{N}}^{\alpha}\right\} + 2\hat{M} \tag{5.59c}$$

for $\alpha = g,\rho$. They depend on g^{K} via $\hat{M}$ and we use the notation

$$\left\{\hat{A};\hat{B}\right\} \equiv \frac{1}{2}(\hat{A}\hat{B} + \hat{B}\hat{A}), \qquad \left[\hat{A};\hat{B}\right] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}. \tag{5.60}$$

We perform the time and space Wigner transformations of Eqs. (5.59) to introduce the bosonic distribution functions $N^{g,\rho}$,

$$\mathcal{N}^{g,\rho}(1,2) = \int\frac{d\omega}{2\pi}e^{-i\omega(t_1-t_2)}\int\frac{d^dq}{(2\pi)^d}e^{i\mathbf{q}\cdot(\mathbf{r}_1-\mathbf{r}_2)}\times \omega\left[2N^{g,\rho}(\omega,\mathbf{q};\mathbf{n}_1,\mathbf{n}_2;t,\mathbf{r}) + \Omega_d\delta(\widehat{\mathbf{n}_1,\mathbf{n}_2})\right], \tag{5.61}$$

where $t = (t_1+t_2)/2$, $\mathbf{r} = (\mathbf{r}_1+\mathbf{r}_2)/2$. Symmetry relation (5.57) translates into the condition

$$N^{g,\rho}(\omega,\mathbf{q};\mathbf{n}_1,\mathbf{n}_2) = -\left[N^{g,\rho}(-\omega,-\mathbf{q};\mathbf{n}_2,\mathbf{n}_1) + \Omega_d\delta(\widehat{\mathbf{n}_1,\mathbf{n}_2})\right]. \tag{5.62}$$

The physical meaning of this relation is the Bose statistics: at $\omega > 0$, $N^{g,\rho}$ corresponds to the occupation numbers entering the probability of the absorption of the bosons, whereas the $\omega < 0$ part describes the boson emission.

The fermionic distribution function f is obtained in two steps: (a) we introduce the gauge-invariant Green's function g (also see the next subsection) and (b) we perform the time Wigner transformation:

$$g^K(t_1, t_2, \mathbf{n}, \mathbf{r}) \equiv \\ \equiv \exp\left(-i\int\limits_{t_2}^{t_1} d\tilde{t}\varphi(\tilde{t}, \mathbf{r})\right) g(t_1, t_2, \mathbf{n}, \mathbf{r}), \quad (5.63\text{a})$$

$$g(t_1, t_2, \mathbf{n}, \mathbf{r}) = \\ = 2\int \frac{d\varepsilon}{2\pi} e^{-i\varepsilon(t_1 - t_2)} \left[1 - 2f(\varepsilon, \mathbf{n}; t, \mathbf{r})\right]. \quad (5.63\text{b})$$

Performing such Wigner transformations of Eqs. (5.56) and (5.38), we find

$$\omega\left[\partial_t \hat{N}^g + \left\{\mathbf{v}; \nabla\hat{N}^g\right\} + i\left[\mathbf{v}\cdot\mathbf{q}; \hat{N}^g\right]\right] = \\ = \widehat{\mathrm{St}}^b\left\{N^g, f\right\}, \quad (5.64\text{a})$$

$$\omega\left[\left\{\frac{1}{1+\hat{F}}; \partial_t \hat{N}^\rho\right\} + \left\{\hat{\boldsymbol{s}}(\omega, \mathbf{q}); \nabla\hat{N}^\rho\right\} + \\ + i\left[\hat{H}_{e-h}(\omega, \mathbf{q}), \hat{N}^\rho\right]\right] = \widehat{\mathrm{St}}^b\left\{N^\rho, f\right\}, \quad (5.64\text{b})$$

where the collective mode velocity operator is

$$\hat{\boldsymbol{s}}(\omega, \mathbf{q}) = \frac{\partial \hat{H}_{e-h}(\omega, \mathbf{q})}{\partial \mathbf{q}} = \mathbf{v} + \omega\frac{\partial}{\partial \mathbf{q}}\left(\frac{\hat{F}}{1+\hat{F}}\right). \quad (5.65)$$

In the left-hand side of Eq. (5.64b), we limited ourselves to the leading Poisson brackets (the equation becomes exact for a short-range interaction because $\partial_{\mathbf{q}}\hat{F} = 0$, and in the unitary limit, $\hat{F} \to \infty$). However, no Poisson brackets arise in the right-hand sides of Eqs. (5.64) as a consequence of the locality of the kinetic equations.

The right-hand sides of Eq. (5.64) describe the decay of an electron–hole pair into an electron and a hole moving in different directions. To write the expression for this collision term, it is convenient to introduce the following object:

$$\Upsilon^{g,\rho}_{ij;kl}(\varepsilon, \omega, \mathbf{q}; t, \mathbf{r}) \equiv N^{g,\rho}(\omega, \mathbf{q}; \mathbf{n}_i, \mathbf{n}_j; t, \mathbf{r}) \times \\ \times\Big\{f(\varepsilon, \mathbf{n}_k; t, \mathbf{r}) - f(\varepsilon - \omega, \mathbf{n}_k; t, \mathbf{r})\Big\} + \Omega_d\delta(\widehat{\mathbf{n}_i\mathbf{n}_j}) \times \\ \times\Big\{f(\varepsilon, \mathbf{n}_l; t, \mathbf{r})\left[1 - f(\varepsilon - \omega, \mathbf{n}_k; t, \mathbf{r})\right]\Big\}. \quad (5.66)$$

It is easy to see that $\Upsilon^{g,\rho} = 0$ in the thermal equilibrium, Eq. (3.1).

In terms of this object and vertex (5.27), we have

$$\widehat{\mathrm{St}}^b\left\{N^{g,\rho}, f\right\}(\omega, \mathbf{q}; \mathbf{n}_1, \mathbf{n}_2; t, \mathbf{r}) = -\int d\varepsilon \int \frac{d\mathbf{n}_3 d\mathbf{n}_4}{\Omega_d^2} \times \\ \times\Big\{\gamma^3_{14}\Upsilon^{g,\rho}_{32;41}(\varepsilon, \omega) + \gamma^2_{34}\Upsilon^{g,\rho}_{13;14}(\varepsilon, \omega)\Big\}, \quad (5.67)$$

where we suppress the spectator arguments t, $\mathbf{r}$, and $\mathbf{q}$ in the right-hand side of the equation. In deriving Eq. (5.67), we used Eqs. (5.28) and (5.38′) and the property

$$\int d\varepsilon \left[f(\varepsilon) - f(\varepsilon - \omega)\right] = -\omega.$$

To understand the physical meaning of the processes described by collision integral (5.67), we use the explicit form of the vertex γ (Eq. (5.27)) for the isotropic impurity scattering $\tau(\theta_{12}) = \tau$. Then the collision integrals can be decomposed into the sum of two contributions

$$\widehat{\mathrm{St}}^b\{N^{g,\rho}, f\} = \widehat{\mathrm{St}}^b_{cl}\{N^{g,\rho}, f\} + \widehat{\mathrm{St}}^b_q\{N^{g,\rho}, f\}.$$

The first term in the right-hand side can be obtained from a simple counting of the probabilities of the processes depicted in Fig. 2*a, b*:

$$\widehat{\mathrm{St}}^b_{cl}\left\{N^{g,\rho}, f\right\}(\omega; \mathbf{n}_1, \mathbf{n}_2) = \\ = \frac{1}{\tau}\int d\varepsilon \int \frac{d\mathbf{n}_3}{\Omega_d}\Big\{\Upsilon^{g,\rho}_{12;32}(\varepsilon, \omega) + \Upsilon^{g,\rho}_{12;13}(\varepsilon, \omega)\Big\}.$$

The second term in the right-hand side originates from the interference of two scattering processes, see Fig. 2*c*. It therefore makes contributions to $\hat{N}$ that are not diagonal in the momentum directions:

$$\widehat{\mathrm{St}}^b_q\left\{N^{g,\rho}, f\right\}(\omega; \mathbf{n}_1, \mathbf{n}_2) = \\ = -\frac{1}{\tau}\int d\varepsilon \int \frac{d\mathbf{n}_3}{\Omega_d}\Big\{\Upsilon^{g,\rho}_{13;12}(\varepsilon, \omega) + \Upsilon^{g,\rho}_{32;21}(\varepsilon, \omega)\Big\}.$$

5.5. The collision integral for electrons

With the bosonic propagators $\mathcal{K}$ at hand, we can proceed with the calculation and include the second-order contributions in the fluctuating fields $K_\pm$ to the collision term of Eilenberger equation (5.22). With the fluctuating fields $K_\pm$ given by Eqs. (5.26′) and (5.35), the Eilenberger equation becomes

$$\left[\tilde{\partial}_t + \mathbf{v}\cdot\tilde{\nabla}\right]\hat{g} = \left[\hat{g} \overset{\circ}{,} \frac{1}{2}\widehat{\mathrm{St}}^\phi_\tau \hat{g} - \\ - i\left(\widehat{\mathrm{St}}_\tau K_+ - 2\tilde{K}_-\right)\hat{\mathbb{1}}_K + i\widehat{\mathrm{St}}_\tau K_- \hat{\sigma}^x_K\right], \quad (5.68)$$

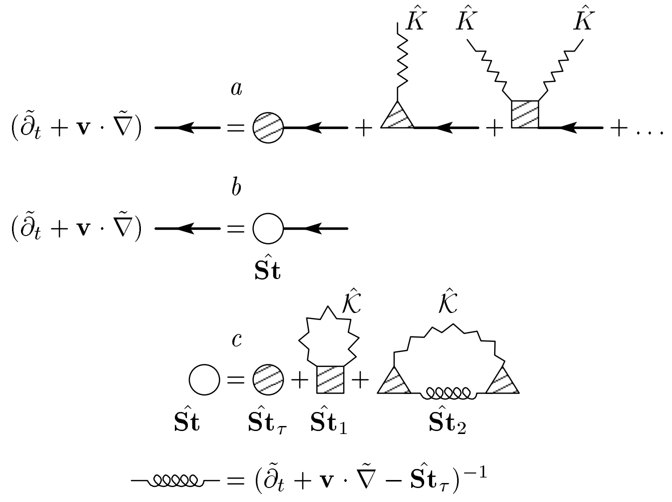

Fig. 3. Schematic representation of averaging over the fluctuation fields $\hat{K}$. *a*) Expansion of Eilenberger equation (5.68) before averaging. *b*) The equation for the Green's function averaged over $\hat{K}$, see Eq. (5.69). *c*) The contributions to the total collision integral in the one-loop approximation

where we use notation (5.11) for the derivatives, $\widehat{\mathrm{St}}_\tau^\phi$ is defined after Eq. (5.22), $\tilde{K}_-$ is defined in Eq. (5.32), and $\hat{\sigma}_K^x$ is the Pauli matrix.

We expand the right-hand side of Eq. (5.68) up to the second order in $\hat{K}$, see Fig. 3*a*; then we average it to obtain Fig. 3*b*. The resulting second-order contributions can have two different origins: (1) they can arise from the expansion of the exponentials truncated at the second order, the term $\widehat{\mathrm{St}}_1$ in Fig. 3*c*, or (2) they are obtained as products of the linear correction δg^K of Eq. (5.36a) and the first-order expansion of the exponentials, the term $\widehat{\mathrm{St}}_2$ in Fig. 3*c*.

The Eilenberger equation for the averaged Green function takes the form

$$\left[\tilde{\partial}_t + \mathbf{v}\cdot\tilde{\nabla}\right]\hat{g} = \hat{\widehat{\mathrm{St}}}\{\hat{g}, N^\rho, N^g\}, \tag{5.69}$$

where $\hat{\widehat{\mathrm{St}}}$ contains both zeroth and second-order contributions. We find (see Appendix D for the details on the cancellation of second-order corrections in the R, A, Z sectors)

$$\hat{\widehat{\mathrm{St}}}\left\{g_0^R, g_0^A, g^K\right\} = \begin{pmatrix} 0 & \widehat{\mathrm{St}}\left\{g^K, N^\rho, N^g\right\} \\ 0 & 0 \end{pmatrix}, \tag{5.70}$$

where $g_0^R = -g_0^A = \delta(t_1 - t_2)$. This means that the matrix Green's function of form (5.23) is still a solution of the Eilenberger equation — the main gain of gauge transformation (5.21) — provided that the kinetic equation for the Keldysh component is satisfied; accordingly, we concentrate on this component only.

Performing gauge transformation (5.63a) of the Keldysh component of the Eilenberger equation, we arrive at the explicitly gauge-invariant form of the kinetic equation

$$\left[\partial_{t_1} + \partial_{t_2} + \mathbf{v}\cdot\nabla + i\mathbf{v}\cdot\int_{t_2}^{t_1} d\tilde{t}\, e\mathbf{E}(\tilde{t}, \mathbf{r})\right] g = \\ = \widehat{\mathrm{St}}\{g, N^\rho, N^g\}. \tag{5.71}$$

In the collision integral $\widehat{\mathrm{St}}\{g, N^\rho, N^g\}$, we use $\widehat{\mathrm{St}}_1\{g, N^\rho, N^g\}$ to denote contributions of type (1) and $\widehat{\mathrm{St}}_2\{g, N^\rho, N^g\}$ for those of type (2)[6],

$$\widehat{\mathrm{St}}\{g, N^\rho, N^g\} = \widehat{\mathrm{St}}_\tau g - 4\widehat{\mathrm{St}}_1\{g, N^\rho, N^g\} - \\ - 4\widehat{\mathrm{St}}_2\{g, N^\rho, N^g\}, \tag{5.72}$$

where $\widehat{\mathrm{St}}_\tau$ is defined in Eq. (5.8). The numerical factors in front of the last two terms are introduced to facilitate the transformation to the canonical form of the kinetic equation in subsequent sections.

The expression for $\widehat{\mathrm{St}}_1$ written in terms of the $\mathcal{K}$ propagators (5.50) and the γ vertex (5.27) is

$$\left[\widehat{\mathrm{St}}_1\right](t_1, t_2; \mathbf{n}_1) = -\frac{i}{16}\int \frac{d\mathbf{n}_2 d\mathbf{n}_3}{\Omega_d^2}\,\gamma_{12}^3 \times \\ \times \Big\{ g(t_1, t_2, \mathbf{n}_2)\left[\tilde{\mathcal{K}}_{32}^K(t_1, t_2) - \tilde{\mathcal{K}}_{31}^K(t_1, t_2)\right] + \\ + \int dt_3 g(t_1, t_3, \mathbf{n}_1) g(t_3, t_2, \mathbf{n}_2) \times \\ \times \left[\mathcal{K}_{32}^A(t_3, t_2) - \mathcal{K}_{31}^A(t_3, t_2)\right] - \\ - g(t_1, t_3, \mathbf{n}_2) g(t_3, t_2, \mathbf{n}_1) \times \\ \times \left[\mathcal{K}_{23}^R(t_1, t_3) - \mathcal{K}_{13}^R(t_1, t_3)\right] \Big\}. \tag{5.73}$$

We introduce the short notation

$$\mathcal{K}_{ij}(t_1, t_2) \equiv \mathcal{K}(t_1, \mathbf{n}_i; t_2, \mathbf{n}_j) \tag{5.74}$$

and

$$\tilde{\mathcal{K}}^K(t_1, t_2) \equiv 2\mathcal{K}^K(t_1, t_2) - \mathcal{K}^K(t_1, t_1) - \mathcal{K}^K(t_2, t_2).$$

We omit the variable $\mathbf{r}$, which always appears in the distribution function as $g(t_1, t_2, \mathbf{n}, \mathbf{r})$ and in the propagators as $\mathcal{K}(t_1, \mathbf{n}_1, \mathbf{r}; t_2, \mathbf{n}_2, \mathbf{r})$. The dependence on the

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[6] This separation has no particular physical meaning, it is just a matter of practicality in the calculations; we return to the physical aspects when we analyze the conservation laws in the next subsection.

electron distribution function g is explicit, whereas the dependence on the bosonic distribution functions is hidden into the propagators, see Eq. (5.58). For reasons to become clear in the next subsection, we split Eq. (5.73) into two parts:

$$\widehat{\mathrm{St}}_1 = \widehat{\mathrm{St}}_1^{in} + \widehat{\mathrm{St}}_1^{el}; \tag{5.75a}$$

$$\begin{aligned}\left[\widehat{\mathrm{St}}_1^{in}\right](t_1,t_2;\mathbf{n}_1) = -\frac{i}{32}\int \frac{d\mathbf{n}_2 d\mathbf{n}_3}{\Omega_d^2}\,\gamma_{12}^3 \times \\ \times\Big\{2g(t_1,t_2,\mathbf{n}_2)\left[\tilde{\mathcal{K}}_{32}^K(t_1,t_2)-\tilde{\mathcal{K}}_{31}^K(t_1,t_2)\right]+ \\ +\int dt_3\,[g(t_1,t_3,\mathbf{n}_1)g(t_3,t_2,\mathbf{n}_2)+ \\ +g(t_1,t_3,\mathbf{n}_2)g(t_3,t_2,\mathbf{n}_1)]\,[\mathcal{K}_{32}^A(t_3,t_2)-\mathcal{K}_{23}^R(t_1,t_3)- \\ -\mathcal{K}_{31}^A(t_3,t_2)+\mathcal{K}_{13}^R(t_1,t_3)]\Big\}.\end{aligned} \tag{5.75b}$$

$$\begin{aligned}\left[\widehat{\mathrm{St}}_1^{el}\right](t_1,t_2;\mathbf{n}_1) = -\frac{i}{32}\int dt_3\int \frac{d\mathbf{n}_2 d\mathbf{n}_3}{\Omega_d^2}\,\gamma_{12}^3 \times \\ \times\Big\{[g(t_1,t_3,\mathbf{n}_1)g(t_3,t_2,\mathbf{n}_2)-g(t_1,t_3,\mathbf{n}_2)g(t_3,t_2,\mathbf{n}_1)]\times \\ \times\,[\mathcal{K}_{32}^A(t_3,t_2)+\mathcal{K}_{23}^R(t_1,t_3)- \\ -\mathcal{K}_{31}^A(t_3,t_2)-\mathcal{K}_{13}^R(t_1,t_3)]\Big\}.\end{aligned} \tag{5.75c}$$

As regards $\widehat{\mathrm{St}}_2$, it is convenient to separate it into two parts, depending on which field, K_+ or K_-, we retain in the expansion,

$$\widehat{\mathrm{St}}_2 = \langle\!\langle\widehat{\mathrm{St}}_+\rangle\!\rangle + \langle\!\langle\widehat{\mathrm{St}}_-\rangle\!\rangle, \tag{5.76a}$$

where $\langle\!\langle\ldots\rangle\!\rangle$ denotes averaging over the fluctuating fields $K_\pm$ with propagators (5.50) and

$$\begin{aligned}\widehat{\mathrm{St}}_-(t_1,t_2;\mathbf{n},\mathbf{r}) = \frac{i}{4}\int dt_3\int\frac{d\mathbf{n}_1}{\Omega_d}K_-(t_3,\mathbf{n}_1,\mathbf{r})\times \\ \times\Bigg[\delta Q(t_1,t_2,\mathbf{n};t_3,\mathbf{n}_1;\mathbf{r})-\frac{i\pi}{2}\delta g_+(t_1,t_2;\mathbf{n},\mathbf{r})\times \\ \times\int_{t_2}^{t_1}dt_5 Q(t_5,t_5,\mathbf{n};t_3,\mathbf{n}_1;\mathbf{r})\Bigg],\end{aligned} \tag{5.76b}$$

$$\begin{aligned}\widehat{\mathrm{St}}_+(t_1,t_2;\mathbf{n}) = \frac{i}{4}\int\frac{d\mathbf{n}_2 d\mathbf{n}_3}{\Omega_d^2}\gamma\begin{pmatrix}\mathbf{n}_3\\ \mathbf{n};\mathbf{n}_2\end{pmatrix}\times \\ \times\Big[K_+(\mathbf{n}_3);\big(\delta g(\mathbf{n}_2)-\delta g(\mathbf{n})\big)\Big](t_1,t_2).\end{aligned} \tag{5.76c}$$

The commutator is defined in Eq. (5.10) and the kernel δQ is the first variation of operator (5.30) with respect to the Keldysh component of the electron Green's function,

$$\begin{aligned}\delta Q(t_1,t_2,\mathbf{n};t_3,\mathbf{n}_1;\mathbf{r}) = \frac{1}{2}\int\frac{d\mathbf{n}_2}{\Omega_d}\gamma\begin{pmatrix}\mathbf{n}_1\\ \mathbf{n};\mathbf{n}_2\end{pmatrix}\times \\ \times\Big\{\Big[\delta g_+(t_1,t_3,\mathbf{n},\mathbf{r})g(t_3,t_2,\mathbf{n}_2,\mathbf{r})+ \\ +g(t_1,t_3,\mathbf{n},\mathbf{r})\delta g_+(t_3,t_2,\mathbf{n}_2,\mathbf{r})\Big]+ \\ +(\mathbf{n}_2\leftrightarrow\mathbf{n})\Big\}.\end{aligned} \tag{5.76d}$$

Finally, the functions $\delta g_\pm$ and $\delta g = \delta g_+ + \delta g_-$ are obtained by solving Eqs. (5.36a) and (5.36b) (after transformation (5.63a)); with the help of Eq. (5.37), we have

$$\begin{aligned}\delta g_+(t_1,t_2;\mathbf{n},\mathbf{r}) = -i\int d3\int\frac{d\mathbf{n}_2 d\mathbf{n}_4}{\Omega_d^2}\,\gamma_{32}^4\times \\ \times\mathcal{L}^g(t_1,\mathbf{n},\mathbf{r};3)\Big[K_+(t_3,\mathbf{n}_4,\mathbf{r}_3)-K_+(t_3-t_{12},\mathbf{n}_4,\mathbf{r}_3)\Big]\times \\ \times\Big[g^K(t_3,t_3-t_{12};\mathbf{n}_2,\mathbf{r}_3)- \\ -g^K(t_3,t_3-t_{12};\mathbf{n}_3,\mathbf{r}_3)\Big],\end{aligned} \tag{5.77}$$

$$\begin{aligned}\delta g_-(t_1,t_2;\mathbf{n},\mathbf{r}) = -i\int d3\int dt_4\int\frac{d\mathbf{n}_4}{\Omega_d}\times \\ \times\mathcal{L}^g(t_1,\mathbf{n},\mathbf{r};3)\,K_-(t_4,\mathbf{n}_4,\mathbf{r}_3)\times \\ \times\Bigg[Q(t_3,t_3-t_{12},\mathbf{n}_3;t_4,\mathbf{n}_4;\mathbf{r}_3)- \\ -\frac{i\pi}{2}g^K(t_3,t_3-t_{12};\mathbf{n}_3,\mathbf{r}_3)\times \\ \times\int_{t_3-t_{12}}^{t_3}dt_5 Q(t_5,t_5,\mathbf{n}_3;t_4,\mathbf{n}_4;\mathbf{r}_3)\Bigg],\end{aligned} \tag{5.78}$$

where $t_{12} = t_1 - t_2$ and notation (5.2) is used7).

For future use, we note the properties (see also the discussion following Eq. (5.36))

$$\begin{aligned}\delta g_-(t,t) = \delta g_+(t,t) = 0, \\ \int d\mathbf{n}\,\delta Q(t_1,t_2,\mathbf{n};t_3,\mathbf{n}_1;\mathbf{r}) = 0,\end{aligned} \tag{5.79}$$

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7) These solutions are exact only in the absence of the electric field, because in its gauge invariant form, the operator acting on $\delta g_\pm$ is the same that appears in the left-hand side of Eq. (5.71). We could perturbatively include field-dependent corrections into our expressions, which would be of the first order in $\mathbf{E}$ for δg_- and of the second order for δg_+ (because δg_+ vanishes in equilibrium). However, as noted above, the first property in Eq. (5.79) implies that these corrections cannot contribute to the physical quantities in which we are interested, and therefore we do not include them in our calculations.

which follow from Eqs. (5.77)–(5.78) and from definitions (5.76d) and (5.27).

The canonical form of the kinetic equation is obtained by performing the time Wigner transformation (5.63b) of both sides of Eq. (5.71). It is clear from the structure of the collision integrals that this procedure leads to the appearance of Poisson brackets in the right-hand side of the kinetic equation. We choose another route, however: we prove the existence of the conservation laws before the Wigner transformation. This then allows us to argue that these Poisson brackets (in our formulation of the kinetic equation) give only small contributions, which can be neglected within the accuracy of the kinetic equation.

5.6. Conservation laws

The derivation of the conservation laws is based on the following properties of the collision integrals in the previous subsection:

$$\langle \widehat{\mathrm{St}}_\tau g\rangle_{\mathbf{n}} = 0, \tag{5.80a}$$

$$\langle \widehat{\mathrm{St}}_-\rangle_{\mathbf{n}} = 0, \tag{5.80b}$$

$$\langle \widehat{\mathrm{St}}_1^{el}\rangle_{\mathbf{n}} = 0. \tag{5.80c}$$

The physical meaning of conditions (5.80) is that the corresponding terms in the collision integral preserve the number of particles within the energy shell. Equations (5.80) follow immediately from definitions (5.8), (5.75b), (5.76b), (5.76d), and (5.79).

The two remaining contributions to the collision integral have the properties

$$\lim_{t_1\to t_2} \widehat{\mathrm{St}}_+(t_1,t_2) = 0, \tag{5.81a}$$

$$\lim_{t_1\to t_2} \widehat{\mathrm{St}}_1^{in}(t_1,t_2) = 0, \tag{5.81b}$$

and

$$\lim_{t_1\to t_2} (\partial_{t_1}-\partial_{t_2})\, \widehat{\mathrm{St}}_+(t_1,t_2) = 0, \tag{5.82a}$$

$$\begin{aligned} &\lim_{t_1\to t_2} (\partial_{t_1}-\partial_{t_2})\, \widehat{\mathrm{St}}_1^{in}(t_1,t_2) = \\ &= \frac{1}{4\pi}\int \frac{d\mathbf{n}_2 d\mathbf{n}_3}{\Omega_d^2}\,\gamma_{12}^3\Big\{ \left[\partial_t\,\mathcal{K}^K\partial_t\right]_{31}(t_1,t_1) - \\ &-\frac{i\pi}{4}\int dt_3 \left[\partial_t\mathcal{K}^R\right]_{13}(t_1,t_3)\big[g(t_1,t_3,\mathbf{n}_1)g(t_3,t_1,\mathbf{n}_2)+ \\ &+ g(t_1,t_3,\mathbf{n}_2)g(t_3,t_1,\mathbf{n}_1)\big] - (\mathbf{n}_1\to\mathbf{n}_2)\Big\}, \end{aligned} \tag{5.82b}$$

where we use notation (5.74) and the vertex is defined in Eq. (5.28). Equations (5.81a) and (5.82a) immediately follow from definition (5.76c) and condition (5.79). Derivations of Eqs. (5.81b) and (5.82b) are given in Appendix E.

Expressions (5.81) mean that while not preserving the number of particles for a given energy shell, the terms $\widehat{\mathrm{St}}_+$ and $\widehat{\mathrm{St}}_1^{in}$ preserve the total number of particles for a given direction (small-angle inelastic scattering). Equation (5.82a) means that the inelastic $\widehat{\mathrm{St}}_+$ term preserves not only the number of particles but also the energy for a given direction. Equation (5.82b) means that the $\widehat{\mathrm{St}}_1^{in}$ term does not preserve the energy for a given direction, thus describing the energy exchange between quasiparticles and electron–hole pairs discussed in Sec. 5.3.

The possibility to find the conserved energy current is based on a certain relation between Eq. (5.82b) and the collision integral for electron–hole pairs. We now turn to the discussion of this relation.

We substitute Eqs. (5.55a) and (5.58) in Eq. (5.82b) and average the result over $\mathbf{n}$; then using the analytical property

$$\mathcal{K}^R(t,t) = \int\frac{d\omega}{2\pi}\mathcal{K}^R(\omega) = 0 \tag{5.83}$$

together with $\mathcal{L}^\rho(1,2) = \bar{\mathcal{L}}^\rho(2,1)$, $\mathcal{L}^g(1,2) = \bar{\mathcal{L}}^g(2,1)$, $\mathcal{N}^{\rho,g}(1,2) = \mathcal{N}^{\rho,g}(2,1)$, and Eq. (5.38), we find

$$\begin{aligned} &\lim_{t_1\to t_2} (\partial_{t_1}-\partial_{t_2})\,\langle \widehat{\mathrm{St}}_1^{in}\rangle_{\mathbf{n}} = \\ &\quad = -\frac{i}{\pi\nu}\mathrm{Tr}_{\mathbf{n}}\left[\hat{\mathcal{L}}^g\left(\hat{\mathcal{N}}^g\widehat{\mathrm{St}}_\tau + \hat{M}\right)\right] + \\ &\quad + \frac{i}{\pi\nu}\mathrm{Tr}_{\mathbf{n}}\left[\hat{\mathcal{L}}^\rho\left(\hat{\mathcal{N}}^\rho\widehat{\mathrm{St}}_\tau + \hat{M}\right)\right], \end{aligned} \tag{5.84}$$

where $\widehat{\mathrm{St}}_\tau$ is defined in Eq. (5.8) and $\mathrm{Tr}_{\mathbf{n}}$ acts as

$$\left[\mathrm{Tr}_{\mathbf{n}}\hat{A}\right](t,\mathbf{r}) \equiv \int\frac{d\mathbf{n}}{\Omega_d}A\left(t,\mathbf{n},\mathbf{r};t,\mathbf{n},\mathbf{r}\right). \tag{5.85}$$

The corresponding traces of the collision integrals for the electron–hole pairs, Eq. (5.59c)[8], are

$$\begin{aligned} \mathrm{Tr}_{\mathbf{n}}\left[\hat{\mathcal{L}}^g\,\widehat{\mathrm{St}}^g\left\{\mathcal{N}^g,g\right\}\right] = \mathrm{Tr}_{\mathbf{n}}\Big[&\left[\hat{\mathcal{L}}^g;\widehat{\mathrm{St}}_\tau\right]\hat{\mathcal{N}}^g + \\ &+ 2\hat{\mathcal{L}}^g\left(\hat{\mathcal{N}}^g\widehat{\mathrm{St}}_\tau + \hat{M}\right)\Big], \end{aligned} \tag{5.86a}$$

$$\begin{aligned} \mathrm{Tr}_{\mathbf{n}}\left[\hat{\mathcal{L}}^\rho\,\widehat{\mathrm{St}}^\rho\left\{\mathcal{N}^\rho,g\right\}\right] = \mathrm{Tr}_{\mathbf{n}}\Big[&\left[\hat{\mathcal{L}}^\rho;\widehat{\mathrm{St}}_\tau\right]\hat{\mathcal{N}}^\rho + \\ &+ 2\hat{\mathcal{L}}^\rho\left(\hat{\mathcal{N}}^\rho\widehat{\mathrm{St}}_\tau + \hat{M}\right)\Big]. \end{aligned} \tag{5.86b}$$

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[8] The operator $\hat{M}$ is gauge-invariant and has the same form in terms of g or g^K.

Comparing Eqs. (5.84) and (5.86) and using Eqs. (5.25) and (5.54) once again, we obtain the desired relation between the collision integrals:

$$\begin{aligned}&2i\pi\nu \lim_{t_1\to t_2}\left(\partial_{t_1}-\partial_{t_2}\right)\langle \widehat{\mathrm{St}}\,\{g,\mathcal{N}^\rho,\mathcal{N}^g\}\rangle_{\mathbf{n}}+\\ &+\mathrm{Tr}_{\mathbf{n}}\left[\hat{\mathcal{L}}^\rho\,\widehat{\mathrm{St}}^\rho\,\{\mathcal{N}^\rho,g\}\right]-\mathrm{Tr}_{\mathbf{n}}\left[\hat{\mathcal{L}}^g\,\widehat{\mathrm{St}}^g\,\{\mathcal{N}^g,g\}\right]=\\ &=\mathrm{Tr}_{\mathbf{n}}\left[\left[\mathbf{v}\cdot\nabla;\hat{\mathcal{L}}^g\right]\hat{\mathcal{N}}^g-\right.\\ &\left.-i\left[\hat{H}_{e-h}(i\partial_t,-i\nabla);\hat{\mathcal{L}}^\rho\right]\hat{\mathcal{N}}^\rho\right].\end{aligned}\tag{5.87}$$

The left-hand side of Eq. (5.87) is the quantum counterpart of relation (2.3b) and Eqs. (5.80)–(5.81) are related to Eq. (2.3a); we now derive the expressions for the electric and energy currents in the spirit of our discussion in Sec. 2.1.

We begin with the conservation of electric charge. According to Eqs. (5.4) and (5.6), the charge density is given by

$$\rho(t,\mathbf{r})=-\frac{e\nu\pi}{2}\lim_{t_1\to t_2\to t}\langle g(t_1,t_2,\mathbf{n},\mathbf{r})\rangle_{\mathbf{n}}.\tag{5.88}$$

Taking the limit $t_1\to t_2\to t$ in both sides of Eq. (5.71) and using Eqs. (5.72), (5.80), and (5.81), we obtain the continuity equation

$$\partial_t\rho+\nabla\cdot\mathbf{j}=0,\tag{5.89}$$

where

$$\mathbf{j}(t,\mathbf{r})=-\frac{ev_F\nu\pi}{2}\lim_{t_1\to t_2\to t}\langle \mathbf{n}g(t_1,t_2,\mathbf{n},\mathbf{r})\rangle_{\mathbf{n}},\tag{5.90}$$

to be compared with Eqs. (2.5)–(2.7).

Having found the usual equation for the electric currents, we turn to the energy conservation. Acting with the operator $(\partial_{t_1}-\partial_{t_2})$ on both sides of Eq. (5.71) and introducing the quantities

$$\begin{aligned}u_e(t,\mathbf{r})&=-\frac{i\pi\nu}{4}\lim_{t_1\to t_2\to t}\left(\partial_{t_1}-\partial_{t_2}\right)\times\\ &\times\langle g(t_1,t_2,\mathbf{n},\mathbf{r})\rangle_{\mathbf{n}},\\ \mathbf{j}^\varepsilon_e(t,\mathbf{r})&=-\frac{i\pi\nu v_F}{4}\lim_{t_1\to t_2\to t}\left(\partial_{t_1}-\partial_{t_2}\right)\times\\ &\times\langle \mathbf{n}g(t_1,t_2,\mathbf{n},\mathbf{r})\rangle_{\mathbf{n}},\end{aligned}\tag{5.91}$$

we find

$$\begin{aligned}&\partial_t u_e+\nabla\cdot\mathbf{j}^\varepsilon_e=\mathbf{j}\cdot\mathbf{E}+\\ &+i\nu\pi\lim_{t_1\to t_2\to t}\left(\partial_{t_1}-\partial_{t_2}\right)\langle\widehat{\mathrm{St}}\,\{g,\mathcal{N}^\rho,\mathcal{N}^g\}\rangle_{\mathbf{n}}.\end{aligned}\tag{5.92}$$

The expression in the left-hand side of Eq. (5.92) has the form of a continuity equation for the energy current of electrons: the first term in the right-hand side is the Joule heat acting as an energy source. The last term in the right-hand side indicates that the electron system is open by itself, due to the energy exchange with electron–hole pairs. As we discussed in Sec. 2.1, this means that the contribution of these degrees of freedom must be taken into account in the definition of the conserved energy and energy current densities. For this, we multiply Eqs. (5.59a) and (5.59b) by $\hat{\mathcal{L}}^g$ and $\hat{\mathcal{L}}^\rho$ respectively. Using Eqs. (5.25), (5.54), and (5.59c) and taking the trace $\mathrm{Tr}_{\mathbf{n}}$ (see Eq. (5.85)) of both sides, we obtain

$$\begin{aligned}&\partial_t u_g+\nabla\cdot\mathbf{j}^\varepsilon_g-\frac{1}{2}\mathrm{Tr}_{\mathbf{n}}\left[\mathbf{v}\cdot\nabla;\hat{\mathcal{L}}^g\right]\hat{\mathcal{N}}^g=\\ &\qquad=\frac{1}{2}\mathrm{Tr}_{\mathbf{n}}\hat{\mathcal{L}}^g\,\widehat{\mathrm{St}}^g,\end{aligned}\tag{5.93a}$$

$$\begin{aligned}&\partial_t u_\rho+\nabla\cdot\mathbf{j}^\varepsilon_\rho-\frac{i}{2}\mathrm{Tr}_{\mathbf{n}}\left[\hat{H}_{e-h}(i\partial_t,-i\nabla);\hat{\mathcal{L}}^\rho\right]\hat{\mathcal{N}}^\rho=\\ &\qquad=\frac{1}{2}\mathrm{Tr}_{\mathbf{n}}\hat{\mathcal{L}}^\rho\,\widehat{\mathrm{St}}^\rho,\end{aligned}\tag{5.93b}$$

where the energy densities $u_{\rho,g}$ and currents $\mathbf{j}^\varepsilon_{\rho,g}$ are defined as

$$\begin{aligned}u_g(t,\mathbf{r})&=\frac{1}{2}\mathrm{Tr}_{\mathbf{n}}\hat{\mathcal{L}}^g\,\hat{\mathcal{N}}^g,\\ u_\rho(t,\mathbf{r})&=\frac{1}{2}\mathrm{Tr}_{\mathbf{n}}\left\{\frac{1}{1+\hat{F}};\hat{\mathcal{L}}^\rho\hat{\mathcal{N}}^\rho\right\},\end{aligned}\tag{5.93c}$$

$$\mathbf{j}^\varepsilon_g=\frac{1}{2}\mathrm{Tr}_{\mathbf{n}}\mathbf{v}\hat{\mathcal{L}}^g\,\hat{\mathcal{N}}^g,\quad \mathbf{j}^\varepsilon_\rho=\frac{1}{2}\mathrm{Tr}_{\mathbf{n}}\left\{\hat{\boldsymbol{s}}\,;\,\hat{\mathcal{L}}^\rho\hat{\mathcal{N}}^\rho\right\}.\tag{5.93d}$$

The velocity operator $\hat{\boldsymbol{s}}$ is defined in Eq. (5.65) and notation (5.60) is used.

We now add Eq. (5.93a) to Eq. (5.92) and subtract Eq. (5.93b). According to Eq. (5.87), all the collision terms and the commutators cancel, and we obtain the energy balance equation (compare with Eq. (2.8)):

$$\partial_t u_{tot}+\nabla\cdot\mathbf{j}^\varepsilon_{tot}=\mathbf{j}\cdot\mathbf{E},\tag{5.94a}$$

$$u_{tot}(t,\mathbf{r})=u_e(t,\mathbf{r})+u_\rho(t,\mathbf{r})-u_g(t,\mathbf{r}),\tag{5.94b}$$

$$\mathbf{j}^\varepsilon_{tot}(t,\mathbf{r})=\mathbf{j}^\varepsilon_e(t,\mathbf{r})+\mathbf{j}^\varepsilon_\rho(t,\mathbf{r})-\mathbf{j}^\varepsilon_g(t,\mathbf{r}).\tag{5.94c}$$

Equations (5.94), (5.91), (5.93c), and (5.93d) constitute the main result in this subsection. They define the conserved currents in terms of quantities to be found from the kinetic equations. We emphasize that the conservation laws thus found are exact (at one loop) in the sense that no approximation has been made beyond the usual Fermi-liquid theory: specifically, no gradient or harmonic expansion has been made and no time or space Poisson brackets have been neglected yet (except those suppressed by the factor q/p_F).

Within the same accuracy with which kinetic equations (5.64) were derived, performing Wigner transformations (5.61)–(5.63) of Eq. (5.90), (5.91), and (5.93d), we find

$$\mathbf{j} = ev_F\nu \int d\varepsilon\, \langle \mathbf{n} f(\varepsilon,\mathbf{n};t,\mathbf{r})\rangle_{\mathbf{n}} \qquad (5.95)$$

for the electric current density and

$$\mathbf{j}_e^\varepsilon = v_F\nu \int d\varepsilon\, \varepsilon \langle \mathbf{n} f(\varepsilon,\mathbf{n};t,\mathbf{r})\rangle_{\mathbf{n}}, \qquad (5.96a)$$

$$\mathbf{j}_\rho^\varepsilon = \int \frac{d\omega}{2\pi}\,\omega \int \frac{d^dq}{(2\pi)^d} \times \\ \times \left\langle \left\{ \hat{\boldsymbol{s}}(\omega,\mathbf{q}); \hat{\mathcal{L}}^\rho(\omega,\mathbf{q})\hat{N}^\rho(\omega,\mathbf{q};t,\mathbf{r}) \right\} \right\rangle_{\mathbf{n}}, \qquad (5.96b)$$

$$\mathbf{j}_g^\varepsilon = \int \frac{d\omega}{2\pi}\,\omega \int \frac{d^dq}{(2\pi)^d} \times \\ \times \left\langle \mathbf{v}\hat{\mathcal{L}}^g(\omega,\mathbf{q})\hat{N}^g(\omega,\mathbf{q};t,\mathbf{r}) \right\rangle_{\mathbf{n}}, \qquad (5.96c)$$

for the energy current densities, in agreement with Eqs. (3.14)–(3.15).

It remains to reduce the expressions found to the usual form of the quantum Boltzmann equation. This is the subject of the next subsection.

5.7. The quantum kinetic equation

After Wigner transformations (5.61)–(5.63b), Eq. (5.71) becomes

$$\left[\partial_t + \mathbf{v}\cdot\nabla + e\mathbf{v}\cdot\mathbf{E}\frac{\partial}{\partial\varepsilon}\right] f(\varepsilon,\mathbf{n};t,\mathbf{r}) = \\ = \widehat{\mathrm{St}}\{f, N^\rho, N^g\}. \qquad (5.97)$$

The collision integral is the sum of the inelastic and elastic parts,

$$\widehat{\mathrm{St}}\{f,N^\rho,N^g\} = \widehat{\mathrm{St}}_{in}\{f,N^\rho,N^g\} + \widehat{\mathrm{St}}_{el}\{f\}. \qquad (5.98)$$

The elastic part is obtained by adding the transform of the «bare» collision integral (the first term in the right-hand side of Eq. (5.72)) to the transforms of $\widehat{\mathrm{St}}_1^{el}$, Eq. (5.75c), and $\widehat{\mathrm{St}}_-$, Eq. (5.76b). The inelastic part is given by the transform of $\widehat{\mathrm{St}}_1^{in}$, Eq. (5.75b), plus the transform of $\widehat{\mathrm{St}}_+$, Eq. (5.76c). However, for the sake of compactness, we do not make such a distinction between elastic and inelastic contributions and, using a notation resembling that in Sec. 5.5, write the collision integral in the form

$$\widehat{\mathrm{St}}\{f,N^\rho,N^g\} = \widehat{\mathrm{St}}_\tau f + \widehat{\mathrm{St}}_1\{f,N^\rho,N^g\} + \\ + \widehat{\mathrm{St}}_-\{f\} + \widehat{\mathrm{St}}_+\{f,N^\rho,N^g\}, \qquad (5.99)$$

where the first term in the right-hand side is the transform of the «bare» collision integral and the other terms are given below.

With the elastic and inelastic parts of collision integral (5.73) kept in a single formula, the corresponding contribution is obtained by first substituting Eqs. (5.58) and then performing Wigner transformations (5.61)–(5.63) (and using their property (5.83)). We decompose the result into distinct contributions due to the two bosonic degrees of freedom:

$$\widehat{\mathrm{St}}_1(\varepsilon,\mathbf{n};t,\mathbf{r}) = \widehat{\mathrm{St}}_1^{e-\rho}(\varepsilon,\mathbf{n};t,\mathbf{r}) - \\ - \widehat{\mathrm{St}}_1^{e-g}(\varepsilon,\mathbf{n};t,\mathbf{r}). \qquad (5.100)$$

As usual, the collisions with the «ghost» particles enter with the opposite signs. In terms of combination (5.66) of distribution functions denoted by Υ and vertex (5.28), these contributions are (we suppress the spectator arguments t and $\mathbf{r}$ in both sides of the equations)

$$\widehat{\mathrm{St}}_1^{e\text{-}\rho}(\varepsilon,\mathbf{n}_1) = \frac{1}{\nu}\int\frac{d\omega}{2\pi}\frac{1}{\omega}\int\frac{d^dq}{(2\pi)^d}\int\frac{d\mathbf{n}_2 d\mathbf{n}_3 d\mathbf{n}_4}{\Omega_d^3} \times \\ \times \{\gamma_{12}^3[\mathcal{L}_{34}^\rho \Upsilon_{41;21}^\rho(\varepsilon,\omega,\mathbf{q}) + \Upsilon_{34;21}^\rho(\varepsilon,\omega,\mathbf{q})\bar{\mathcal{L}}_{41}^\rho] + \\ + \gamma_{21}^3[\mathcal{L}_{34}^\rho \Upsilon_{42;21}^\rho(\varepsilon,\omega,\mathbf{q}) + \\ + \Upsilon_{34;21}^\rho(\varepsilon,\omega,\mathbf{q})\bar{\mathcal{L}}_{42}^\rho]\}, \qquad (5.101a)$$

$$\widehat{\mathrm{St}}_1^{e\text{-}g}(\varepsilon,\mathbf{n}_1) = \frac{1}{\nu}\int\frac{d\omega}{2\pi}\frac{1}{\omega}\int\frac{d^dq}{(2\pi)^d}\int\frac{d\mathbf{n}_2 d\mathbf{n}_3 d\mathbf{n}_4}{\Omega_d^3} \times \\ \times \{\gamma_{12}^3[\mathcal{L}_{34}^g \Upsilon_{41;21}^g(\varepsilon,\omega,\mathbf{q}) + \Upsilon_{34;21}^g(\varepsilon,\omega,\mathbf{q})\bar{\mathcal{L}}_{41}^g] + \\ + \gamma_{21}^3[\mathcal{L}_{34}^\rho \Upsilon_{42;21}^g(\varepsilon,\omega,\mathbf{q}) + \\ + \Upsilon_{34;21}^g(\varepsilon,\omega,\mathbf{q})\bar{\mathcal{L}}_{42}^g]\}. \qquad (5.101b)$$

Here and below, the short notation

$$\mathcal{L}_{ij}^\alpha = \mathcal{L}^\alpha(\omega,\mathbf{q};\mathbf{n}_i,\mathbf{n}_j), \quad \alpha = g,\rho$$

is used. It is readily seen that these contributions coincide with the local electron–boson collision integral in Eq. (3.19b).

Proceeding as above, we obtain the transform of Eq. (5.76b) as

$$\widehat{\mathrm{St}}_-(\varepsilon,\mathbf{n}_1) = \widehat{\mathrm{St}}_{el}(\varepsilon,\mathbf{n}_1) + \widehat{\mathrm{St}}_{-,l}(\varepsilon,\mathbf{n}_1), \qquad (5.102a)$$

$$\widehat{\mathrm{St}}_{el} = \frac{2}{\nu}\mathrm{Re}\int\frac{d\omega}{2\pi}\,\frac{1}{\omega}\int\frac{d^dq}{(2\pi)^d}\int\frac{d\mathbf{n}_2\ldots d\mathbf{n}_6}{\Omega_d^5}\gamma_{13}^2\gamma_{46}^5\times$$
$$\times\left[\mathcal{L}^\rho-\mathcal{L}^g\right]_{52}\left[f(\varepsilon-\omega,\mathbf{n}_6)-f(\varepsilon-\omega,\mathbf{n}_4)\right]\times$$
$$\times\left[\mathcal{L}^g_{14}f(\varepsilon,\mathbf{n}_3)+\mathcal{L}^g_{34}f(\varepsilon,\mathbf{n}_1)\right],\qquad(5.102\mathrm{b})$$

$$\widehat{\mathrm{St}}_{-,l} = \frac{1}{\nu}\int\frac{d\omega}{2\pi}\,\frac{1}{\omega^2}\int\frac{d^dq}{(2\pi)^d}\int\frac{d\mathbf{n}_2\ldots d\mathbf{n}_6}{\Omega_d^5}\gamma_{13}^2\gamma_{46}^5\times$$
$$\times\Big[2f(\varepsilon,\mathbf{n}_4)-f(\varepsilon-\omega,\mathbf{n}_4)-f(\varepsilon+\omega,\mathbf{n}_4)-(\mathbf{n}_4\to\mathbf{n}_6)\Big]\times$$
$$\times\int d\varepsilon_1\Big[f(\varepsilon_1,\mathbf{n}_1)[1-f(\varepsilon_1-\omega,\mathbf{n}_3)]+(\mathbf{n}_1\leftrightarrow\mathbf{n}_3)\Big]\times$$
$$\times\left[\mathcal{L}^\rho-\mathcal{L}^g\right]_{52}\mathcal{L}^g_{14}+$$
$$+\frac{1}{2}\left[\mathcal{L}^\rho-\mathcal{L}^g\right]_{52}\left[\mathcal{L}^g_{14}-\mathcal{L}^g_{34}\right]\qquad(5.102\mathrm{c})$$

Equation (5.102b) is (the singlet part of) the elastic electron–electron collision integral, Eq. (3.20b). To obtain Eq. (5.102c) in the given form, we used the analytic properties of the propagators and changed the variable as $\varepsilon_1\to\varepsilon_1+\omega$ in some of the terms.

We finally transform Eq. (5.76c) and obtain

$$\widehat{\mathrm{St}}_+(\varepsilon,\mathbf{n}_1)=\widehat{\mathrm{St}}_n^{e\text{-}e}(\varepsilon,\mathbf{n}_1)+$$
$$+\widehat{\mathrm{St}}_{+,n}(\varepsilon,\mathbf{n}_1)+\widehat{\mathrm{St}}_{+,l}(\varepsilon,\mathbf{n}_1).\qquad(5.103\mathrm{a})$$

The first term is given by (the singlet part of) Eq. (3.20d). The second term is

$$\widehat{\mathrm{St}}_{+,n}=\widehat{\mathrm{St}}_{+,n}^{e\text{-}\rho}-\widehat{\mathrm{St}}_{+,n}^{e\text{-}g}\qquad(5.103\mathrm{b})$$

with $\widehat{\mathrm{St}}_{+,n}^{e\text{-}\alpha}$ given by Eq. (3.19d) excluding the last line. The third term is

$$\widehat{\mathrm{St}}_{+,l} = \frac{1}{2\nu}\int\frac{d\omega}{2\pi}\,\frac{1}{\omega}\int\frac{d^dq}{(2\pi)^d}\int\frac{d\mathbf{n}_2\ldots d\mathbf{n}_6}{\Omega_d^5}\gamma_{13}^2\gamma_{46}^5\times$$
$$\times\Big[2f(\varepsilon,\mathbf{n}_4)-f(\varepsilon-\omega,\mathbf{n}_4)-f(\varepsilon+\omega,\mathbf{n}_4)-(\mathbf{n}_4\to\mathbf{n}_6)\Big]\times$$
$$\times\left[\mathcal{L}^\rho\left(2N^\rho+1\right)-\mathcal{L}^g\left(2N^g+1\right)\right]_{52}\times$$
$$\times\left[\mathcal{L}^g_{34}-\mathcal{L}^g_{14}\right].\qquad(5.103\mathrm{c})$$

Adding Eqs. (5.103c) and (5.102c), we recover the last line in Eq. (3.19d) and (the singlet part of) the local electron–electron collision integral, Eq. (3.20f). This concludes the derivation of the quantum kinetic equation for the singlet channel.

5.8. The triplet channel

Inclusion of the interaction in the triplet channel is straightforward; in Eilenberger equation (5.7), we add the term

$$i\left[\hat{\boldsymbol{\phi}}\cdot\boldsymbol{\sigma};\hat{g}\right]\qquad(5.104)$$

to the left-hand side, where σ_i are the Pauli matrices ($i=x,y,z$) and the fluctuating field $\hat{\phi}_i$ is a 3-component vector in the $L=1$ angular momentum space. Therefore, all the triplet channel propagators, polarization operators, and density matrices should be considered 3×3 matrices; for example, we have

$$\left[D_0\right]_{ij}(1,2)=-\frac{F^\sigma(\theta_{12})\,\delta(\mathbf{r}_{12})}{\nu}\delta(t_{12})\delta_{ij}$$

(cf. Eq. (5.15)) and the retarded polarization operator is given by

$$\Pi^R_{ij}(1,2)=\nu\left\{\delta_{12}+\frac{\pi}{4}\mathrm{Tr}\left[\sigma_i\frac{\delta g^K(t_1,t_1,\mathbf{n}_1,\mathbf{r}_1)}{\delta\phi_{+j}(t_2,\mathbf{r}_2,\mathbf{n}_2)}\right]\right\},$$

where the trace is over spin indices.

The trace of triplet channel operators includes the sum over the indices i,j. In the absence of the magnetic field, all the operators are diagonal, e.g.,

$$\left[\mathcal{L}^\sigma\right]_{ij}=\mathcal{L}^\sigma\delta_{ij},$$

and the trace results in extra factors of 3 in comparison to the singlet channel. The derivation can therefore be repeated with simple modifications and it gives the quantum kinetic equation presented in Sec. 3. We only note one main difference in the derivation for the triplet channel: the gauge transformation, which has the form

$$\hat{g}\to e^{-i\hat{\mathbf{K}}\cdot\boldsymbol{\sigma}}\hat{g}\,e^{i\hat{\mathbf{K}}\cdot\boldsymbol{\sigma}},$$

does not commute with interaction term (5.104). Additional second-order terms arise due to commutators of the Pauli matrices; however, these terms vanish in the one-loop approximation and we can neglect them.

In the next section, we use the quantum kinetic equation to calculate the interaction corrections to the transport coefficients and specific heat.

6. DERIVATION OF TRANSPORT COEFFICIENTS AND SPECIFIC HEAT

In this section, we calculate the transport coefficients for quasi one-dimensional, two-dimensional, and three-dimensional systems; the evaluation of the interaction correction to the specific heat is in the final subsection.

To calculate the currents in the presence of an external field (electric field $\mathbf{E}$ or temperature gradient ∇T), we need to solve the kinetic equations. We assume that the external fields are weak, i.e.,

$$eEL_T\ll T,\quad \nabla TL_T\ll T,$$

with the temperature length defined in Eq. (5.53). These conditions ensure that the deviations from the equilibrium distribution functions are small and we can solve the equations by iteration.

In the lowest order, the distribution functions should turn the collision integrals in Eqs. (3.4) and (3.6) to zero, and the sought corrections $\delta f, \delta N^\alpha$ are linear in the electric field or in the gradients of the distribution functions. In other words, we seek a solution of the kinetic equations of the form

$$\begin{aligned} &f(\varepsilon, \mathbf{n}; \mathbf{r}) = f_F(\varepsilon; \mathbf{r}) + \delta f(\varepsilon, \mathbf{n}; \mathbf{r}), \\ &\hat{N}^\alpha(\omega, \mathbf{q}; \mathbf{r}) = N_P(\omega; \mathbf{r})\mathbb{1} + \delta\hat{N}^\alpha(\omega, \mathbf{q}; \mathbf{r}), \end{aligned} \tag{6.1}$$

where the Fermi and Planck distribution functions (2.2) depend on the spatial coordinate only through the temperature $T(\mathbf{r})$. For compactness, we consider only the singlet channel explicitly and indicate how to include the triplet channel.

We start from the electron part of the kinetic equation. The bare impurity collision part $\widehat{\mathrm{St}}_\tau$ is larger than the other terms, and it therefore suffices to calculate the latter in the first order of perturbation theory. Considering short-range impurities, such that τ is independent of the scattering angle, we find

$$\begin{aligned} &\delta f = \delta f_0 + \delta f_1, \\ &\delta f_0 = \tau \mathbf{v} \cdot \left(e\mathbf{E} - \frac{\varepsilon \nabla T}{T}\right)\left(-\frac{\partial f_F(\varepsilon)}{\partial\varepsilon}\right), \\ &\delta f_1 = \tau\delta\widehat{\mathrm{St}}\left\{f_F + \delta f_0, N_P + \delta N^\rho, N_P + \delta N^g\right\}, \end{aligned} \tag{6.2}$$

where $\delta\widehat{\mathrm{St}}$ is the linearized collision integral. We note that according to the discussion of the conservation laws in Sec. 3, we need to consider only the local electron–boson contribution, Eq. (3.19b), and the elastic electron–electron one, Eq. (3.20b).

Expression (6.2) is to be substituted in Eqs. (3.14) and (3.15b) to find the electric current and the electron component of the energy current. Integration of the δf_0 term is straightforward. Due to the structure of collision integrals (3.19)–(3.20), the integration over ε can be performed before the ω and $\mathbf{q}$ integrations in the δf_1 term. For the combination of distribution functions entering Eq. (3.16a), we find

$$\begin{aligned} &\delta\Upsilon_{ij,kl}(\varepsilon,\omega) = \\ &\quad = \delta N(\omega, \mathbf{q}; \mathbf{n}_i, \mathbf{n}_j)\left\{f_F(\varepsilon) - f_F(\varepsilon-\omega)\right\} + \\ &+ \Omega_d\delta(\widehat{\mathbf{n}_i\mathbf{n}_j})N_P(\omega)\left\{\delta f(\varepsilon, \mathbf{n}_k) - \delta f(\varepsilon-\omega, \mathbf{n}_k)\right\} + \\ &\quad + \Omega_d\delta(\widehat{\mathbf{n}_i\mathbf{n}_j})\Big[\delta f(\varepsilon, \mathbf{n}_l)\big(1 - f_F(\varepsilon-\omega)\big) - \\ &\qquad\qquad - f_F(\varepsilon)\delta f(\varepsilon-\omega, \mathbf{n}_k)\Big], \end{aligned} \tag{6.3}$$

where we next use the identities

$$\begin{aligned} &\int d\varepsilon\, f_F(\varepsilon)\frac{\partial f_F(\varepsilon-\omega)}{\partial\varepsilon} = \frac{\partial}{\partial\omega}\Big[\omega N_P(\omega)\Big], \\ &\int d\varepsilon \frac{\partial f_F(\varepsilon)}{\partial\varepsilon}\left[1 - f_F(\varepsilon-\omega)\right] = \frac{\partial}{\partial\omega}\Big[\omega N_P(\omega)\Big], \end{aligned} \tag{6.4a}$$

$$\begin{aligned} &\int d\varepsilon\, \varepsilon \frac{\partial f_F(\varepsilon)}{\partial\varepsilon}\left[1 - f_F(\varepsilon-\omega)\right] = \\ &\qquad = \frac{1}{2}\omega^2\frac{\partial N_P(\omega)}{\partial\omega}, \\ &\int d\varepsilon\, (\varepsilon-\omega)\frac{\partial f_F(\varepsilon-\omega)}{\partial\varepsilon} f_F(\varepsilon) = \\ &\qquad = -\frac{1}{2}\omega^2\frac{\partial N_P(\omega)}{\partial\omega}, \\ &\int d\varepsilon\, \varepsilon^2 \frac{\partial f_F(\varepsilon)}{\partial\varepsilon}\Big[1 - f_F(\varepsilon-\omega)\Big] = \\ &\qquad = \frac{\pi^2}{3}T^2\frac{\partial}{\partial\omega}\Big[\omega N_P(\omega)\Big] + \frac{1}{3}\omega^3\frac{\partial N_P(\omega)}{\partial\omega}, \\ &\int d\varepsilon\, \varepsilon(\varepsilon-\omega) f_F(\varepsilon)\frac{\partial f_F(\varepsilon-\omega)}{\partial\varepsilon} = \\ &\qquad = \frac{\pi^2}{3}T^2\frac{\partial}{\partial\omega}\Big[\omega N_P(\omega)\Big] - \frac{1}{6}\omega^3\frac{\partial N_P(\omega)}{\partial\omega} \end{aligned} \tag{6.4b}$$

to obtain

$$\begin{aligned} &\frac{1}{2}\int d\varepsilon\big[\delta\Upsilon_{ij,kl}(\varepsilon,\omega,\mathbf{q}) - \delta\Upsilon_{ij,kl}(\varepsilon,-\omega,-\mathbf{q})\big] = \\ &\qquad = e v_F \tau \mathbf{E}\cdot(\mathbf{n}_k - \mathbf{n}_l)\frac{\partial}{\partial\omega}\left(\omega N_P^{ij}\right), \end{aligned} \tag{6.5a}$$

$$\begin{aligned} &\frac{1}{2}\int d\varepsilon\, \varepsilon\big[\delta\Upsilon_{ij,kl}(\varepsilon,\omega,\mathbf{q}) - \delta\Upsilon_{ij,kl}(\varepsilon,-\omega,-\mathbf{q})\big] = \\ &\quad = -\frac{\omega^2}{2}\delta N(\omega,\mathbf{q};\mathbf{n}_i,\mathbf{n}_j) - \frac{v_F\tau\nabla T}{4T}\cdot(\mathbf{n}_l+\mathbf{n}_k)\times \\ &\qquad \times\omega^3\frac{\partial N_P^{ij}}{\partial\omega} + \frac{v_F\tau\nabla T}{T}\cdot(\mathbf{n}_l - \mathbf{n}_k)\times \\ &\qquad\quad \times\left[\frac{\pi^2T^2}{3}\frac{\partial\big(\omega N_P^{ij}\big)}{\partial\omega} + \frac{\omega^3}{12}\frac{\partial N_P^{ij}}{\partial\omega}\right], \end{aligned} \tag{6.5b}$$

where

$$N_P^{ij} \equiv \Big[\hat{N}_P\Big](\mathbf{n}_i, \mathbf{n}_j) \equiv N_P(\omega)\Omega_d\delta(\widehat{\mathbf{n}_i\mathbf{n}_j}). \tag{6.6}$$

We here retained only the contribution odd in ω because the even part vanishes after the ω integration in the relevant collision integral, see Eq. (3.19b).

The combination of the distribution functions entering the elastic collision part, Eq. (3.20b), gives

$$\begin{aligned} &\frac{1}{2}\int d\varepsilon f_F(\varepsilon)\big[\delta f(\varepsilon-\omega,\mathbf{n}) - \delta f(\varepsilon+\omega,\mathbf{n})\big] = \\ &\qquad = -e v_F\tau\mathbf{E}\cdot\mathbf{n}\frac{\partial}{\partial\omega}\Big[\omega N_P(\omega)\Big], \end{aligned} \tag{6.7a}$$

$$\frac{1}{2}\int d\varepsilon\, \varepsilon f_F(\varepsilon)\big[\delta f(\varepsilon-\omega,\mathbf{n})-\delta f(\varepsilon+\omega,\mathbf{n})\big] = \\ = \frac{v_F\tau\mathbf{n}\cdot\nabla T}{T}\times \\ \times\left[\frac{\pi^2T^2}{3}\frac{\partial\big(\omega N_P(\omega)\big)}{\partial\omega}-\frac{\omega^3}{6}\frac{\partial N_P(\omega)}{\partial\omega}\right], \quad (6.7b)$$

where we again retained only the part odd in ω, nonvanishing after the ω integration in Eq. (3.20b).

Using Eq. (3.14), we find the electric current $\mathbf{j}=\hat{\sigma}\mathbf{E}$ with the conductivity tensor $\hat{\sigma}$ given by

$$\hat{\sigma}=\sigma_D\left\{\hat{\mathbb{1}}+\int d\omega\left[\hat{\mathcal{S}}^{el}(\omega)+\hat{\mathcal{E}}(\omega)\right]\times \right.\\ \left.\times\frac{\partial}{\partial\omega}\Big[\omega N_P(\omega)\Big]\right\}, \quad (6.8)$$

where $\sigma_D = \tau v_F^2 e^2\nu/d$ is the Drude conductivity. With the spatial indices denoted by $\mu,\nu=1,\ldots,d$, the elastic kernels $\mathcal{S}^{el}_{\mu\nu}$ and $\mathcal{E}_{\mu\nu}$, which originate from Eq. (3.19b) and Eq. (3.20b) respectively, are given by

$$\mathcal{S}^{el}_{\mu\nu}(\omega)=\mathcal{S}^{11}_{\mu\nu}(\omega)+\mathcal{S}^{12}_{\mu\nu}(\omega), \\ \mathcal{S}^{11}_{\mu\nu}(\omega)=\frac{d}{\pi\omega\nu}\int\frac{d^dq}{(2\pi)^d}\int\frac{d\mathbf{n}_1 d\mathbf{n}_2}{(\Omega_d)^2}n_{1\mu}n_{1\nu}\times \\ \times\,\mathrm{Re}\big[\mathcal{L}^{\rho}_{11}-\mathcal{L}^{g}_{11}\big]+\mathrm{Re}\,[\mathcal{L}^{\rho}_{22}-\mathcal{L}^{g}_{22}]- \\ -\,2\mathrm{Re}\,[\mathcal{L}^{\rho}_{12}-\mathcal{L}^{g}_{12}], \\ \mathcal{S}^{12}_{\mu\nu}(\omega)=\frac{2d}{\pi\omega\nu}\int\frac{d^dq}{(2\pi)^d}\times \\ \times\int\frac{d\mathbf{n}_1 d\mathbf{n}_2}{(\Omega_d)^2}n_{1\mu}n_{2\nu}\mathrm{Re}[\mathcal{L}^{\rho}_{12}-\mathcal{L}^{g}_{12}], \\ \mathcal{E}_{\mu\nu}(\omega)=-\frac{d\tau}{\pi\omega\nu}\int\frac{d^dq}{(2\pi)^d}\int\frac{d\mathbf{n}_1\ldots d\mathbf{n}_6}{(\Omega_d)^6}\gamma^6_{12}\gamma^5_{43}\times \\ \times\,(n_{1\mu}n_{3\nu}-n_{2\mu}n_{3\nu})\times \\ \times\,\mathrm{Re}\Big\{[\mathcal{L}^{\rho}_{56}-\mathcal{L}^{g}_{56}][\mathcal{L}^{g}_{13}+\mathcal{L}^{g}_{14}]\Big\}, \quad (6.9)$$

where we keep only the singlet channel correction for compactness; inclusion of the triplet channel contribution is straightforward[9]. We show in Appendix F that our expression for the conductivity coincides with the one in Ref. [17]. It is natural that the conductivity does not involve any bosonic distribution function (cf. Eq. (6.5a)), because the inelastic electron collision with such bosons changes the energy of the electron but not the direction of its motion.

In contrast, even the electron contribution to the thermal conductivity tensor $\hat{\kappa}$, such that $\mathbf{j}^{\varepsilon}_{tot}=-\hat{\kappa}\nabla T$, is sensitive to the bosonic distribution functions. We represent the total thermal conductivity as

$$\hat{\kappa}=\hat{\kappa}_{WF}+\delta\hat{\kappa}+\hat{\kappa}^{\rho}-\hat{\kappa}^{g}. \quad (6.10)$$

The first term in this expression obeys the Wiedemann–Franz law with the interaction corrections to the electric conductivity included, i.e., $\hat{\kappa}_{WF}=L\hat{\sigma}T$, with the Lorentz number given by Eq. (1.1). The second term represents the (electronic) correction to the Wiedemann–Franz law due to the energy dependence of the elastic scattering and due to the inelastic electron scattering on bosons. Finally, the third and the fourth terms represent the contribution of the ρ and g bosons to the thermal transport. These additional contributions are given by

$$\delta\hat{\kappa} \;=\delta\hat{\kappa}_{el}+\delta\hat{\kappa}_{in}, \quad (6.11a)$$

$$[\delta\kappa_{el}]_{\mu\nu}= \\ =\frac{\sigma_D}{e^2T}\int d\omega\Big[\mathcal{S}^{el}_{\mu\nu}(\omega)-2\mathcal{E}_{\mu\nu}(\omega)\Big]\left[\frac{\omega^3}{12}\frac{\partial N_P}{\partial\omega}\right], \quad (6.11b)$$

$$[\delta\kappa_{in}]_{\mu\nu}=\frac{\sigma_D}{e^2T}\int d\omega\Big[\mathcal{S}^{12}_{\mu\nu}(\omega)-\mathcal{S}^{11}_{\mu\nu}(\omega)\Big]\left[\frac{\omega^3}{4}\frac{\partial N_P}{\partial\omega}\right]+ \\ +\,v_F\int\frac{d\omega}{2\pi}\,\omega\int\frac{d^dq}{(2\pi)^d}\int\frac{d\mathbf{n}_1 d\mathbf{n}_2 d\mathbf{n}_3}{(\Omega_d)^3}\,n_{1\mu}\times \\ \times\Big\{\mathrm{Re}\Big[\mathcal{L}^{\rho}_{12}\delta_\nu N^{\rho}_{23}+\mathcal{L}^{\rho}_{32}\delta_\nu N^{\rho}_{21}-\mathcal{L}^{\rho}_{12}\delta_\nu N^{\rho}_{21}\Big]- \\ -\,\mathrm{Re}\Big[\mathcal{L}^{g}_{12}\delta_\nu N^{g}_{23}+\mathcal{L}^{g}_{32}\delta_\nu N^{g}_{21}-\mathcal{L}^{g}_{12}\delta_\nu N^{g}_{21}\Big]\Big\}, \quad (6.11c)$$

$$\kappa^{\alpha}_{\mu\nu}=-\int\frac{d\omega}{2\pi}\,\omega\int\frac{d^dq}{(2\pi)^d}\times \\ \times\int\frac{d\mathbf{n}_1 d\mathbf{n}_2}{(\Omega_d)^2}\left\{\hat{s}^{\alpha}_{\mu};\mathcal{L}^{\alpha}_{12}\delta_\nu N^{\alpha}_{21}\right\}, \quad (6.12)$$

where

$$\delta_\mu N^{\alpha}_{ij}=\frac{\delta}{\delta(\nabla_\mu T)}\big[\delta N^{\alpha}(\omega,\mathbf{q};\mathbf{n}_i,\mathbf{n}_j)\big]. \quad (6.13)$$

Equations (6.8)–(6.12) are the complete expressions for the electric and thermal transport coefficients. To obtain the explicit result, we must solve Eqs. (3.6) to find the distribution functions δN^{α}. We do this by restricting ourselves to the diffusive $T\tau\ll 1$ regime, except for two-dimensional systems for which we con-

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[9] By the simple substitution $\mathcal{L}^{\rho}-\mathcal{L}^{g}\to\mathcal{L}^{\rho}+3\mathcal{L}^{\sigma}-4\mathcal{L}^{g}$ in the kernels.

sider an arbitrary temperature range[10]. Moreover, for the Coulomb interaction, we consider the unitary limit (for infrared-finite momentum integrals), which enables us to drop all the terms that depend on $\partial_{\mathbf{q}}\hat{F}$.

From now on, we retain only the zeroth harmonic of the Fermi-liquid constants, for which we use the notation F_0^α. For the singlet channel, this means

$$F_0^\rho \equiv \int \frac{d\theta}{2\pi} F^\rho(\theta) + \nu V(\mathbf{q}) \tag{6.14a}$$

and for the triplet channel,

$$F_0^\sigma \equiv \int \frac{d\theta}{2\pi} F^\sigma(\theta). \tag{6.14b}$$

We recall that $F^g = 0$ and that the Coulomb interaction potential is given by

$$V(\mathbf{q}) = \begin{cases} \dfrac{4\pi e^2}{q^2}, & d=3, \\ \dfrac{2\pi e^2}{q}, & d=2, \\ e^2 \ln \dfrac{1}{(qa)^2}, & d=1, \end{cases} \tag{6.14c}$$

where a is a length of the order of the quasi one-dimensional wire width.

6.1. Diffusive regime

We first consider the distribution function N^g; substituting expression (6.1) in Eq. (3.6), in the linear order in ∇T, we obtain

$$\begin{aligned} -\frac{\omega}{T}\mathbf{v}_1\cdot\nabla T\frac{\partial N_P^{12}(\omega)}{\partial\omega} + i\Big[\mathbf{v}\cdot\mathbf{q};\delta N^g\Big] = \\ = 2\Big\{\widehat{\mathrm{St}}_\tau;\delta N^g\Big\} + v_F\frac{\omega}{T}\frac{\partial N_P(\omega)}{\partial\omega}\times \\ \times \nabla T\cdot\big(\Omega_d\delta(\widehat{\mathbf{n}_1,\mathbf{n}_2})\mathbf{n}_1 - \mathbf{n}_1 - \mathbf{n}_2\big), \end{aligned} \tag{6.15}$$

with N_P^{ij} defined in Eq. (6.6). The (exact at this order) solution for $\delta N^g(\omega;\mathbf{n}_1,\mathbf{n}_2)$ is

$$\delta N^g = \delta N^0 \equiv v_F\tau\mathbf{n}_1\cdot\nabla T\frac{\omega}{T}\frac{\partial N_P^{12}(\omega)}{\partial\omega}. \tag{6.16}$$

For the distribution function N^ρ, the above is only the starting point for the iterative solution:

$$\delta N^\rho = \delta N^0 + \delta N^1. \tag{6.17}$$

The equation for δN^1 is

$$\begin{aligned} \left(-\frac{\omega}{T}\frac{\partial\hat{N}_P(\omega)}{\partial\omega}\nabla T\right)\cdot\left(\omega\frac{\partial}{\partial\mathbf{q}}\frac{\hat{F}^\rho}{1+\hat{F}^\rho}\right) + \\ + i\left[\omega\frac{\hat{F}^\rho}{1+\hat{F}^\rho};\delta\hat{N}^g\right] + i\Big[\hat{H}_{e-h};\delta\hat{N}^1\Big] = \\ = 2\Big\{\widehat{\mathrm{St}}_\tau;\delta\hat{N}^1\Big\}. \end{aligned} \tag{6.18}$$

In the diffusive limit $T\tau \ll 1$, the (first-iteration) solution would be of the form

$$\delta N^1 \approx \omega\tau\left(v_F\tau\frac{\omega}{T}\frac{\partial N_P(\omega)}{\partial\omega}\nabla T\right)\cdot\mathbf{V}$$

for a vector $\mathbf{V}$ with a magnitude of the order one. However, contributions from frequencies ω larger than the temperature T are exponentially suppressed, i.e., $\omega\tau \lesssim T\tau \ll 1$; therefore, δN^1 can be neglected in comparison to δN^g. Thus, in the diffusive limit,

$$\delta_\mu N_{ij}^\alpha = v_F\tau[\mathbf{n}_i]_\mu\frac{\omega}{T}\frac{\partial N_P^{ij}(\omega)}{\partial\omega}. \tag{6.19}$$

For the propagators $\mathcal{L}^\alpha$ in Eq. (3.11), the diffusive approximation amounts to the substitution

$$n_\mu n_\nu \to \frac{\delta_{\mu\nu}}{d},$$

which leads to the expression

$$\begin{aligned} \mathcal{L}^\alpha(\mathbf{n}_1,\mathbf{n}_2) = \tau\big(\delta(\widehat{\mathbf{n}_1,\mathbf{n}_2}) - 1\big) + L_0^\alpha + \\ + (n_1+n_2)_\mu L_{1\mu}^\alpha + n_{1\mu}n_{2\nu}L_{2\mu\nu}^\alpha, \end{aligned} \tag{6.20}$$

where the functions L_i^α depend on ω, $\mathbf{q}$ only and are explicitly given by

$$L_0^\alpha = \frac{1}{\dfrac{-i\omega}{1+F_0^\alpha} + Dq^2}, \tag{6.21a}$$

$$L_{1\mu}^\alpha = -i\tau v_F q_\mu L_0^\alpha, \tag{6.21b}$$

$$L_{2\mu\nu}^\alpha = -d\tau D q_\mu q_\nu L_0^\alpha. \tag{6.21c}$$

Here, $D = \tau v_F^2/d$ is the diffusion constant. These formulas are valid whenever ω, $Dq^2 \ll 1/\tau$.

Within this approximation, we have

$$\mathcal{S}_{\mu\nu}^{12}(\omega) \pm \mathcal{S}_{\mu\nu}^{11}(\omega) \propto L_{2\mu\nu}^\rho - L_{2\mu\nu}^g.$$

This means that in both $\hat{\sigma}$ and $\delta\hat{\kappa}$ (see Eqs. (6.8) and (6.11)), we can neglect the contributions of the

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[10] The Boltzmann equation description of strictly one-dimensional systems is not applicable and considering the quasi-one-dimensional ballistic case within our scheme is meaningless because of the effects of boundary scattering. The ballistic regime in three dimensions also cannot be considered within our scheme because the main effect on the thermal conductivity is due to the inelastic scattering processes with momentum transfer of the order of k_F.

$\hat{\mathcal{S}}$ kernels (we note that by inserting solution (6.19) in Eq. (6.11c), $\delta\hat{\kappa}_{in}$ is given by twice the first line of that equation). Indeed, the leading contribution is given by the kernel $\hat{\mathcal{E}}$ in Eq. (6.11b). This kernel has the approximate form

$$\mathcal{E}_{\mu\nu}(\omega) = \frac{4}{\pi\nu\tau}\,\frac{1}{d\omega}\times \\ \times\int\frac{d^dq}{(2\pi)^d}\mathrm{Re}\left[L_0^g\left(L^{\rho}_{2\mu\nu}-L^{g}_{2\mu\nu}\right)\right]. \tag{6.22}$$

Finally, the bosonic contributions (cf. Eq. (6.12)) to thermal conductivity (6.10) can be written as[11]

$$\hat{\kappa}^{\rho}-\hat{\kappa}^{g} = \frac{\sigma_D}{e^2T}\int d\omega\,\hat{\mathcal{B}}(\omega)\left[\frac{\omega^3}{4}\,\frac{\partial N_P}{\partial\omega}\right] \tag{6.23}$$

with

$$\mathcal{B}_{\mu\nu}(\omega) = -\frac{2\delta_{\mu\nu}}{\pi\omega\nu}\int\frac{d^dq}{(2\pi)^d}\mathrm{Re}\left[L_0^{\rho}-L_0^{g}\right]. \tag{6.24}$$

The next step is to evaluate the momentum integrals; we first give the results for the short-range interaction described by (the zeroth harmonic of) the Fermi-liquid constant F_0, and then we indicate the modifications needed to account for the long-range part of the Coulomb interaction in the singlet channel. The triplet channel contributions are obtained by multiplying the obtained results by three and identifying F_0 with F_0^{σ}.

For the elastic kernel, we find

$$\mathcal{E}_{\mu\nu}(\omega) = \frac{\delta_{\mu\nu}}{d}\,\frac{e^2}{\sigma_D}\,\frac{\Omega_d}{(2\pi)^d}\left(\frac{|\omega|}{D}\right)^{d/2-1}\frac{1}{\omega}\,\frac{1}{\cos\dfrac{\pi d}{4}}\times \\ \times\left[\frac{d}{2}-\frac{1}{F_0}\left(1+F_0-(1+F_0)^{1-d/2}\right)\right]. \tag{6.25}$$

The expression for the Coulomb interaction is obtained by taking the unitary limit $F_0\to+\infty$.

The result for the bosonic kernel is

$$\mathcal{B}_{\mu\nu}(\omega) = \frac{\delta_{\mu\nu}}{2}\,\frac{e^2}{\sigma_D}\,\frac{\Omega_d}{(2\pi)^d}\left(\frac{|\omega|}{D}\right)^{d/2-1}\frac{1}{\omega}\times \\ \times\frac{1}{\cos\dfrac{\pi d}{4}}\left[1-(1+F_0)^{1-d/2}\right]. \tag{6.26}$$

For $d=3$, the limit $F_0\to+\infty$ gives the correct formula for the long-range contribution, but for $d=1$, the limit diverges. But with the full form of the interaction potential retained, this infrared divergence is cut off at the inverse screening radius. With logarithmic accuracy, the result for the Coulomb interaction is found by substituting $[\ldots]\to -ak\ln^{1/2}(Dk^2/|\omega|)$, where a is a length of the order of the wire width and $k^2=4\pi e^2\nu$ is the square of the inverse screening radius (in the bulk).

We can now proceed with the calculation of the integrals over ω in Eqs. (6.8), (6.11), and (6.23) using the identity

$$\int d\omega\,\omega^m\partial_\omega N_P(\omega) = -2T^m\zeta(m)\Gamma(m+1). \tag{6.27}$$

Here, $\zeta(x)$ is the Riemann zeta function, whose values at the points relevant for our discussion are

$$\zeta\left(-\frac{1}{2}\right)\approx-0.208,\quad \zeta(0)=-\frac{1}{2},\quad \zeta\left(\frac{1}{2}\right)\approx-1.460,$$

$$\zeta\left(\frac{3}{2}\right)\approx 2.612,\quad \zeta(2)=\frac{\pi^2}{6}\approx 1.645,\quad \zeta\left(\frac{5}{2}\right)\approx 1.341,$$

and $\Gamma(x)$ is the Euler gamma function, with the values

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi},\quad \Gamma(1)=1,\quad \Gamma\left(\frac{3}{2}\right)=\frac{1}{2}\sqrt{\pi},$$

$$\Gamma\left(\frac{5}{2}\right)=\frac{3}{4}\sqrt{\pi},\quad \Gamma(3)=2,\quad \Gamma\left(\frac{7}{2}\right)=\frac{15}{8}\sqrt{\pi}.$$

Performing the final ω integrations, we obtain

$$\sigma = \sigma_D + e^2\frac{\Omega_d}{(2\pi)^d}\,\frac{2}{d}\zeta\left(\frac{d}{2}-1\right)\left(\frac{T}{D}\right)^{d/2-1}\times \\ \times\Gamma\left(\frac{d}{2}\right)\frac{d-4}{2-d}\,\frac{1}{\cos\dfrac{\pi d}{4}}\times \\ \times\left[\frac{d}{2}-\frac{1}{F_0}\left(1+F_0-(1+F_0)^{1-d/2}\right)\right], \tag{6.28}$$

$$\delta\kappa = \frac{1}{3d}\,\frac{\Omega_d}{(2\pi)^d}\left(\frac{T}{D}\right)^{d/2}D\,\zeta\left(\frac{d}{2}+1\right)\times \\ \times\Gamma\left(\frac{d}{2}+2\right)\frac{1}{\cos\dfrac{\pi d}{4}}\times \\ \times\left[\frac{d}{2}-\frac{1}{F_0}\left(1+F_0-(1+F_0)^{1-d/2}\right)\right], \tag{6.29}$$

$$\kappa^{\rho}-\kappa^{g} = -\frac{1}{4}\,\frac{\Omega_d}{(2\pi)^d}\left(\frac{T}{D}\right)^{d/2}D\,\zeta\left(\frac{d}{2}+1\right)\times \\ \times\Gamma\left(\frac{d}{2}+2\right)\frac{1}{\cos\dfrac{\pi d}{4}}\left[1-(1+F_0)^{1-d/2}\right], \tag{6.30}$$

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[11] We choose to collect the bosonic contributions into a single kernel such that the resulting momentum integral is convergent.

where in the absence of the magnetic field, $\sigma_{\mu\nu} = \sigma\delta_{\mu\nu}$, and a similar relation holds for the thermal conductivity. According to the previous discussion, for the Coulomb interaction, the correct expressions are given by the limit $F_0 \to +\infty$, with the exception of the term $\kappa^\rho - \kappa^g$ in the case where $d = 1$ for which, with logarithmic accuracy, the result is obtained by substituting $[\ldots] \to -ak\ln^{1/2}(Dk^2/T)$. The final answers for the corrections to the thermal conductivity are given in Eqs.(4.2)–(4.3).

We note that for $d = 3$, the ω integration in $\hat\sigma$ and $\hat\kappa_{WF}$ is actually ultraviolet-divergent. This divergence can be incorporated as a renormalization into the Drude results; this renormalization, however, does not invalidate the Wiedemann – Franz law.

6.2. Two-dimensional system

To evaluate the interaction corrections for the entire temperature range, we need the exact form of the propagators. In two dimensions, they are given by

$$\mathcal{L}^\alpha(\omega,\mathbf{q};\mathbf{n}_1,\mathbf{n}_2) = \Omega_d\delta(\widehat{\mathbf{n}_1,\mathbf{n}_2})\mathcal{L}_0(\omega,\mathbf{q};\mathbf{n}_1) + \\ + \mathcal{L}_0(\omega,\mathbf{q};\mathbf{n}_1)\mathcal{L}_0(\omega,\mathbf{q};\mathbf{n}_2)\times \\ \times \frac{\left(-i\omega\dfrac{F_0^\alpha}{1+F_0^\alpha}+\dfrac{1}{\tau}\right)\mathcal{C}}{\mathcal{C}-\left(-i\omega\dfrac{F_0^\alpha}{1+F_0^\alpha}+\dfrac{1}{\tau}\right)}, \quad (6.31)$$

where

$$\mathcal{L}_0(\omega,\mathbf{q};\mathbf{n}) = \frac{1}{-i\omega + i\mathbf{v}\cdot\mathbf{q} + 1/\tau}, \\ \mathcal{C} = \sqrt{(-i\omega+1/\tau)^2 + (v_F q)^2}. \quad (6.32)$$

We note that the variables ω and $v_F q$ are now bounded only by the Fermi energy E_F.

As before, we need to find the nonequilibrium corrections $\delta N^{\rho,g}$ to the bosonic distribution functions. These are again given by Eqs. (6.16), (6.17), but to obtain the thermal conductivity in an arbitrary temperature range, we calculate the solution of Eq. (6.18) exactly (in the linear order in ∇T):

$$\delta N^1 = \tilde{N}\left\{\frac{\mathbf{q}\cdot\nabla T}{v_F q^2} + \frac{i}{iv_F\mathbf{q}\cdot(\mathbf{n}_1-\mathbf{n}_2)+2/\tau}\times \right. \\ \left. \times\left[\bar\lambda(\mathbf{n}_1)\mathbf{n}_1^\perp - \lambda(\mathbf{n}_2)\mathbf{n}_2^\perp\right]\cdot\nabla T\right\}, \quad (6.33)$$

where the bar denotes complex conjugation and we introduce the quantities

$$\tilde{N} = v_F\tau\frac{\omega^2}{T}\frac{\partial N_P}{\partial\omega}\frac{F_0}{1+F_0}, \\ \mathbf{n}^\perp = \mathbf{n} - \frac{(\mathbf{n}\cdot\mathbf{q})\mathbf{q}}{q^2}, \\ \lambda(\mathbf{n}) = \frac{a(\mathbf{n})}{a(\mathbf{n})-b}, \quad b = \frac{-i\omega F_0}{1+F_0} + \frac{1}{\tau}, \\ a(\mathbf{n}) = \sqrt{(v_F q)^2 + \left(-i\mathbf{q}\cdot\mathbf{v} + \frac{2}{\tau}\right)^2}. \quad (6.34)$$

We do not calculate the corrections to the electric conductivity, which would reproduce the results in Ref. [17], as shown in Appendix F (see also Ref. [18] for the generalization to arbitrary disorder). For convenience in the calculations, we separate the contributions due to δN^0 and δN^1 in $\delta\hat\kappa_{in}$ and $\hat\kappa^\rho$ (cf. Eqs. (6.11c) and (6.23)):

$$\delta\hat\kappa_{in} = \delta\hat\kappa_{in}^0 + \delta\hat\kappa_{in}^1, \quad (6.35a)$$

$$\delta\hat\kappa_{in}^0 = \frac{2\sigma_D}{e^2T}\int d\omega\left[\hat{\mathcal{S}}^{12}(\omega) - \hat{\mathcal{S}}^{11}(\omega)\right]\times \\ \times\left[\frac{\omega^3}{4}\frac{\partial N_P}{\partial\omega}\right], \quad (6.35b)$$

$$\left[\delta\kappa_{in}^1\right]_{\mu\nu} = v_F\int\frac{d\omega}{2\pi}\,\omega\int\frac{d^2q}{(2\pi)^2}\int\frac{d\mathbf{n}_1d\mathbf{n}_2d\mathbf{n}_3}{(\Omega_2)^3}\times \\ \times n_{1\mu}\mathrm{Re}\Big\{\mathcal{L}_{12}^\rho\delta_\nu N_{23}^1 + \mathcal{L}_{32}^\rho\delta_\nu N_{21}^1 - \\ - \mathcal{L}_{12}^\rho\delta_\nu N_{21}^1\Big\}, \quad (6.35c)$$

$$\hat\kappa^\rho = \hat\kappa_0^\rho + \hat\kappa_1^\rho, \quad (6.36a)$$

$$\hat\kappa_0^\rho - \hat\kappa^g = \frac{\sigma_D}{e^2T}\int d\omega\,\hat{\mathcal{B}}^0(\omega)\left[\frac{\omega^3}{4}\frac{\partial N_P}{\partial\omega}\right], \quad (6.36b)$$

$$\hat\kappa_1^\rho = -\int\frac{d\omega}{2\pi}\,\omega\int\frac{d^2q}{(2\pi)^2}\int\frac{d\mathbf{n}_1d\mathbf{n}_2}{(\Omega_2)^2}\times \\ \times\left\{\hat s_\mu;\mathcal{L}_{12}^\rho\delta_\nu N_{21}^1\right\}, \quad (6.36c)$$

where, as in Eq. (6.13), we use $\delta_\nu N^1$ to denote the variational derivative with respect to the temperature gradient and

$$\mathcal{B}_{\mu\nu}^0(\omega) = -\frac{4}{\pi\omega\nu}\int\frac{d^2q}{(2\pi)^2}\times \\ \times\int\frac{d\mathbf{n}_1}{\Omega_2}n_{1\mu}n_{1\nu}\mathrm{Re}\left[\mathcal{L}_{11}^\rho - \mathcal{L}_{11}^g\right]. \quad (6.36d)$$

Expressions (6.8) for the electric conductivity and (6.11b) for the elastic correction to the thermal conductivity and definition (6.9) for the kernels remain unchanged. The momentum and angular integrals in (6.9) can be calculated exactly; for the singlet channel in the unitary limit, we find

$$\mathcal{E}_{\mu\nu}(\omega) = -\frac{e^2\delta_{\mu\nu}}{\sigma_D 2\pi^2}\,\frac{1}{2\omega}\left[2 - 2\omega\tau H(\omega\tau)\operatorname{arctg}\frac{1}{\omega\tau} + \right.$$

$$+ \frac{(\omega\tau)^2}{2}\left(\frac{1}{2} - H(\omega\tau)\right)\ln\left(1 + \frac{1}{(\omega\tau)^2}\right) -$$

$$\left. - (\omega\tau)^2 H(\omega\tau)\ln 2\right], \tag{6.37a}$$

$$\mathcal{S}^{11}_{\mu\nu}(\omega) = -\frac{e^2\delta_{\mu\nu}}{\sigma_D 2\pi^2}\,\tau\,\frac{\pi}{2}\operatorname{sign}\omega, \tag{6.37b}$$

$$\mathcal{S}^{12}_{\mu\nu}(\omega) = -\frac{e^2\delta_{\mu\nu}}{\sigma_D 2\pi^2}\,\tau\left\{\left(2H(\omega\tau) - 1\right)\operatorname{arctg}\frac{1}{\omega\tau} + \right.$$

$$+ \frac{\pi}{2}\operatorname{sign}\omega + \omega\tau H(\omega\tau)\,\times$$

$$\left.\times\left[\frac{1}{2}\ln\left(1 + \frac{1}{(\omega\tau)^2}\right) + \ln 2\right]\right\} \tag{6.37c}$$

with the function H defined as

$$H(x) = \frac{1}{4 + x^2}. \tag{6.37d}$$

To perform the momentum integral in Eq. (6.36d), we must keep the full form of the propagator in order to avoid the infrared divergence that we would obtain in the unitary limit,

$$\mathcal{B}^0_{\mu\nu}(\omega) = -\frac{e^2\delta_{\mu\nu}}{\sigma_D 2\pi^2}\,\frac{1}{\omega}\left\{\ln\left(\frac{Dk^2}{|\omega|}\right) + \pi|\omega|\tau - \right.$$

$$\left. - \frac{1}{2}\ln\left[1 + (\omega\tau)^2\right] - \omega\tau\operatorname{arctg}\left(\frac{1}{\omega\tau}\right)\right\}, \tag{6.37e}$$

where $k = 2\pi e^2\nu$ is the inverse screening radius. Next, we calculate the angular integrals in Eqs. (6.35c) and (6.36c) as well as the angular part of the momentum integrals, with the result

$$\int\frac{d\theta_q}{2\pi}\int\frac{d\mathbf{n}_1 d\mathbf{n}_2 d\mathbf{n}_3}{(\Omega_2)^3}\,\times$$

$$\times\, n_{1\mu}\mathrm{Re}\left\{\mathcal{L}^{\rho}_{12}\delta_\nu N^1_{23} + \mathcal{L}^{\rho}_{32}\delta_\nu N^1_{21}\right\} =$$

$$= \frac{i}{4}\tilde{N}\delta_{\mu\nu}\left\{\frac{1}{(vq)^2}\,\frac{1}{\mathcal{C}\bar{\mathcal{C}}}\left[\frac{\mathcal{C}\bar{b}'}{\mathcal{C} - b} + \frac{\bar{\mathcal{C}}b'}{\bar{\mathcal{C}} - \bar{b}}\right]\times\right.$$

$$\times\left[\bar{\mathcal{C}} - \bar{b}' - \mathcal{C} + b'\right] + \frac{1}{(vq)^2}\left[\frac{\mathcal{C} - b'}{\mathcal{C}} - \frac{\bar{\mathcal{C}} - \bar{b}'}{\bar{\mathcal{C}}}\right] -$$

$$\left. - \frac{1}{\mathcal{C}\bar{\mathcal{C}}}\left[\frac{\mathcal{C}}{\mathcal{C} - b} - \frac{\bar{\mathcal{C}}}{\bar{\mathcal{C}} - \bar{b}}\right]\right\}, \tag{6.38a}$$

$$\int\frac{d\theta_q}{2\pi}\int\frac{d\mathbf{n}_1 d\mathbf{n}_2}{(\Omega_2)^2}\,n_{1\mu}\mathrm{Re}\left\{\mathcal{L}^{\rho}_{12}\delta_\nu N^1_{21}\right\} =$$

$$= \frac{i}{4}\tilde{N}\delta_{\mu\nu}\frac{\tau}{2}\left\{\left[\frac{\bar{\mathcal{C}}}{\bar{\mathcal{C}} - \bar{b}} - \frac{\mathcal{C}}{\mathcal{C} - b}\right]\left[\frac{1}{\mathcal{C}} + \frac{1}{\bar{\mathcal{C}}}\right] - \right.$$

$$\left. - \frac{1}{(vq)^2}\left[\frac{\mathcal{C}\bar{b}'}{\mathcal{C}-b} + \frac{\bar{\mathcal{C}}b'}{\bar{\mathcal{C}}-\bar{b}}\right]\left[\frac{\mathcal{C}-b'}{\mathcal{C}} - \frac{\bar{\mathcal{C}}-\bar{b}'}{\bar{\mathcal{C}}}\right]\right\}. \tag{6.38b}$$

The function $\mathcal{C}$ is defined in Eq. (6.32) and b' is given by the $F_0 \to +\infty$ limit of b in Eq. (6.34), where $\tilde{N}$ is also defined. The remaining integrals over the magnitude of the momentum can be evaluated approximately; the result can be written as

$$\delta\hat{\kappa}^1_{in} + \hat{\kappa}^{\rho}_1 = \frac{\sigma_D}{e^2 T}\int d\omega\,\hat{\mathcal{B}}^1(\omega)\left[\frac{\omega^3}{4}\,\frac{\partial N_P}{\partial\omega}\right], \tag{6.39a}$$

with

$$\mathcal{B}^1_{\mu\nu}(\omega) = \frac{e^2\delta_{\mu\nu}}{\sigma_D 2\pi^2}\,\tau\left\{\frac{2\omega\tau}{1 + (\omega\tau)^2}\ln\left(\frac{v_F k}{2|\omega|}\right) - \right.$$

$$\left. - \operatorname{arctg}\omega\tau - \omega\tau\ln\left(\frac{E_F}{\sqrt{\omega^2 + \tau^{-2}}}\right)\right\}. \tag{6.39b}$$

In the above kernel, the first term in the curly brackets originates from $\hat{\kappa}^{\rho}_1$ only: as discussed in Sec. 5.2, no long-range terms can be present in the electron contribution to the thermal conductivity. We note that the second term in the above expression is beyond the logarithmic accuracy of our approximate calculation and must be dropped. Similarly, most of the terms in the other kernels can be neglected, and collecting the logarithmic contributions, we obtain

$$\Delta\kappa_s = -\frac{1}{2\pi^2 T}\int d\omega\left[\frac{\omega^2}{4}\,\frac{\partial N_P}{\partial\omega}\right]\times$$

$$\times\left\{(\omega\tau)^2\ln\left(\frac{E_F}{|\omega|}\right) + \frac{2}{1 + (\omega\tau)^2}\ln\left(\frac{v_F k}{|\omega|}\right) + \right.$$

$$+ \left[\frac{7}{12}(\omega\tau)^2 - \frac{5}{6} + \frac{16}{3}\,\frac{1}{4 + (\omega\tau)^2}\right]\times$$

$$\left.\times\ln\left(1 + \frac{1}{(\omega\tau)^2}\right)\right\}, \tag{6.40}$$

where we define the singlet and triplet channel corrections as

$$[\kappa - \kappa_{WF}]_{\mu\nu} = (\Delta\kappa_s + 3\Delta\kappa_t)\,\delta_{\mu\nu}.$$

The final integration can now be performed within the logarithmic accuracy; we find (cf. Eq. (4.4b))

$$\Delta\kappa_s = -\frac{\pi^2}{15}T(T\tau)^2\ln\left(\frac{E_F}{T}\right) + \frac{T}{6}g_1(2\pi T\tau)\times \\ \times\ln\left(\frac{v_F k}{T}\right) - \frac{T}{24}g_2(\pi T\tau)\ln\left(1+\frac{1}{(T\tau)^2}\right), \quad (6.41)$$

where

$$g_1(x) = \frac{3}{x^2}\left\{\frac{1}{x}\left[2\psi'\left(\frac{1}{x}\right) - x^2\right] - 2\right\}, \quad (6.42a)$$

$$g_2(x) = \frac{14}{15}x^2 + \frac{8}{3}g_1(x) - \frac{5}{3}, \quad (6.42b)$$

and ψ' is the derivative of the digamma function. Because the asymptotic behavior of $g_1(x)$ is

$$g_1(x) = \begin{cases} 1 - \dfrac{1}{5}x^2 + \dfrac{1}{7}x^4 + \ldots, & x \ll 1, \\ \dfrac{3}{x} - \dfrac{6}{x^2} + \dfrac{\pi^2}{x^3} + \ldots, & x \gg 1, \end{cases} \quad (6.42c)$$

both these functions tend to 1 as $T\tau \to 0$; therefore, in the diffusive limit, the main contribution is $T\ln(Dk^2/T)/12$. On the other hand, for $T\tau \gg 1$, the first term in Eq. (6.41) is the dominant one.

Turning to the triplet channel, we restrict ourselves to the limiting diffusive and quasiballistic cases for simplicity, although one can extend the calculation to the entire temperature range, as is done in Ref. [17] for the electric conductivity.

In the diffusive limit $T\tau \ll 1$, we know from our previous analysis that we can discard the $\mathcal{B}^1$ term as well as the $\mathcal{S}$ terms. The relevant kernels are then[12]

$$\mathcal{E}_{\mu\nu}(\omega) = -\frac{e^2\delta_{\mu\nu}}{\sigma_D 2\pi^2\omega}\left[1 - \frac{1}{F_0^\sigma}\ln\left(1+F_0^\sigma\right)\right], \quad (6.43)$$

$$\mathcal{B}^0_{\mu\nu}(\omega) = -\frac{e^2\delta_{\mu\nu}}{\sigma_D 2\pi^2\omega}\ln\left(1+F_0^\sigma\right). \quad (6.44)$$

Their substitution in Eqs. (6.11)–(6.36b) gives

$$\Delta\kappa_t = -\frac{T}{18}\left[1 - \frac{1}{F_0^\sigma}\ln\left(1+F_0^\sigma\right)\right] + \\ + \frac{T}{12}\ln\left(1+F_0^\sigma\right). \quad (6.45)$$

In the opposite limit $T\tau \gg 1$, the main contribution comes, as for the singlet channel, from the logarithmic divergence at large momenta in the kernel $\mathcal{B}^1$:

$$\mathcal{B}^1_{\mu\nu}(\omega) = -\frac{e^2\delta_{\mu\nu}}{\sigma_D 2\pi^2}\times \\ \times\tau\left\{\omega\tau\ln\left(\frac{E_F}{\sqrt{\omega^2+\tau^{-2}}}\right)\right\}\left(\frac{F_0^\sigma}{1+F_0^\sigma}\right)^2. \quad (6.46)$$

Then the correction to the thermal conductivity is

$$\Delta\kappa_t = -\frac{\pi^2}{15}T(T\tau)^2\ln\left(\frac{E_F}{T}\right)\left(\frac{F_0^\sigma}{1+F_0^\sigma}\right)^2, \quad (6.47)$$

which concludes the derivation of Eq. (4.4c).

This correction to the thermal conductivity (and the corresponding one in the singlet channel) is the contribution of inelastic processes to the energy relaxation rate[13]. In a clean system, such inelastic processes cannot relax momentum (because of the Galilean invariance), and hence they do not affect the electric conductivity, but they can contribute to the energy relaxation rate Γ_ε. In the kinetic theory, the thermal conductivity can be written, up to a numerical coefficient, as

$$\kappa \propto TE_F/\Gamma_\varepsilon,$$

and the rate is given by the sum of the rates for the relevant processes, namely the electron–impurity and electron–electron scattering rates,

$$\Gamma_\varepsilon = \Gamma_{imp} + \Gamma_{e\text{-}e}$$

with $\Gamma_{imp} = 1/\tau$ and

$$\Gamma_{e\text{-}e} = a\frac{T^2}{E_F}\ln\left(\frac{E_F}{T}\right).$$

Here, a is a constant whose exact value is irrelevant for our argument. In the limit $(T^2/E_F)\ln(E_F/T) \ll 1/\tau$, we can expand the expression for the total rate, substitute the result into the above formula for κ and obtain

$$\kappa \propto T\tau E_F - aT(T\tau)^2\ln\left(\frac{E_F}{T}\right).$$

The first term in the right-hand side is the usual Drude result for the thermal conductivity and the second term has the form of correction (6.47). We note that in the opposite limit (clean system), the result is

$$\kappa \propto \frac{E_F^2}{T\ln\left(\dfrac{E_F}{T}\right)},$$

in agreement with the result in Ref. [30].

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[12] They can be obtained from Eqs. (6.25), (6.26) in the limit $d \to 2$.

[13] A similar argument is presented in Ref. [21].

6.3. Specific heat

Recalling our discussion on the structure of the kinetic equation in Sec. 2, we write the total specific heat as the sum of the electronic and bosonic contributions,

$$c_V = c_V^0 + \delta c_V, \tag{6.48a}$$

$$c_V^0 = \frac{\partial u_e}{\partial T} = \frac{\pi^2}{3}\nu T, \tag{6.48b}$$

$$\delta c_V = \frac{\partial}{\partial T}\Big(u_\rho - u_g\Big), \tag{6.48c}$$

where, in accordance with Eq. (5.93c), the bosonic energy densities (in the equilibrium (3.1)) are

$$u_\alpha = \int d\omega\, \omega\, b^\alpha(\omega) N_P(\omega) \tag{6.49}$$

with[14]

$$b^\alpha(\omega) = \frac{\mathrm{Re}}{2\pi}\int \frac{d^dq}{(2\pi)^d}\mathrm{Tr}_{\mathbf{n}}\left\{\frac{1}{1+\hat{F}^\alpha}; \mathcal{L}^\alpha(\omega,\mathbf{q})\right\}. \tag{6.50}$$

As before, we explicitly consider the singlet-channel, short-range interaction in the zeroth harmonic approximation for the Fermi-liquid constant (denoted by F_0). The results for the long-range interaction in the unitary limit are obtained by letting $F_0^\rho \to +\infty$. For the triplet channel, we must substitute F_0^ρ with F_0^σ and multiply by an overall factor of 3. The final answer with the correct coefficients is given in Sec. 4.2.

In the diffusive limit, to which we restrict our attention for $d = 1, 3$, we have

$$b^\rho(\omega) - b^g(\omega) = \frac{\mathrm{Re}}{2\pi}\int \frac{d^dq}{(2\pi)^d} \times \\ \times \left[\frac{1}{1+F_0^\rho}L_0^\rho(\omega,\mathbf{q}) - L_0^g(\omega,\mathbf{q})\right], \tag{6.51}$$

with the functions L_0^α defined in Eqs. (6.20) and (6.21). After the momentum integration, we find

$$b^\rho(\omega) - b^g(\omega) = \frac{\Omega_d}{(2\pi)^d}\left(\frac{|\omega|}{D}\right)^{d/2}\frac{1}{4\omega} \times \\ \times \frac{\cos\frac{\pi}{4}(d-2)}{\sin\frac{\pi}{2}(d-2)}\left[1 - \frac{1}{(1+F_0^\rho)^{d/2}}\right]. \tag{6.52}$$

Inserting this in Eq. (6.48c) gives

$$\delta c_V = \frac{1}{2}\frac{\Omega_d}{(2\pi)^d}\left(\frac{T}{D}\right)^{d/2}\zeta\left(\frac{d}{2}+1\right)\Gamma\left(\frac{d}{2}+2\right) \times \\ \times \frac{\cos\frac{\pi}{4}(d-2)}{\sin\frac{\pi}{2}(d-2)}\left[1 - \frac{1}{(1+F_0^\rho)^{d/2}}\right]. \tag{6.53}$$

The relevant numerical values for the zeta and gamma functions are given after Eq. (6.27), which has been used to evaluate the ω integral.

For $d = 2$, we can keep the full form of the propagators to find the singular contribution to the specific heat at an arbitrary value of $T\tau$,

$$b^\rho(\omega) - b^g(\omega) = -\frac{\mathrm{Re}}{2\pi}\int \frac{d^2q}{(2\pi)^2}\left[\frac{F_0^\rho}{1+F_0^\rho}\frac{1}{\mathcal{C}-b} - \right. \\ \left. - \frac{(-i\omega+1/\tau)}{\mathcal{C}}\left(\frac{1}{\mathcal{C}-b} - \frac{1}{\mathcal{C}-1/\tau}\right)\right], \tag{6.54}$$

with $\mathcal{C}$ and b defined respectively in Eqs. (6.32) and (6.34). The first term in the integral is formally divergent as $|q| \to \infty$; this divergence gives a linear-in-T contribution to the specific heat that does not depend on disorder. This term must be disregarded because all the linear terms are included in the definition of the effective electron mass, and taking it into account would lead to a double counting. To regularize the integral, we replace $1/(\mathcal{C}-b) \to 1/(\mathcal{C}-b) - 1/\mathcal{C}$ in the first line.

Evaluating the momentum integral, we obtain

$$b^\rho(\omega) - b^g(\omega) = -\frac{1}{8\pi^2 D}\left[\frac{F_0^\rho}{1+F_0^\rho}\ln\left(\frac{E_F}{|\omega|}\right) + \right. \\ \left. + \left(\frac{F_0^\rho}{1+F_0^\rho}\right)^2\frac{\pi}{2}\tau|\omega| - \frac{1}{1+F_0^\rho}\ln(1+F_0^\rho)\right]. \tag{6.55}$$

The final answer for the correction to the specific heat is then

$$\delta c_V = -\frac{1}{12}\frac{T}{D}\frac{1}{1+F_0^\rho}\left[F_0^\rho\ln\left(\frac{E_F}{T}\right) - \ln(1+F_0^\rho)\right] - \\ - \frac{1}{4\pi^2}\Big[(2\gamma-3)\zeta(2) - 2\zeta'(2)\Big]\frac{T}{D}\frac{F_0^\rho}{1+F_0^\rho} - \\ - \frac{3}{4\pi}\zeta(3)\frac{T}{D}(T\tau)\left(\frac{F_0^\rho}{1+F_0^\rho}\right)^2, \tag{6.56}$$

where $\zeta(2) \approx 1.645$, $\zeta'(2) \approx -0.938$, and $\zeta(3) \approx 1.202$. In the quasiballistic limit $\tau \to +\infty$, only the last line is relevant:

$$\delta c_V = -\frac{3}{2\pi}\zeta(3)\left(\frac{F_0^\rho}{1+F_0^\rho}\right)^2\frac{T^2}{v_F^2}.$$

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[14] This definition of the density of states is half the one in Eqs. (2.19) because of the different limits for the ω integration in the energy density and the specific heat.

This T^2-correction to the specific heat has the same form as the correction found for two-dimensional Fermi liquids [28] and agrees (in the appropriate limit) with the result in Ref. [29].

As discussed before, the long-range interaction can be taken into account by passing to the limit as $F_0^\rho \to +\infty$, while the triplet channel contribution is three times larger (see also Sec. 4.2).

7. CONCLUSIONS

Locality at the scale determined by the temperature and the validity of the conservation laws are two main requirements for a proper kinetic description of any system. In the present paper, we derived such a description for the interaction effects in disordered metals (assuming that the clean counterpart of the system is a stable Fermi liquid).

We showed that this description requires the introduction of bosonic distribution functions in addition to the usual fermionic quasiparticle distribution function. These neutral bosons are of two types: (i) the ones describing oscillations in charge density (singlet) or spin density (triplet) and (ii) fictitious (ghost) bosons that prevent overcounting the degrees of freedoms (electron–hole pairs) already included in the fermionic part. The obtained conservation laws together with gauge invariance allow an unambiguous definition of the corresponding electric and energy currents.

For the electric transport, the neutral bosons are not important and our description reproduces the known results for the correction to the conductivity obtained in Ref. [11] for the diffusive regime and in Ref. [17] in the ballistic and crossover regimes.

The neutral bosons, however, are crucial for the thermal properties of the system. Namely, their contributions to the energy density are responsible for nonanalytic corrections to the specific heat, see Eqs. (4.6) and (4.8). Our kinetic equation approach reproduces the results for the interaction corrections to the specific heat previously calculated within the equilibrium diagram technique [12]. Moreover, the neutral boson contributions to the energy current violate the Wiedemann–Franz law, see Eq. (4.1) and the discussion that follows it. The violation is stronger for lower-dimensional systems ($d = 1, 2$) in the diffusive regime, see Eqs. (4.2) and (4.4b). Other effects contributing to the violation of the Wiedemann–Franz law are the energy dependence of the electron elastic scattering and the inelastic scattering of the electrons on the neutral bosons. The latter effect was found to be relevant in the quasiballistic regime $T\tau \gg \hbar$ for two-dimensional systems, see Eqs. (4.4).

The violation of the Wiedemann–Franz law was investigated before in the diffusive regime in Refs. [20] and [23] within the «quantum kinetic equation» approach and by the Kubo formula in Ref. [22]. Ironically, even though the forms of the energy current operator used in those references are wrong, the final results for the thermal conductivity are consistent with our Eqs. (4.2)–(4.4). We think that this agreement is accidental.

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APPENDIX A

Correction to the thermodynamic potential

A standard analytic continuation of Eq. (2.17) gives

$$\delta\Omega = \int \frac{d\omega}{4\pi} \operatorname{cth}\left(\frac{\omega}{2T}\right) \times$$
$$\times \int \frac{d^d q}{(2\pi)^d} \mathrm{Im}\,\ln\left(1 + \frac{F}{\nu}\Pi^R(\omega, \mathbf{q})\right) =$$
$$= \int \frac{d\omega}{2\pi}\frac{1}{2} \operatorname{cth}\left(\frac{\omega}{2T}\right) \mathrm{Im}\,\mathrm{Tr}\,\ln\left(1 + \frac{\hat{F}}{\nu}\hat{\Pi}^R\right). \quad \text{(A.1)}$$

In the second line, we use the operator notation (see Eqs. (3.3) and (3.13)), which gives the correct generalization for the momentum-dependent Fermi-liquid parameter. Substituting the transform of the explicit expression (5.43a) for the polarization operator, we rewrite the argument of the logarithm as

$$\left(1 + \hat{F}\right)\left[\left(\hat{\mathcal{L}}^g\right)^{-1} + \frac{\hat{F}}{1+\hat{F}} i\omega\right]\hat{\mathcal{L}}^g.$$

According to definition (3.11), the term in the square brackets is $\left(\hat{\mathcal{L}}^\rho\right)^{-1}$. Using the property

$$\mathrm{Tr}\ln\left(\hat{A}\hat{B}\right) = \mathrm{Tr}\ln\hat{A} + \mathrm{Tr}\ln\hat{B}$$

and the fact that $\ln\left(1 + \hat{F}\right)$ does not contribute to the imaginary part, we conclude that

$$\mathrm{Im}\mathrm{Tr}\ln\left(1 + \frac{\hat{F}}{\nu}\hat{\Pi}^R\right) = -\mathrm{Im}\mathrm{Tr}\left[\ln\hat{\mathcal{L}}^\rho - \ln\hat{\mathcal{L}}^g\right].$$

Substituting this identity in Eq. (A.1), we finally obtain Eq. (2.18).

APPENDIX B

The microscopic form of the energy current operator

The action entering the partition function that describes the electron gas in the presence of an external electric field is given by

$$S = \int dt\, d^d r \left[i\psi^\dagger \partial_t \psi - \psi^\dagger \hat{H}_{gi} \psi - \psi^\dagger \varphi \psi \right] \qquad \text{(B.1)}$$

with the condition $\nabla \times \mathbf{A} = 0$ that ensures the absence of the magnetic field. The variables $(t, \mathbf{r})$ on which all the fields depend are suppressed. The gauge-invariant part of the Hamiltonian for the noninteracting system has the usual form

$$\psi^\dagger \hat{H}_{gi} \psi = \frac{1}{2m} \left(i\nabla + \mathbf{A} \right) \psi^\dagger \left(-i\nabla + \mathbf{A} \right) \psi + \\ + \psi^\dagger V_{imp} \psi, \qquad \text{(B.2)}$$

where V_{imp} is the impurity potential and the potentials φ and $\mathbf{A}$ describe the external electric field:

$$e\mathbf{E}_{ext} = -\nabla\varphi + \partial_t \mathbf{A}.$$

As usual, the charge conservation law

$$\partial_t \rho + \nabla \cdot \mathbf{j} = 0 \qquad \text{(B.3)}$$

follows from the requirement of gauge invariance, with the charge and current densities given by

$$\rho = e\psi^\dagger \psi, \\ \mathbf{j} = \frac{e}{2m} \left[\psi^\dagger \left(-i\nabla + \mathbf{A} \right) \psi + \left(i\nabla + \mathbf{A} \right) \psi^\dagger \psi \right]. \qquad \text{(B.4)}$$

The invariance of the action under the replacement

$$\psi(t, \mathbf{r}) \to \psi(t + \alpha(t, \mathbf{r}), \mathbf{r})$$

(and a similar replacement for $\psi^\dagger$) underlies the derivation of the energy conservation law. A straightforward calculation gives

$$\frac{\delta S}{\delta \alpha} = 0 = \partial_t'(\psi^\dagger \hat{H}_{gi} \psi) + \varphi \partial_t (\psi^\dagger \psi) - \\ - \frac{i}{2m} \nabla \cdot \left[\partial_t \psi^\dagger (-i\nabla + \mathbf{A}) \psi - (i\nabla + \mathbf{A}) \psi^\dagger \partial_t \psi \right], \qquad \text{(B.5)}$$

where the prime means that the derivative acts on ψ, $\psi^\dagger$ only. By adding and subtracting terms proportional to φ in the last bracket and to $\partial_t \mathbf{A}$ in the first term, we find the energy conservation law

$$\partial_t u_0 + \nabla \cdot \mathbf{j}_0^\varepsilon = \mathbf{j} \cdot \mathbf{E}_{ext} - \frac{1}{e} \varphi \left[\partial_t \rho + \nabla \cdot \mathbf{j} \right], \qquad \text{(B.6)}$$

where

$$u_0 = \psi^\dagger \hat{H}_{gi} \psi, \qquad \text{(B.7a)}$$

$$\mathbf{j}_0^\varepsilon = -\frac{1}{2m} \Big[(i\partial_t + \varphi) \psi^\dagger (-i\nabla + \mathbf{A}) \psi - \\ - (i\nabla + \mathbf{A}) \psi^\dagger (i\partial_t - \varphi) \psi \Big]. \qquad \text{(B.7b)}$$

The first term in the right-hand side of Eq. (B.6) is the usual Joule heat; the last term in that equation is not gauge invariant, but it vanishes because it is proportional to the left-hand side of continuity equation (B.3).

We now consider the generalization to the interacting case. The Hamiltonian then contains the additional term

$$\frac{1}{2} \int d^d r_1\, \psi^\dagger \psi(\mathbf{r}) V(\mathbf{r} - \mathbf{r}_1) \psi^\dagger \psi(\mathbf{r}_1),$$

where $V(\mathbf{r}) = e^2/|\mathbf{r}|$ describes the density–density Coulomb interaction, which can be decoupled by the Hubbard–Stratonovich transformation. This amounts to introducing the quantum fields ϕ and $\mathbf{A}$ in the action by adding the term

$$-\psi^\dagger \phi \psi + \frac{1}{2} \mathbf{E}_{fl}^2, \quad e\mathbf{E}_{fl} = -\nabla\phi + \partial_t \mathbf{A} \qquad \text{(B.8)}$$

and redefining the vector potential as the sum of the external and fluctuating ones:

$$\mathbf{A} \to \mathbf{A}_{ext} + \mathbf{A}. \qquad \text{(B.9)}$$

The variation of the action with respect to the fluctuating potentials results in the first and fourth Maxwell equations relating the fluctuating electric field to the charge and current densities,

$$\nabla \cdot \mathbf{E}_{fl} = \rho, \quad 0 = \mathbf{j} + \partial_t \mathbf{E}_{fl}, \qquad \text{(B.10)}$$

where the electric current is defined in Eq. (B.4), but with substitution (B.9) performed.

To obtain the energy conservation law, we must consider the further transformation

$$\phi(t, \mathbf{r}) \to \phi(t + \alpha(t, \mathbf{r}), \mathbf{r})$$

and a similar one for $\mathbf{A}$. Proceeding as before, we find the conservation law

$$\frac{\partial u_?}{\partial t} + \nabla \cdot \mathbf{j}_?^\varepsilon = \mathbf{j} \cdot \mathbf{E}_{ext}, \\ u_? = u_0 - \frac{1}{2} \mathbf{E}_{fl}^2 + \frac{1}{e} \left[\rho\phi + \frac{\partial \mathbf{A}}{\partial t} \cdot \mathbf{E}_{fl} \right], \qquad \text{(B.11)} \\ \mathbf{j}_?^\varepsilon = \mathbf{j}_0^\varepsilon - \frac{1}{e} \phi \mathbf{j} + \frac{1}{e} \left[\phi \mathbf{j} - \frac{\partial \phi}{\partial t} \mathbf{E}_{fl} \right],$$

where u_0, $\mathbf{j}_0^\varepsilon$ are defined in eq. (B.7) (with substitution (B.9)). Given the form of Eq. (B.11), one might be tempted to call $u_?$ and $\mathbf{j}_?^\varepsilon$ the energy and energy current densities, in terms of which the conservation law takes exactly the same form as in the noninteracting case. But such a redefinition would result in gauge-dependent expressions for the densities, because the terms in the square brackets taken separately are not gauge invariant. Hence, this naive redefinition of the conserved quantities is nonphysical, because any physical perturbation can be coupled to only gauge-invariant quantities. To find gauge-invariant definitions, we rewrite the contribution of those gauge-noninvariant terms as

$$\begin{aligned}\frac{\partial}{\partial t}\left[\rho\phi + \frac{\partial \mathbf{A}}{\partial t}\cdot\mathbf{E}_{fl}\right] + \nabla\left[\phi\mathbf{j} - \frac{\partial\phi}{\partial t}\mathbf{E}_{fl}\right] &= \\ = \frac{1}{e}\phi\left[\partial_t\rho + \nabla\cdot\mathbf{j}\right] + \frac{1}{e}\frac{\partial\phi}{\partial t}\left[\rho - \nabla\cdot\mathbf{E}_{fl}\right] &+ \\ + \frac{1}{e}\frac{\partial}{\partial t}\left(\partial_t\mathbf{A} - \nabla\phi\right)\cdot\mathbf{E}_{fl} + \frac{1}{e}\left[\nabla\phi\cdot\mathbf{j} + \partial_t\mathbf{A}\,\partial_t\mathbf{E}_{fl}\right]&.\end{aligned}$$

Here, the second line vanishes because of charge conservation, Eq. (B.3), and because of the first Maxwell equation (B.10). In the third line, we use the second Maxwell equation to eliminate the current; in the result, we substitute the definition of the fluctuating field given in Eq. (B.8) and obtain that the third line of the above equation is equal to $\partial_t\mathbf{E}_{fl}^2$. This enables us to conclude that the correct, gauge-invariant expressions for the energy and energy current densities are

$$u = \psi^\dagger \hat{H}_{gi}\psi + \frac{1}{2}\mathbf{E}_{fl}^2, \tag{B.12}$$

$$\begin{aligned}\mathbf{j}^\varepsilon = -\frac{1}{2m}\Big[(i\partial_t + \varphi)\psi^\dagger(-i\nabla + \mathbf{A})\psi &- \\ - (i\nabla + \mathbf{A})\psi^\dagger(i\partial_t - \varphi)\psi\Big]&, \end{aligned}\tag{B.13}$$

where the potentials are the total ones:

$$\mathbf{A} = \mathbf{A}_{ext} + \mathbf{A}, \quad \varphi = \varphi_{ext} + \phi$$

We note that these expressions are gauge-invariant with respect to gauge transformations of both the external and fluctuating potentials. We believe that only such quantities can be coupled to the «gravitational field» in the Luttinger scheme for the calculation of the thermal conductivity [5].

The same final answer is obtained if the interaction is decoupled in the «gauge-fixed» form $\mathbf{A} = 0$. In this case, which is most widely used in the literature, there are two contributions to the energy current vertex in the diagram approach, see Fig. 4: in addition to the usual vertex of the noninteracting case, which arises from the terms $\partial_\tau\psi^\dagger\nabla\psi$, there is a vertex from the $\phi\psi^\dagger\nabla\psi$ terms. These vertices were not taken into account in Refs. [20, 22, 23]. However, analogous vertices were previuosly considered in the calculations of the thermoelectric coefficient[15] with the inclusion of the electron–electron interaction in the particle–hole channel [31] and in the Cooper channel [32] and for the electron–phonon interaction [33].

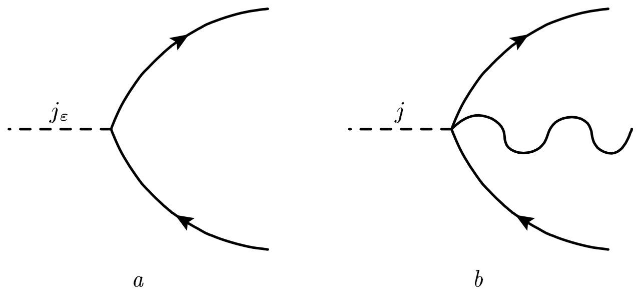

Fig. 4. *a*) The energy current vertex for the noninteracting case; *b*) the additional vertex induced by the interaction. The solid lines with arrows are the electron Green's functions, the wavy line is the interaction propagator, the dashed lines are the «standard» (noninteracting case) energy and electric current operators defined respectively in Eqs. (B.7b) and (B.4)

APPENDIX C

Alternative parameterization

The operator $\hat{H}_{e-h}$ defined in Eq. (5.54) is clearly not a standard Hamiltonian. However, we can introduce a different definition of the propagator $\mathcal{L}^\rho$,

$$\left[\frac{\partial}{\partial t_1} + i\hat{H}_{e-h}(-i\nabla_1) - \widehat{\mathrm{St}}_s^{\,out}\right]\mathcal{L}^\rho = \delta_{12}, \tag{C.1}$$

such that the (new) $\hat{H}_{e-h}$ operator is indeed a Hamiltonian:

$$\begin{gathered}\hat{H}_{e-h}(\mathbf{q}) \equiv \hat{\mathcal{F}}\,\mathbf{v}\!\cdot\!\mathbf{q}\,\hat{\mathcal{F}}\,, \quad \widehat{\mathrm{St}}_s^{\,out} \equiv \hat{\mathcal{F}}\,\widehat{\mathrm{St}}_\tau\hat{\mathcal{F}}\,, \\ \hat{\mathcal{F}} \equiv \left(1 + \hat{F}\right)^{1/2}\end{gathered}\tag{C.2}$$

---

[15] A derivation similar to that in this appendix was performed independently by M. Yu. Reyzer (private communication).

(the action of the operator $\hat{F}$ is defined in Eq. (5.49)). Proceeding as in Sec. 5.4, we obtain the following expressions for the $\mathcal{K}$ propagators:

$$\begin{aligned} \nu\hat{\mathcal{K}}^R &= (\partial_t)^{-1}\left[\hat{\mathcal{L}}^g - \hat{\mathcal{F}}\hat{\mathcal{L}}^\rho\hat{\mathcal{F}}\right], \\ \nu\hat{\mathcal{K}}^A &= -\left[\hat{\bar{\mathcal{L}}}^g - \hat{\mathcal{F}}\hat{\bar{\mathcal{L}}}^\rho\hat{\mathcal{F}}\right](\partial_t)^{-1}, \end{aligned} \tag{C.3a}$$

$$\begin{aligned} \nu\hat{\mathcal{K}}^K = {} & -i\,(\partial_t)^{-1}\left[\hat{\mathcal{L}}^g\hat{\mathcal{N}}^g + \hat{\mathcal{N}}^g\hat{\bar{\mathcal{L}}}^g\right](\partial_t)^{-1} + \\ & + i\,(\partial_t)^{-1}\,\hat{\mathcal{F}}\left[\hat{\mathcal{L}}^\rho\hat{\mathcal{N}}^\rho + \hat{\mathcal{N}}^\rho\hat{\bar{\mathcal{L}}}^\rho\right]\hat{\mathcal{F}}\,(\partial_t)^{-1}. \end{aligned} \tag{C.3b}$$

The «kinetic equation» for $\mathcal{N}^g$, Eq. (5.64a), remains unchanged, while $\mathcal{N}^\rho$ now satisfies the equation

$$\left[\partial_t + \hat{H}_{e-h}(-i\nabla);\hat{\mathcal{N}}^\rho\right] = \widehat{\mathrm{St}}^\rho\left\{\mathcal{N}^\rho, g^K\right\}, \tag{C.4}$$

where

$$\widehat{\mathrm{St}}^\rho\left\{\mathcal{N}^\rho, g^K\right\} \equiv 2\left\{\widehat{\mathrm{St}}_s^{out};\hat{\mathcal{N}}^\rho\right\} + 2\hat{\mathcal{F}}\hat{M}\hat{\mathcal{F}} \tag{C.5}$$

or, after Wigner transformations (5.61)–(5.63),

$$\begin{aligned} \omega\left[\partial_t\hat{N}^\rho + \left\{\hat{\boldsymbol{s}}, \nabla\hat{N}^\rho\right\} + i\left[\hat{H}_{e-h}(\mathbf{q}), \hat{N}^\rho\right]\right] = \\ = \widehat{\mathrm{St}}^\rho\left\{N^\rho, f\right\}, \end{aligned} \tag{C.6}$$

$$\begin{aligned} \widehat{\mathrm{St}}^\rho\left\{N^\rho, f\right\}(\omega, \mathbf{n}_1, \mathbf{n}_2) = \int d\varepsilon \int \frac{d\mathbf{n}_3\ldots d\mathbf{n}_6}{\Omega_d^4} \times \\ \times \left[\mathcal{F}_{16}\gamma_{34}^6\Upsilon_{52;43}^\rho(\varepsilon,\omega)\mathcal{F}_{45} + \right. \\ \left. + \mathcal{F}_{56}\gamma_{34}^6\Upsilon_{15;34}^\rho(\varepsilon,\omega)\mathcal{F}_{42}\right], \end{aligned} \tag{C.7}$$

with definitions (5.27) and (5.66) for γ and Υ.

We can then proceed as in Sec. 5.6 and obtain conservation laws (5.89) and (5.94); the only formal difference is in the definition of the bosonic energy density, which is now

$$u_\alpha(t,\mathbf{r}) = \frac{1}{2}\mathrm{Tr}_{\mathbf{n}}\left[\hat{\mathcal{L}}^\alpha\hat{\mathcal{N}}^\alpha\right].$$

In the alternative parameterization, the formalism can be developed with not more difficulties than in the original one. However, the evaluation of the thermal conductivity becomes cumbersome. In the original parameterization, it is also easier to include (at least perturbatively) the effects due to higher harmonics of the Fermi-liquid parameters.

APPENDIX D

Derivation of the electron collision integral

The calculation of the matrix collision integral is simplified by the introduction of two functions $A(t,\mathbf{r},\mathbf{n},\tilde{\mathbf{n}})$ and $B(t,\mathbf{r},\mathbf{n},\tilde{\mathbf{n}})$ such that

$$e^{i\hat{K}(t,\mathbf{n},\mathbf{r})}e^{-i\hat{K}(t,\tilde{\mathbf{n}},\mathbf{r})} = \left(A\hat{\mathbb{1}}_K + B\hat{\sigma}_K^x\right)_{\mathbf{n},\tilde{\mathbf{n}}}. \tag{D.1}$$

We recall that $\hat{K} = K_+\hat{\mathbb{1}}_K + K_-\hat{\sigma}_K^x$ and

$$\hat{\sigma}_K^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The collision integral (i.e., the right-hand side of Eq. (5.22)) in the matrix notation is then

$$\begin{aligned} \frac{1}{2\tau}\left\langle\left[\hat{g}(\mathbf{n})\circ\left(A\hat{\mathbb{1}}_K + B\hat{\sigma}_K^x\right)_{\mathbf{n},\tilde{\mathbf{n}}}\hat{g}(\tilde{\mathbf{n}})\left(A\hat{\mathbb{1}}_K + B\hat{\sigma}_K^x\right)_{\tilde{\mathbf{n}},\mathbf{n}} - \right.\right. \\ - \left(A\hat{\mathbb{1}}_K + B\hat{\sigma}_K^x\right)_{\mathbf{n},\tilde{\mathbf{n}}}\hat{g}(\tilde{\mathbf{n}}) \times \\ \left.\left. \times\left(A\hat{\mathbb{1}}_K + B\hat{\sigma}_K^x\right)_{\tilde{\mathbf{n}},\mathbf{n}}\circ\hat{g}(\mathbf{n})\right]\right\rangle_{\tilde{\mathbf{n}}}, \end{aligned} \tag{D.2}$$

where the open dot indicates the time convolution (cf. Eq. (5.9)) and the time argument of the functions A and B is the first (second) time argument of the Green's function on their right (left), e.g.,

$$BgB\circ g \equiv \int dt_3 B(t_1)g^K(t_1,t_3)B(t_3)g^K(t_3,t_2).$$

Substituting the matrix Green's function of form (5.23), we find that the collision integral becomes

$$\left\langle\begin{pmatrix} \widehat{\mathrm{St}}^R & \widehat{\mathrm{St}}^K \\ \widehat{\mathrm{St}}^Z & \widehat{\mathrm{St}}^A \end{pmatrix}\right\rangle_{\tilde{\mathbf{n}}}. \tag{D.3}$$

The explicit expressions for the retarded, advanced, and «Z» components are

$$\begin{aligned} \widehat{\mathrm{St}}^R &= g^K(\mathbf{n})\left[B;A\right] + g^K(\mathbf{n})\circ B(\mathbf{n},\tilde{\mathbf{n}})g^K(\tilde{\mathbf{n}})B(\tilde{\mathbf{n}},\mathbf{n}), \\ \widehat{\mathrm{St}}^A &= \left[A;B\right]g^K(\mathbf{n}) - B(\mathbf{n},\tilde{\mathbf{n}})g^K(\tilde{\mathbf{n}})B(\tilde{\mathbf{n}},\mathbf{n})\circ g^K(\mathbf{n}), \\ \widehat{\mathrm{St}}^Z &= 2\delta(t_1-t_2)\left[A;B\right] - 2B(\mathbf{n},\tilde{\mathbf{n}})g^K(\tilde{\mathbf{n}})B(\tilde{\mathbf{n}},\mathbf{n}), \end{aligned}$$

where the (equal-time) commutator is

$$\left[A;B\right] \equiv A(\mathbf{n},\tilde{\mathbf{n}})B(\tilde{\mathbf{n}},\mathbf{n}) - B(\mathbf{n},\tilde{\mathbf{n}})A(\tilde{\mathbf{n}},\mathbf{n}).$$

The calculations performed so far are exact. But at the one-loop level, we are interested in terms up to the second order in the fluctuating fields. Then expansion of the exponentials in Eq. (D.1) shows that any product of two functions B is proportional to terms of the form K_-K_-, which vanish after averaging over the fluctuating fields; accordingly, we drop such terms. The remaining terms are all commutators, whose explicit (approximate) form is

$$\big[A;B\big] = 2i\Big(K_-(\tilde{\mathbf{n}}) - K_-(\mathbf{n})\Big). \tag{D.4}$$

We note that the second-order terms cancel each other exactly. The surviving first-order terms lead to Eq. (5.24) for K_-.

For reference, we present the expression for $\widehat{\mathrm{St}}^K$ from which Eq. (5.73) and Eq. (5.76) are derived (with the exception of the last line in Eq. (5.76b), which is a consequence of requirement (5.34) for K_+):

$$\begin{aligned}\widehat{\mathrm{St}}^K &= 2\delta(t_1-t_2)\big[A;B\big]+2A(\mathbf{n},\tilde{\mathbf{n}})g^K(\tilde{\mathbf{n}})A(\tilde{\mathbf{n}},\mathbf{n})-\\ &- g^K(\mathbf{n})A(\mathbf{n},\tilde{\mathbf{n}})A(\tilde{\mathbf{n}},\mathbf{n}) - A(\mathbf{n},\tilde{\mathbf{n}})A(\tilde{\mathbf{n}},\mathbf{n})g^K(\mathbf{n}) +\\ &+ g^K(\mathbf{n})B(\mathbf{n},\tilde{\mathbf{n}})B(\tilde{\mathbf{n}},\mathbf{n}) + B(\mathbf{n},\tilde{\mathbf{n}})B(\tilde{\mathbf{n}},\mathbf{n})g^K(\mathbf{n}) +\\ &+g^K(\mathbf{n})\circ B(\mathbf{n},\tilde{\mathbf{n}})g^K(\tilde{\mathbf{n}})A(\tilde{\mathbf{n}},\mathbf{n})-\\ &- A(\mathbf{n},\tilde{\mathbf{n}})g^K(\tilde{\mathbf{n}})B(\tilde{\mathbf{n}},\mathbf{n})\circ g^K(\mathbf{n}).\end{aligned} \tag{D.5}$$

APPENDIX E

Derivation of Eqs. (5.81b) and (5.82b)

We start the derivation by separating the contributions of the Keldysh and retarded/advanced propagators to $\widehat{\mathrm{St}}_1^{in}$ in Eq. (5.75b):

$$\begin{aligned}\widehat{\mathrm{St}}_1^K &= -\frac{i}{16}\int\frac{d\mathbf{n}_2d\mathbf{n}_3}{(\Omega_d)^2}\,\gamma_{12}^3\, g(\bar t,\delta t,\mathbf{n}_2)\times\\ &\times\Big[2\mathcal{K}^K(\bar t,\delta t,\mathbf{n}_3,\mathbf{n}_2) - \mathcal{K}^K\left(\bar t+\frac{\delta t}{2},0,\mathbf{n}_3,\mathbf{n}_2\right) -\\ &- \mathcal{K}^K\left(\bar t-\frac{\delta t}{2},0,\mathbf{n}_3,\mathbf{n}_2\right) - (\mathbf{n}_2\to\mathbf{n}_1)\Big],\end{aligned} \tag{E.1}$$

$$\begin{aligned}\widehat{\mathrm{St}}_1^{RA} &= -\frac{i}{32}\int dt_3\int\frac{d\mathbf{n}_2d\mathbf{n}_3}{(\Omega_d)^2}\,\gamma_{12}^3\times\\ &\times\big[\mathcal{K}^A(t_3,t_2,\mathbf{n}_3,\mathbf{n}_2)-\mathcal{K}^R(t_1,t_3,\mathbf{n}_2,\mathbf{n}_3)-(\mathbf{n}_2\to\mathbf{n}_1)\big]\times\\ &\times\big[g(t_1,t_3,\mathbf{n}_1)g(t_3,t_2,\mathbf{n}_2) +\\ &+ g(t_1,t_3,\mathbf{n}_2)g(t_3,t_2,\mathbf{n}_1)\big].\end{aligned} \tag{E.2}$$

For convenience, we rewrite the Keldysh part in terms of the new time variables $\bar t, \delta t$:

$$\begin{aligned}&\bar t = \frac{t_1+t_2}{2}, \quad \delta t = t_1 - t_2\,,\\ &g(t_1,t_2)\to g(\bar t,\delta t),\\ &\mathcal{K}^K(t_1,t_1)\to\mathcal{K}^K(\bar t+\delta t/2,0),\ldots\end{aligned} \tag{E.3}$$

We then consider the limit of Eq. (E.1) as $t_2\to t_1$; clearly, as $\delta t\to 0$, the square bracket vanishes. But we know that in this limit, $g\to -2i/\pi\delta t$ (cf. Eq. (5.31)): in principle, there could be a nonvanishing contribution from the first-order expansion of the propagators in δt. The last two Keldysh propagators depend on δt in their first variable, but with opposite signs, and hence their respective first-order terms cancel each other. As for the first propagator, the property $\mathcal{K}^K(1,2)=\mathcal{K}^K(2,1)$ translates into $\mathcal{K}^K(t,\delta t) = \mathcal{K}^K(t,-\delta t)$, which ensures the absence of first-order terms. We conclude that in the limit as $\delta t\to 0$, the Keldysh propagator terms vanish. Similarly, from the property $\mathcal{K}^A(1,2)=\mathcal{K}^R(2,1)$, it follows that Eq. (E.2) vanishes for $t_2=t_1$; this concludes the proof of Eq. (5.81b).

We now turn to Eq. (5.82b). Because $\partial_{t_1}-\partial_{t_2} = 2\partial_{\delta t}$, we must expand the Keldysh propagators to the second order in δt. At this order, the square bracket in Eq. (E.1) is (up to the proper combination of angular variables)

$$\begin{aligned}\delta t^2\left(\partial_{\delta t}^2 - \frac{1}{4}\partial_{\bar t}^2\right)\mathcal{K}^K(\bar t,0) &=\\ &= -\delta t^2\lim_{t_2\to t_1}\partial_{t_1}\partial_{t_2}\mathcal{K}^K(t_1,t_2),\end{aligned}$$

where we restored the original time variables. In the operator notation, this is

$$\delta t^2\Big[\partial_t\,\mathcal{K}^K\partial_t\Big]_{t_2=t_1}.$$

Therefore,

$$\begin{aligned}\lim_{t_2\to t_1}(\partial_{t_1}-\partial_{t_2})\,\widehat{\mathrm{St}}_1^K(t_1,t_2) &= \frac{1}{4\pi}\int\frac{d\mathbf{n}_2d\mathbf{n}_3}{\Omega_d^2}\gamma_{12}^3\times\\ &\times\Big\{\Big[\partial_t\,\mathcal{K}^K\partial_t\Big](t_1,t_1,\mathbf{n}_3,\mathbf{n}_1) -\\ &- \Big[\partial_t\,\mathcal{K}^K\partial_t\Big](t_1,t_1,\mathbf{n}_3,\mathbf{n}_2)\Big\},\end{aligned}$$

which proves the first part of Eq. (5.82b).

As regards Eq. (E.2), using the analytic property $\mathcal{K}^A(1,2)=\mathcal{K}^R(2,1)$ again, we conclude that when the derivatives ∂_{t_1}, ∂_{t_2} act on the distribution functions g, the terms in the second line cancel each other. However, there are nonvanishing contributions when a derivative acts on the propagators, such as

$$\begin{aligned}&\int dt_3\int\frac{d\mathbf{n}_2d\mathbf{n}_3}{\Omega_d^2}\,\gamma_{12}^3\, g(t_1,t_3,\mathbf{n}_1)g(t_3,t_1,\mathbf{n}_2)\times\\ &\times\Big\{\Big[\partial_t\mathcal{K}^R\Big](t_1,t_3,\mathbf{n}_1,\mathbf{n}_3) - \Big[\partial_t\mathcal{K}^R\Big](t_1,t_3,\mathbf{n}_2,\mathbf{n}_3)\Big\}.\end{aligned}$$

Collecting all the terms, we arrive at

$$\lim_{t_2\to t_1}(\partial_{t_1}-\partial_{t_2})\widehat{\mathrm{St}}_1^{RA}(t_1,t_2)=$$
$$=\frac{i}{16}\int dt_3\int\frac{d\mathbf{n}_3d\mathbf{n}_2}{\Omega_d^2}\gamma_{12}^3\times$$
$$\times\left\{\Big[\partial_t\mathcal{K}^R\Big](t_1,t_3,\mathbf{n}_2,\mathbf{n}_3)-\Big[\partial_t\mathcal{K}^R\Big](t_1,t_3,\mathbf{n}_1,\mathbf{n}_3)\right\}\times$$
$$\times\big[g(t_1,t_3,\mathbf{n}_1)g(t_3,t_1,\mathbf{n}_2)+g(t_1,t_3,\mathbf{n}_2)g(t_3,t_1,\mathbf{n}_1)\big],$$

which concludes the derivation of Eq. (5.82b).

APPENDIX F

Elastic kernels in terms of the interaction propagator D

To compare the kernels in Eq. (6.9) with the corresponding expressions in Ref. [17], we use the Fourier transforms of Eqs. (5.51) and (5.55a) to obtain

$$\mathrm{Re}\Big[\hat{\mathcal{L}}^\rho-\hat{\mathcal{L}}^g\Big]=-\nu\omega\,\mathrm{Im}\Big[\hat{\mathcal{L}}^g\hat{D}^R\hat{\mathcal{L}}^g\Big].\qquad(\mathrm{F}.1)$$

If we assume, as is done in Ref. [17], that the Fermi-liquid parameters are independent of the momentum direction, then the interaction propagators $D^{R,A}$ are also independent of it and the above equation becomes

$$\mathrm{Re}\Big[\mathcal{L}^\rho-\mathcal{L}^g\Big]=-\nu\omega\,\mathrm{Im}\Big[\mathcal{L}^g\rangle D^R\langle\mathcal{L}^g\Big],\qquad(\mathrm{F}.2)$$

where we generalized the angular integral notation such that

$$\langle\mathcal{L}^g=\int\frac{d\mathbf{n}_1}{\Omega_d}\mathcal{L}^g(\mathbf{n}_1,\mathbf{n}_2),\quad \mathcal{L}^g\rangle=\int\frac{d\mathbf{n}_2}{\Omega_d}\mathcal{L}^g(\mathbf{n}_1,\mathbf{n}_2).$$

We recall that our ghost propagator $\mathcal{L}^g$ coincides with the diffuson propagator D in Ref. [17].

By substituting Eq. (F.2), we rewrite kernels (6.9) as

$$\mathcal{S}_{\mu\nu}^{11}(\omega)=\frac{2}{\pi}\delta_{\mu\nu}\times$$
$$\times\int\frac{d^dq}{(2\pi)^d}\Big(\langle\mathcal{L}^g\rangle\langle\mathcal{L}^g\rangle-\langle\mathcal{L}^g\mathcal{L}^g\rangle\Big)D^R,\qquad(\mathrm{F}.3)$$

$$\mathcal{S}_{\mu\nu}^{12}(\omega)=-\frac{2d}{\pi}\int\frac{d^dq}{(2\pi)^d}\langle n_\alpha\mathcal{L}^g\rangle\langle\mathcal{L}^g n_\beta\rangle D^R,\qquad(\mathrm{F}.4)$$

$$\mathcal{E}_{\mu\nu}(\omega)=\frac{d}{\pi\tau}\,\mathrm{Im}\int\frac{d^dq}{(2\pi)^d}\,D^R\Big[\langle\mathcal{L}^g n_\alpha\mathcal{L}^g n_\beta\rangle\langle\mathcal{L}^g\rangle-$$
$$-\langle\mathcal{L}^g n_\alpha\mathcal{L}^g\rangle\langle n_\beta\mathcal{L}^g\rangle+\langle\mathcal{L}^g n_\alpha\rangle\langle\mathcal{L}^g n_\beta\rangle\langle\mathcal{L}^g\rangle-$$
$$-\langle\mathcal{L}^g\rangle\langle n_\alpha\mathcal{L}^g n_\beta\rangle\langle\mathcal{L}^g\rangle\Big]+D^R\Big[\langle\mathcal{L}^g\rangle\langle n_\alpha\mathcal{L}^g n_\beta\mathcal{L}^g\rangle-$$
$$-\langle\mathcal{L}^g n_\alpha\rangle\langle\mathcal{L}^g n_\beta\mathcal{L}^g\rangle-\langle\mathcal{L}^g n_\alpha\mathcal{L}^g n_\beta\mathcal{L}^g\rangle\Big].\qquad(\mathrm{F}.5)$$

The first square bracket in the kernel $\mathcal{E}$ can be expressed as

$$\tau\Big[\langle\mathcal{L}^g n_\alpha\mathcal{L}^g\Big[n_\beta;\widehat{\mathrm{St}}_\tau\Big]\mathcal{L}^g\rangle+$$
$$+\langle\mathcal{L}^g\Big[n_\alpha;\widehat{\mathrm{St}}_\tau\Big]\mathcal{L}^g n_\beta\rangle\langle\mathcal{L}^g\rangle\Big].\qquad(\mathrm{F}.6)$$

Using the identity

$$\mathcal{L}^g\Big[\mathbf{n};\widehat{\mathrm{St}}_\tau\Big]\mathcal{L}^g=\Big[\mathbf{n};\mathcal{L}^g\Big],\qquad(\mathrm{F}.7)$$

we rewrite it as

$$\tau\Big[\frac{\delta_{\mu\nu}}{d}\Big(\langle\mathcal{L}^g\mathcal{L}^g\rangle-\langle\mathcal{L}^g\rangle\langle\mathcal{L}^g\rangle\Big)+$$
$$+\langle\mathcal{L}^g\rangle\langle n_\alpha\mathcal{L}^g n_\beta\rangle-\langle\mathcal{L}^g n_\alpha\mathcal{L}^g n_\beta\rangle\Big].\qquad(\mathrm{F}.8)$$

In the second square bracket, we use the identity

$$\mathcal{L}^g\mathbf{n}\mathcal{L}^g=\frac{i}{v_F}\partial_{\mathbf{q}}\mathcal{L}^g\qquad(\mathrm{F}.9)$$

to obtain

$$\frac{i}{v_F}\Big[\langle\mathcal{L}^g\rangle\partial_{q_\beta}\langle n_\alpha\mathcal{L}^g\rangle-\langle\mathcal{L}^g n_\alpha\rangle\partial_{q_\beta}\langle\mathcal{L}^g\rangle-$$
$$-\langle\mathcal{L}^g n_\alpha\partial_{q_\beta}\mathcal{L}^g\rangle\Big].\qquad(\mathrm{F}.10)$$

Finally, the identity

$$\mathcal{L}^g\mathcal{L}^g=-i\partial_\omega\mathcal{L}^g\qquad(\mathrm{F}.11)$$

enables us to conclude that the sum of the three kernels

$$\mathcal{S}^{11}+\mathcal{S}^{12}+\mathcal{E}$$

that determines the correction to the conductivity, Eq. (6.8), coincides with the combination $(K_0-K_1-L_0/v_F\tau)$ in the expression for the conductivity in Ref. [17].

APPENDIX G

Inelastic kernel for the phase relaxation time

We consider a uniform system in which the bosons are assumed to be in equilibrium with the electrons. In other words, the distribution function f is independent of $\mathbf{r},\mathbf{n}$ and the boson–electron collision integral (3.18) must vanish. The latter condition enables us to express the bosonic distributions N^α in terms of f and obtain

$$\Upsilon_{ij}(\varepsilon,\omega)=-\Omega_d\delta(\widehat{\mathbf{n}_i\mathbf{n}_j})\frac{1}{\omega}\int d\varepsilon_1\,\Psi(\varepsilon,\varepsilon_1;\omega)\qquad(\mathrm{G}.1)$$

(from now on, irrelevant angular and momentum variables are omitted; all relevant definitions can be found in Sec. 3). The former condition implies that collision integrals (3.19d), (3.20b), and (3.20f) vanish, and therefore the kinetic equation for (the zeroth harmonic of) f reduces to

$$\partial_t f(\varepsilon;t) = \int d\omega \int d\varepsilon_1\, A(\omega)\Psi(\varepsilon,\varepsilon_1;\omega), \qquad (G.2)$$

where

$$A(\omega) = \frac{-2}{\nu\pi\omega^2} \times$$
$$\times \operatorname{Re}\operatorname{Tr}\left\{\left[2\widehat{\mathrm{St}}_\tau \hat{\bar{\mathcal{L}}}^g + \hat{1}\right]\widehat{\mathrm{St}}_\tau\left[\hat{\mathcal{L}}^\rho + 3\hat{\mathcal{L}}^\sigma - 4\hat{\mathcal{L}}^g\right]\right\}. \qquad (G.3)$$

We substitute Eq. (F.1) and a similar relation for the triplet channel (D_T^R being the triplet channel propagator) in the expression for $A(\omega)$; we then use identity (5.44) and obtain

$$A(\omega) = -\frac{2}{\pi\omega}\operatorname{Im}\operatorname{Tr}\left[\widehat{\mathrm{St}}_\tau \hat{\bar{\mathcal{L}}}^g \left(\hat{D}^R + \hat{D}_T^R\right)\hat{\mathcal{L}}^g\right]. \qquad (G.4)$$

Using Eq. (5.44) again, we immediately recover the form of the inelastic kernel $A(\omega)$ given in Ref. [27].

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