ЖЭТФ, 2005, том 128, вып. 5 (11), стр. 1078–1084

A KINETIC APPROACH TO BOSE – EINSTEIN CONDENSATES: SELF-PHASE MODULATION AND BOGOLIUBOV OSCILLATIONS

J. T. Mendonça*, R. Bingham**

Rutherford Appleton Laboratory, Chilton, Didcot, Oxon OX11 OQX, U.K.

P. K. Shukla*****

Institut für Theoretische Physik IV, Fakultät für Physik und Astronomie, Ruhr-Universität Bochum, D-44780 Bochum, Germany

Submitted 22 April 2005

A kinetic approach to Bose – Einstein condensates (BECs) is proposed, based on the Wigner – Moyal equation (WME). In the semiclassical limit, the WME reduces to the particle-number conservation equation. Two examples of applications are i) a self-phase modulation of a BE condensate beam, where we show that a part of the beam is decelerated and eventually stops as a result of the gradient of the effective self-potential; ii) the derivation of a kinetic dispersion relation for sound waves in BECs, including collisionless Landau damping.

PACS: 02.70.Uu, 32.80.Pj, 67.40.Db

1. INTRODUCTION

Presently, the Bose – Einstein condensates (BECs) provide one of the most active and creative areas of research in physics [1, 2]. The dynamics of BECs are usually described by a nonlinear Schrödinger equation (known in this field as the Gross – Pitaevskii equation (GPE) [3, 4]), which determines the evolution of a collective wave function of ultra-cold atoms in BECs, evolving in the mean field self-potential.

In this paper, we propose the use of an alternative but nearly equivalent approach to the physics of BECs, based on a kinetic equation for the condensate. We also show that this kinetic theory can lead to a more complete understanding of the physical processes occurring in BECs, not only by providing an alternative method for describing the system but also by improving our global view of the physical phenomena. It is our hope that this will also lead to the discovery of new aspects of BECs.

The key point of our approach is the use of the Wigner – Moyal equation (WME) for BECs, describing the spatio-temporal evolution of the appropriate Wigner function [5]. Wigner functions for BECs were discussed in the past [6, 7] and the WME has been sporadically used [8]. But no systematic application of the WME to BECs has previously been considered. In the semiclassical limit, this equation reduces to the particle-number conservation equation, which is a kinetic equation formally analogous to the Liouville equation, but with a nonlinear potential. A description of BECs in terms of the kinetic equation is adequate in a series of problems, as is exemplified here, and can be seen as intermediate (in accuracy) between the GPE and the hydrodynamic equations usually found in the literature.

This paper is organized as follows. In Sec. 2, we establish the WME and discuss its approximate version as a kinetic equation for the Wigner function. We then apply the kinetic equation to two distinct physical problems. The first one, considered in Sec. 3, is the self-phase modulation of a BEC beam. A similar problem has been studied numerically in the past [9]. Here, we

*E-mail: T.Mendonca@rl.ac.uk. On leave from the Instituto Superior Técnico, 1049-001 Lisboa, Portugal.

**E-mail: r.bingham@rl.ac.uk. Also at the Department of Physics, University of Strathclyde, Glasgow G4 0NG, Scotland.

***E-mail: ps@tp4.rub.de. Also at the Department of Physics, Umeå University, SE-90187 Umeå, Sweden.

derive explicit analytical results and show that a part of the BEC beam is decelerated and eventually comes to a complete halt as a result of the collective forces acting on the condensate. The second example is considered in Sec. 4, where we establish a kinetic dispersion relation for sound waves in BECs, giving a kinetic correction to the usual Bogoliubov sound speed [10, 11] and predicting the occurrence of Landau damping [12, 13]. Our description of Landau damping is significantly different from that previously considered for transverse oscillations of BECs [14]. Finally, in Sec. 5, the virtues and limitations of the present kinetic approach are briefly discussed.

2. WIGNER – MOYAL EQUATION FOR THE BOSE CONDENSATE

It is known that for an ultra-cold atomic ensemble, and in particular for BECs, the ground-state atomic quantum field can be replaced by a macroscopic atomic wave function ψ. In a large variety of situations, the evolution of ψ is determined by the GPE

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + (V_0 + V_{eff})\psi, \tag{1}$$

where $V_0 \equiv V_0(\mathbf{r})$ is the confining potential and V_{eff} is the effective potential that takes the inter-atomic interactions inside the condensate into account; in the simplest form,

$$V_{eff}(\mathbf{r}, t) = g|\psi(\mathbf{r}, t)|^2,$$

where g is a constant [3, 4].

We consider the situation where this wave equation can be replaced by a kinetic equation. To construct such an equation, we introduce the Wigner function associated with ψ via [5]

$$W(\mathbf{r}, \mathbf{k}, t) = \int \psi\left(\mathbf{r} + \frac{\mathbf{s}}{2}, t\right)\psi^*\left(\mathbf{r} - \frac{\mathbf{s}}{2}, t\right) \times \times \exp(-i\mathbf{k}\cdot\mathbf{s})\, d\mathbf{s}. \tag{2}$$

It is then possible to derive (see the Appendix) the evolution equation for the Wigner function:

$$\left(\frac{\hbar^2}{2m}\mathbf{k}\cdot\nabla - i\hbar\frac{\partial}{\partial t}\right) W = -2V(\sin\Lambda)\, W, \tag{3}$$

where

$$\Lambda = \leftarrow \left(\frac{\partial}{\partial \mathbf{r}} \cdot \frac{\partial}{\partial \mathbf{p}}\right) \rightarrow \tag{4}$$

is a bi-directional differential operator that acts to the left on V and to the right on W [5]. In this equation, the potential is

$$V = V_0 + g\int W(\mathbf{r}, \mathbf{k}, t)\frac{d\mathbf{k}}{(2\pi)^3} + \delta V, \tag{5}$$

where

$$\delta V = g\left(|\psi(\mathbf{r}, t)|^2 - \int W(\mathbf{r}, \mathbf{k}, t)\frac{d\mathbf{k}}{(2\pi)^3}\right) \tag{6}$$

can be considered a noise term associated with the square mean deviations of the quasiprobability, determined by the Wigner function W with respect to the local quantum probability, determined by the wave function ψ.

Equation (3) can be seen as the WME describing the space and time evolution of BECs, and it is exactly equivalent to GPE (1). However, it is of little use in the above exact form, and it is convenient to introduce some simplifying assumptions. This is justified in the important case of slowly varying potentials. In this case, we can neglect the higher-order spatial derivatives and introduce the approximation $\sin\Lambda \sim \Lambda$. This corresponds to the semiclassical approximation, where the quantum potential fluctuations can also be neglected, viz. $\delta V \to 0$. Introducing these two simplifying assumptions, valid in the semiclassical limit, we reduce the WME to the much simpler form

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla + \mathbf{F}\cdot\frac{\partial}{\partial \mathbf{k}}\right) W = 0, \tag{7}$$

where $\mathbf{v} = \hbar\mathbf{k}/m$ is the velocity of the condensate atoms corresponding to the wave vector state $\mathbf{k}$, and $\mathbf{F} = -\nabla V$ is a force associated with the inhomogeneity of the condensate self-potential. The nonlinear term in GPE (1) is hidden inside this force $\mathbf{F}$. As we see in what follows, this nonlinear term looks very much like a ponderomotive force term, similar to radiation pressure.

We note that this new equation is a closed kinetic equation for the Wigner function W. In this semiclassical limit, W is just the particle occupation number for translational states with the momentum $\mathbf{p} = \hbar\mathbf{k}$. Equation (7) is equivalent to a conservation equation, stating the conservation of the quasiprobability W in the six-dimensional classical phase space $(\mathbf{r}, \mathbf{k})$, and can also be written as

$$\frac{d}{dt}W(\mathbf{r}, \mathbf{k}, t) = 0. \tag{8}$$

This kinetic equation can then be used to describe physical processes occurring in a BEC, as long as the semiclassical approximation of slowly varying potentials is justified. The interest in such kinetic descriptions is illustrated with the aid of two simple and different examples, to be presented in the next two sections. Many other applications can be envisaged, and will be explored in the future.

3. SELF-PHASE MODULATION OF A BEAM CONDENSATE

We first consider the kinetic description of self-phase modulation of a BEC gas moving with respect to the confining potential $V_0(\mathbf{r})$. Here, we can explore the similarity of this problem to that of self-phase modulation of short laser pulses moving in a nonlinear optical medium, which is well known in the literature [15]. To simplify our description, we consider the one-dimensional problem of a beam moving along the z axis and neglect the axial variation of the background potential, $\partial V_0/\partial z \approx 0$. The radial structure of the beam can easily be introduced later, and does not essentially modify the results obtained here. Kinetic equation (7) can then be written as

$$\left(\frac{\partial}{\partial t} + v_z\frac{\partial}{\partial z} + F_z\frac{\partial}{\partial k}\right) W(z,k,t) = 0, \qquad (9)$$

with v_z and F_z given by

$$v_z = \frac{\hbar k}{m} + g\frac{\partial}{\partial t}I(z,t), \quad F_z = \frac{dk}{dt} = -g\frac{\partial}{\partial z}I(z,t), \qquad (10)$$

where we have used the intensity of the beam condensate defined by

$$I(z,t) = \int W(z,k,t)\frac{dk}{2\pi}. \qquad (11)$$

We assume that an ultra-cold atomic beam has the mean velocity $v_0 = \hbar k_0/m$. This suggests the use of the new space coordinate $\eta = z - v_0 t$. In terms of this new coordinate, the semiclassical equations of motion of a cold atom in the beam can be written as

$$\begin{aligned} \frac{d\eta}{dt} &= \frac{\partial h}{\partial k} = \frac{1}{m}(k-k_0), \\ \frac{dk}{dt} &= -\frac{\partial h}{\partial \eta} = -\frac{g}{\hbar}\frac{\partial}{\partial \eta}I(\eta,t), \end{aligned} \qquad (12)$$

where we have introduced the Hamiltonian function

$$\begin{aligned} h(\eta,k,t) = \omega(\eta,k,t) - kv_0 = \\ = \frac{k}{m}\left(\frac{k}{2} - k_0\right) + \frac{g}{\hbar}I(\eta,t). \end{aligned} \qquad (13)$$

Here, $\omega(\eta,k,t)$ is the Hamiltonian in the rest frame expressed in the new coordinate. A straightforward integration of the equations of motion leads to

$$k(t) = k_0 - \frac{g}{\hbar}\int_0^t \frac{\partial}{\partial \eta}I(\eta,t')\,dt'. \qquad (14)$$

At this point, it is useful to introduce the concept of the beam energy chirp, $\langle\epsilon(\eta,t)\rangle$, in analogy with the frequency chirp of short laser pulses [15]. By definition, it is the beam mean energy at a given position and a given time,

$$\langle\epsilon(\eta,t)\rangle = \hbar\int W(\eta,k,t)\omega(\eta,k,t)\frac{dk}{2\pi}, \qquad (15)$$

where the weight function $W(\eta,k,t)$ is the solution of one-dimensional kinetic equation (9). A formal solution of this equation can be written as

$$W(\eta,k,t) = W(\eta_0(\eta,k,t), k_0(\eta,k,t), t_0), \qquad (16)$$

where η_0 and k_0 are the initial conditions corresponding to the observed values at time t, as determined by dynamical equations (12). With (16) used in Eq. (15), we obtain

$$\langle\epsilon(\eta,t)\rangle = \hbar\int W(\eta_0,k_0,t_0)\left[\frac{k^2}{2m} + \frac{g}{\hbar}I(\eta,t)\right]\frac{dk}{2\pi}. \qquad (17)$$

From Eq. (14), we see that $dk = dk_0$. Neglecting higher-order nonlinearities, we can then rewrite the above expression as [15]

$$\langle\epsilon(\eta,t)\rangle = \langle\epsilon(0)\rangle - \frac{k_0}{m}g\int_0^t \frac{\partial}{\partial \eta}I(\eta,t')\,dt', \qquad (18)$$

where $\langle\epsilon(0)\rangle \equiv \langle\epsilon(\eta_0,t_0)\rangle$ is the initial beam energy chirp.

We first consider the case where the beam profile $I(\eta)$ is independent of time. This is, of course, only valid for very short time intervals where the beam velocity dispersion is negligible. In this simple case, we have

$$\langle\epsilon(\eta,t)\rangle = \langle\epsilon(0)\rangle - \hbar v_0 g\frac{\partial I}{\partial \eta}t. \qquad (19)$$

The maximum energy shift is attained at some position inside the beam profile, $\eta = \eta_{max}$, determined by the stationarity condition

$$\frac{\partial}{\partial \eta}\langle\epsilon(\eta,t)\rangle = \frac{\partial^2 I}{\partial \eta^2} = 0. \qquad (20)$$

To deduce more specific answers, we assume a Gaussian beam profile

$$I(\eta) = I_0 \exp\left(-\eta^2/\sigma^2\right), \tag{21}$$

where σ determines the beam width. For this profile, we have $\eta_{max} = \pm\sigma/\sqrt{2}$, which leads to the maximum energy shift

$$\Delta\epsilon(t) \equiv \langle\epsilon(t)\rangle_{max} - \langle\epsilon(0)\rangle = \\ = \pm\frac{\hbar\sqrt{2}}{\sigma} g v_0 I_0 \exp\left(-\frac{1}{2}\right) t. \tag{22}$$

This is similar to the well-known result in nonlinear optics stating that the maximum energy chirp due to a self-phase modulation is proportional to t, or to the distance traveled by the beam, $d = v_0 t$. This result clearly indicates that the initial beam eventually splits into two parts, one being accelerated to higher translational speeds and the other being decelerated. This corresponds to the red-shift and blue-shift observed in nonlinear optics. The decelerated beam eventually stops after a time $t \approx \tau$, such that $\Delta\epsilon(\tau) = \langle\epsilon(0)\rangle$. This determines the condition for translational beam freezing.

We note that the same result could also be obtained directly from GPE (1). But the present derivation is interesting because it demonstrates the irrelevance of the phase of the wave function ψ, which was ignored in our kinetic calculation. Therefore, instead of the self-phase modulation, it would be more appropriate to call it the beam self-deceleration.

Another interesting aspect of our kinetic approach is that it can be easily refined, as is briefly shown here. We can improve the above calculation by considering the beam dispersion. It inevitably becomes relevant because of the linear velocity dispersion of the atomic beam. Such a dispersion decreases the chirping effect, because of the decrease of $\partial I/\partial\eta$ in time. To model it, we can assume a time-varying Gaussian beam shape, as described by

$$I(\eta, t) = I_0 \left(\frac{\sigma_0}{\sigma(t)}\right)^{1/2} \exp\left(-\frac{\eta^2}{\sigma^2(t)}\right). \tag{23}$$

If we now assume that

$$\sigma(t) = \sigma_0(1 + \delta t^2),$$

where

$$\delta = \frac{2m}{\hbar^2}\frac{\Delta\epsilon_0}{\sigma_0}$$

is proportional to the initial energy spread $\Delta\epsilon_0$, we obtain a new expression for the maximum energy shift, of the form

$$\Delta\epsilon_d(t) = \frac{\ln t}{t}\Delta\epsilon(t), \tag{24}$$

where $\Delta\epsilon(t)$ is determined by Eq. (22). It is clear that the linear beam velocity dispersion decreases the maximum attainable chirp, by changing the linearity with time into a logarithmic law. However, this only occurs for very long times, $t \sim 1/\sqrt{\delta}$, which are not relevant for ultra-cold atomic beams with a very low translational energy dispersion $\Delta\epsilon_0$.

The other cause of the beam dispersion is the nonlinear process itself, which eventually breaks the initial pulse into two distinct pulses. In this case, the self-phase modulation process is not attenuated because the beam width is conserved, but the two secondary pulses suffer self-phase modulation themselves, and eventually break up later, resulting in the formation of several secondary pulses with different mean energies. However, the nonlinear dispersion is also negligible whenever

$$\sigma_0^2 > \frac{4m}{\hbar^2}|\Delta\epsilon(t)|t^2.$$

A more complete description of all these dispersion regimes can be obtained by solving kinetic equation (9) numerically.

4. KINETIC DESCRIPTION OF BOGOLIUBOV OSCILLATIONS

The second example of an application of the kinetic equation for BECs deals with the dispersion relation of sound waves. For simplicity, we again consider the one-dimensional model and neglect the radial structure of the oscillations. This allows us to treat the lowest-order oscillating modes of the condensate. We assume some given equilibrium distribution $W_0(z, k, t)$, for instance, corresponding to the Thomas – Fermi equilibrium solution in a given confining potential $V_0(\mathbf{r}_\perp, z)$ [16], and after linearization of the one-dimensional kinetic equation (9) with respect to the perturbation $\tilde{W}$, we obtain

$$\left(\frac{\partial}{\partial t} + v_z\frac{\partial}{\partial z}\right)\tilde{W}(z, k, t) + \tilde{F}\frac{\partial}{\partial k}W_0(z, k, t) = 0, \tag{25}$$

where the perturbed force is determined by

$$\tilde{F} = -\frac{g}{\hbar}\frac{\partial}{\partial z}\tilde{I}(z, t) = -\frac{g}{\hbar}\frac{\partial}{\partial z}\int \tilde{W}(z, k, t)\frac{dk}{2\pi}. \tag{26}$$

We now assume perturbations of the form $\tilde{W}, \tilde{I} \sim \sim \exp(ikz - i\omega t)$. From the above equations, we then obtain a relation between the perturbation amplitude of the Wigner function $\tilde{W}$ and the perturbed beam intensity $\tilde{I}$,

$$\tilde{W} = -\frac{gk}{\hbar(\omega - kv')}\tilde{I}\frac{\partial}{\partial k'}W_0(k'), \qquad (27)$$

where we now specify the particle wavenumber state with k' in order to avoid confusion with the wavenumber k of the oscillation that we intend to study. The velocity corresponding to this particle state is $v' = \hbar k'/m$. Integration over the momentum spectrum of the particle condensate then leads to the equation

$$1 + \frac{g}{\hbar}k\int \frac{\partial W_0(k')/\partial k'}{\omega - \hbar k k'/m}\frac{dk'}{2\pi} = 0. \qquad (28)$$

This is the kinetic dispersion relation for axial perturbations in BECs. We illustrate this result by considering the simple case of a condensate beam with no translational dispersion, or with a translational temperature exactly equal to zero. The equilibrium of the beam can then be described by

$$W_0(k') = 2\pi n_0 \delta(k' - k'_0), \qquad (29)$$

where

$$n_0 = \frac{1}{2\pi}\int W_0(k')dk'$$

is the particle number density in the condensate. Replacing this in dispersion relation (28), we have

$$1 - \frac{gk^2}{m}\frac{n_0}{(\omega - kv'_0)^2} = 0, \qquad (30)$$

where $v'_0 = \hbar k'_0/m = p'_0/m$ is the beam velocity. This can also be written as

$$(\omega - kv'_0)^2 = k^2 c_s^2, \qquad (31)$$

where

$$c_s = \sqrt{gn_0/m} \qquad (32)$$

is nothing but the Bogoliubov sound speed. Obviously, Eq. (31) is the Doppler-shifted dispersion relation of sound waves in the BEC gas. In its reference frame, it reduces to $\omega = kc_s$.

We now consider the situation where, instead of distribution (29), we have a beam with a small translational velocity spread, such that the number of particles with a velocity $v' \sim c_s$ is small but nonzero. In this case, the resonant contribution in the integral of Eq. (28) has to be retained, although it is still possible to neglect the kinetic corrections in the principal part of the integral. The dispersion relation can then be written, in the condensate frame of reference, as

$$1 - \frac{k^2 c_s^2}{\omega^2} - \frac{i}{2}\frac{gm}{\hbar^2}\left(\frac{\partial W_0}{\partial k'}\right)_{k'=k'_s} = 0, \qquad (33)$$

where $k'_s = mc_s/\hbar$ is the resonant momentum. The imaginary term in this equation can lead to damping of sound waves. Writing $\omega = kc_s + i\gamma$, with $|\gamma| \ll kc_s$, we obtain the expression for the damping coefficient

$$\gamma = \frac{\omega}{4}\frac{gm}{\hbar^2}\left(\frac{\partial W_0}{\partial k'}\right)_{k'=k'_s}. \qquad (34)$$

This expression corresponds to the noncollisional Landau damping of Bogoliubov oscillations in BECs. The present approach can also be generalized in a straightforward way to higher-order oscillations of the condensate, where the radial structure has to be taken into account [11, 17].

5. CONCLUSIONS

We have proposed a kinetic view of the Bose–Einstein condensate physics, based on the Wigner–Moyal equation. In the semiclassical limit, the latter can be reduced to a closed kinetic equation for the corresponding Wigner function. The kinetic approach to BECs can be seen as an intermediate step between the GPE and the hydrodynamical equations for the condensate gas, often found in the literature.

We have discussed two different physical problems, in order to illustrate the versatility of the kinetic theory. One is a self-phase modulation of a BEC beam. The other is the dispersion relation of the Bogoliubov oscillations in the condensate gas. The first example shows that due to the influence of its own inhomogeneous self-potential, nearly half of the beam is accelerated, while the other half is decelerated. Under certain conditions, the decelerated part of the beam tends to a state of complete halt. The second example shows that a kinetic dispersion relation for sound waves in BECs can be established, where Landau damping is automatically included. The present results only involve the lowest-order modes, but the same approach can be used to describe higher-order oscillations of BECs, including their radial structures, as well as the coupling to a background thermal gas. This investigation is beyond the scope of the present work, however.

Several other different problems relevant to BECs can also be considered in the framework of the kinetic theory, such as modulational instabilities [18] and the wakefield generation. This indicates that the kinetic theory is a very promising approach to the physics of BECs, which will eventually allow introducing new ideas in this stimulating area of research and suggesting new configurations to the experimentalists. However, the present work also clearly states that the present theory is only valid in the semiclassical limit, and therefore some relevant problems where the phase of the BEC wave function plays an important role can only be treated by means of the GPE. Surprisingly, the self-phase modulation is not one of them, as demonstrated here.

One of the authors (J. T. M.) appreciates the hospitality of the Rutherford Appleton Laboratory. This work was done within the frame of the Centre for Fundamental Physics.

APPENDIX

Derivation of the Wigner – Moyal equation

In this derivation, we follow a procedure already used in other cases, for instance, in the case of electromagnetic waves moving in a space- and time-dependent dielectric medium [19]. For a different but nearly equivalent derivation of the WME, see the appendix in Ref. [8]. We consider two distinct sets of values for space and time coordinates, $(\mathbf{r}_1, t_1)$ and $(\mathbf{r}_2, t_2)$, and use the notation $\psi_j = \psi(\mathbf{r}_j, t_j)$ and $V_j = V(\mathbf{r}_j, t_j)$, for $j = 1, 2$. This allows us to write two versions of GPE (1) as

$$\left(\frac{\hbar^2}{2m}\nabla_j^2 - i\hbar\frac{\partial}{\partial t_j}\right)\psi_j = -V_j\psi_j. \tag{35}$$

Multiplying the $j = 1$ equation by ψ_2^* and the conjugate of the $j = 2$ equation by ψ_1, and subtracting the resulting equations, we obtain

$$\left[\frac{\hbar^2}{2m}(\nabla_1^2 - \nabla_2^2) - i\hbar\left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2}\right)\right]C_{12} = = -(V_1 - V_2)C_{12}, \tag{36}$$

where we set $C_{12} = \psi_1\psi_2^*$. The above equation suggests the use of two pairs of space and time variables,

$$\begin{aligned} \mathbf{r}_1 &= \mathbf{r} - \mathbf{s}/2, \quad t_1 = t - \tau/2, \\ \mathbf{r}_2 &= \mathbf{r} + \mathbf{s}/2, \quad t_2 = t + \tau/2. \end{aligned} \tag{37}$$

We can then rewrite the above equation as

$$\left[\frac{\hbar^2}{m}\frac{\partial}{\partial\mathbf{r}}\cdot\frac{\partial}{\partial\mathbf{s}} - i\hbar\frac{\partial}{\partial t}\right]C_{12} = -(V_1 - V_2)C_{12}. \tag{38}$$

It can also easily be shown, by expanding the potentials V_j around $V(\mathbf{r}, t)$, that

$$(V_1 - V_2) = 2\sinh\left(\frac{\mathbf{s}}{2}\cdot\frac{\partial}{\partial\mathbf{r}} + \frac{\tau}{2}\frac{\partial}{\partial t}\right)V(\mathbf{r}, t). \tag{39}$$

We now introduce the double Fourier transform of the function $C_{12} \equiv C(\mathbf{r}, \mathbf{s}, t, \tau)$ in the variables $\mathbf{s}$ and τ, defined by

$$W(\mathbf{r}, t, \omega, \mathbf{k}) = \int d\mathbf{s}\int d\tau\, C(\mathbf{r}, \mathbf{s}, t, \tau) \times \times \exp(-i\mathbf{k}\cdot\mathbf{s} + i\omega\tau). \tag{40}$$

It can be rewritten in terms of the wave function ψ as

$$W(\mathbf{r}, t, \omega, \mathbf{k}) = \int d\mathbf{s}\int d\tau\, \psi\left(\mathbf{r} + \frac{\mathbf{s}}{2}, t + \frac{\tau}{2}\right) \times \times \psi^*\left(\mathbf{r} - \frac{\mathbf{s}}{2}, t - \frac{\tau}{2}\right)\exp(-i\mathbf{k}\cdot\mathbf{s} + i\omega\tau). \tag{41}$$

Using this in Eq. (38), we obtain the equation

$$\left(\frac{\hbar^2}{m}\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{r}} - \hbar\frac{\partial}{\partial t}\right)W = -2V(\sin\Lambda')W \tag{42}$$

for the Fourier transform, where we use the differential operator

$$\Lambda' = \frac{1}{2}\leftarrow\left(\frac{\partial}{\partial\mathbf{r}}\cdot\frac{\partial}{\partial\mathbf{k}} - \frac{\partial}{\partial t}\frac{\partial}{\partial\omega}\right)\rightarrow \tag{43}$$

acting to the left on the potential $V(\mathbf{r}, t)$ and to the right on W.

This is a formidable equation for W, which can be simplified by noting that the GPE implies the existence of a well-defined relation between energy and momentum. This means that ω must be equal to some function of $\mathbf{k}$, or $\omega = \omega(\mathbf{k})$. Hence, we can state that

$$W(\mathbf{r}, t, \omega, \mathbf{k}) = W(\mathbf{r}, \mathbf{k}, t)\delta(\omega - \omega(\mathbf{k})). \tag{44}$$

This leads to a much simpler evolution equation for $W(\mathbf{r}, \mathbf{k}, t)$. Before writing it, we also note that the nonlinear term in V depends on $|\psi|^2$, and not on the function W. Thus, we can finally write

$$\left(\frac{\hbar^2}{2m}\mathbf{k}\cdot\nabla - i\hbar\frac{\partial}{\partial t}\right)W = -2(V_0 + g|\psi|^2)(\sin\Lambda)W, \tag{45}$$

where Λ is the simpler differential operator

$$\Lambda = \leftarrow\left(\frac{\partial}{\partial\mathbf{r}}\cdot\frac{\partial}{\partial\mathbf{p}}\right)\rightarrow. \tag{46}$$

The function $W(\mathbf{r}, \mathbf{k}, t)$ can be seen as the Wigner function associated with the GPE, and Eq. (45) as the

WME equation that describes its spatio-temporal behavior. This equation is equivalent to the initial wave equation (1), but it is not a closed equation for the quasiprobability function W. Therefore, some simplifying assumptions have to be introduced in order to make it more tractable, as explained in Sec. 2.

REFERENCES

1. A. J. Legget, Rev. Mod. Phys. **73**, 307 (2001).

2. F. Dalfovo et al., Rev. Mod. Phys. **71**, 463 (1999).

3. E. P. Gross, Nuovo Cim. **20**, 454 (1961); J. Math. Phys. **4**, 195 (1963).

4. L. P. Pitaevskii, Zh. Eksp. Teor. Fiz. **40**, 646 (1961).

5. M. Hillary, R. F. O'Connel, M. O. Scully, and E. P. Wigner, Phys. Rep. **106**, 121 (1984).

6. M. J. Steel et al., Phys. Rev. A **58**, 4824 (1998).

7. C. W. Gardiner and M. J. Davis, J. Phys. B: Atom. Mol. Opt. Phys. **36**, 4731 (2003).

8. S. A. Gardiner et al., Phys. Rev. A **62**, 023612 (2000).

9. W. Zhang and D. F. Walls, Phys. Rev. A **49**, 3799 (1994).

10. M. R. Andrews et al., Phys. Rev. Lett. **79**, 553 (1997).

11. E. Zaremba, Phys. Rev. A **57**, 518 (1998).

12. L. P. Pitaevskii, Phys. Lett. A **235**, 398 (1997).

13. B. Jackson and E. Zaremba, New J. Phys. **5**, 88 (2003).

14. M. Guilleumas and L. P. Pitaevskii, Phys. Rev. A **67**, 053607 (2003).

15. L. O. Silva and J. T. Mendonça, Opt. Comm. **196**, 285 (2001).

16. G. Byam and C. J. Pethick, Phys. Rev. Lett. **76**, 6 (1996).

17. S. Stringari, Phys. Rev. Lett. **77**, 2360 (1996).

18. V. V. Konotop and M. Salerno, Phys. Rev. A **65**, 021602 (2002).

19. J. T. Mendonça and N. L. Tsintsadze, Phys. Rev. E **62**, 4276 (2000).