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THE INFLUENCE OF SPIN ON THERMODYNAMICAL QUANTITIES

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We use the brick-wall method to investigate thermodynamical quantities around a static Gibbons – Maeda dilaton black hole and show that each of these quantities contains an additional spin-dependent term and that the usual result that the entropy density, energy density, and pressure take the same forms as in flat space–time holds only for the leading term. Our results are compatible with the early conclusions that the black hole entropy is not exactly proportional to the horizon area and that Hawking radiation is not to be purely thermal.

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In theoretical physics, the thermodynamics of black holes remains an enigma; it turns out to be a junction of general relativity, quantum mechanics, and statistical physics. Logarithmic corrections to the Bekenstein – Hawking entropy due to spin fields have been extensively investigated [1–5]. The Hawking radiation via tunneling from the black hole has also been studied widely and was proved not to be exactly thermal [6–10]. This implies that other information in addition to temperature could be preserved in formation and evaporation of a black hole, as argued by Hawking in Ref. [11].

It is generally assumed that the entropy density, the energy density, and the pressure of an ideal relativistic gas in curved space–time have the same forms as in Minkowski space–time [12], where they are independent of the spin of the field, except that different fields obey different statistics. The physical reason can be traced back to the equivalence principle [13].

Applying the quantization procedure referred to as the Boulware vacuum state and Killing time t, Li [14, 15] studied the thermodynamical quantities around the Schwarzschild black hole and the Reissner – Nordström black hole and found that the corrected expressions for these quantities include additional spin-dependent terms. Obviously, these results are important and helpful for further investigation in related subjects such as black hole entropy and black hole radiation. Our purpose in this paper is to extend this method to the Gibbons – Maeda dilaton black hole and to investigate the influence of spin on the thermodynamical quantities by the brick-wall method [16]. Doubts regarding the validity of the brick-wall method are expressed in some references [17, 18], but these objections are shown to be overcome [19] when the ground state is correctly identified and the local description of the statistical mechanics is equivalent to that of a quantized field in the curved background, which is defined globally and whose ground state is the Boulware state [20].

The metric of static Gibbons – Maeda dilaton black hole is given by [21, 22]

$$ds^2 = \frac{(r-r_+)(r-r_-)}{r^2-D^2}\,dt^2 - \frac{r^2-D^2}{(r-r_+)(r-r_-)}\,dr^2 - (r^2-D^2)(d\theta^2+\sin^2\theta\,d\varphi^2), \quad (1)$$

where

$$r_\pm = M \pm \sqrt{M^2+D^2-P^2-Q^2}$$

are the outer and inner horizons. Here, M, P, Q, and D are the mass, electric charge, magnetic charge, and dilaton parameter, related by

$$D = \frac{P^2-Q^2}{2M}.$$

In space–time (1), the area of the spherical surface at a point r outside the horizon is

$$A(r) = 4\pi(r^2-D^2).$$

We introduce the null tetrad vectors

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$$l^\mu = \left(\frac{r^2 - D^2}{(r - r_+)(r - r_-)}, 1, 0, 0\right),$$
$$n^\mu = \frac{1}{2}\left(1, -\frac{(r - r_+)(r - r_-)}{r^2 - D^2}, 0, 0\right), \qquad (2)$$
$$m^\mu = \frac{1}{\sqrt{2(r^2 - D^2)}}\left(0, 0, 1, \frac{i}{\sin\theta}\right).$$

The nonvanishing spin coefficients and the only nonvanishing component of the Weyl tensor can then be obtained using the Newman–Penrose formula [23] as

$$\alpha = -\beta = -\frac{\operatorname{ctg}\theta}{2\sqrt{2(r^2 - D^2)}}, \quad \rho = -\frac{r}{r^2 - D^2},$$

$$\mu = -\frac{r(r - r_+)(r - r_-)}{2(r^2 - D^2)^2},$$
$$\gamma = \frac{1}{4}\left[\frac{2r - r_+ + r_-}{r^2 - D^2} - \frac{2r(r - r_+)(r - r_-)}{(r^2 - D^2)^2}\right], \qquad (3)$$

$$\Psi_2 = \frac{r(r_+ - r_-) - 2r^2}{2(r^2 - D^2)^2} + \frac{(2r^2 + D^2)(r - r_+)(r - r_-)}{2(r^2 - D^2)^3}.$$

Equations (3) tell us that metric (1) is of the Petrov type D. Using the result of Teukolsky [24], the field equations for the neutrino ($s = 1/2$), electromagnetic ($s = 1$), and gravitational ($s = 2$) fields in the source-free case can be combined into the equations

$$\begin{aligned}&\{[D - (2p + 1)\rho][\Delta - 2p\gamma + \mu] -\\ &- [\delta + 2(p - 1)\alpha][\overline{\delta} - 2s\alpha] -\\ &- (2s - 1)(s - 1)\Psi_2\}\Omega_q = 0,\\ &\{[\Delta - 2(p + 1)\gamma + (1 - 2p)\mu][D - \rho] -\\ &- [\overline{\delta} - 2(p + 1)\alpha][\delta - 2s\alpha] -\\ &- (2s - 1)(s - 1)\Psi_2\}\Omega_q = 0,\end{aligned} \qquad (4)$$

where D, Δ, and δ are the directional derivatives given by $D = l^\mu\partial_\mu$, $\Delta = n^\mu\partial_\mu$, and $\delta = m^\mu\partial_\mu$, Ω_q are the mode functions, and q represents the set of quantum numbers. The first equation is for spin states $p = s$ and the other is for $p = -s$.

In the quantization procedure referred to as the Boulware vacuum state and Killing time t, the mode functions of these fields in the vicinity of the Gibbons–Maeda dilaton black hole can be written as

$$\Omega_q = \Omega_{Elmp} = r^{p-s}{}_pR_{lE}(r){}_pY_{lm}(\theta, \varphi)e^{-iEt}. \qquad (5)$$

Substituting Eqs. (3) and (5) in Eqs. (4), we obtain the radial equation

$$\left\{\frac{d^2}{dr^2} + \left[\frac{(p+1)(2r - r_+ - r_-)}{(r - r_+)(r - r_-)} + \frac{pr}{r^2 - D^2}\right]\frac{d}{dr} + \right.$$
$$+ \frac{(r^2 - D^2)^2E^2}{(r - r_+)^2(r - r_-)^2} +$$
$$\left. + \frac{iEC(r, p) + B(r, p) - \lambda^2}{(r - r_+)(r - r_-)}\right\}{}_pR_{lE}(r) = 0, \qquad (6)$$

where

$$B(r, p) = -\frac{2pD^2 - 2p(4p - 5)r^2 + (4p^2 + 3)(r_+ + r_-)r + (2p + 1)r_+r_-}{r^2 - D^2} +$$
$$+ \frac{(r - r_+)(r - r_-)}{(r^2 - D^2)^2}\left[(4p^2 + 3p + 7)r^2 + (2p^2 - 3p + 1)D^2\right], \qquad (7)$$

$$C(r, p) = 4pr - \frac{p(r^2 - D^2)(2r - r_+ + r_-)}{(r - r_+)(r - r_-)},$$

and the angular equation

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta)\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + \right.$$
$$\left. + \frac{2ip\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\varphi} - p^2\operatorname{ctg}^2\theta - p + \lambda^2\right]{}_pY_{lm}(\theta, \varphi) = 0. \qquad (8)$$

Equation (8) shows that ${}_pY_{lm}(\theta, \varphi)$ is a spin-weighted spherical harmonic, and the separation constant λ satisfies the relation

$$\lambda^2 = (l - p)(l + p + 1), \qquad (9)$$

where l and m are integers satisfying the inequalities

$$l \geq s \quad \text{and} \quad -l \leq m \leq l. \qquad (10)$$

Writing

$${}_pR_{lE}(r) = \exp[iS(r, p, l, E)]$$

and using the Wentzel–Kramers–Brillouin (WKB) approximation, we obtain the radial wave number $k \equiv \partial_r S$:

$$k = \left[\frac{(r^2-D^2)^2E^2}{(r-r_+)^2(r-r_-)^2} + \right. \left. + \frac{B(r,p)-(l-p)(l+p+1)}{(r-r_+)(r-r_-)}\right]^{1/2} \quad (11)$$

under the brick-wall boundary conditions

$$\Omega_q = 0 \quad \text{at} \quad r = r_+ + \varepsilon \quad \text{and} \quad r = r_+ + \varepsilon + L,$$

where ε is the distance of the brick wall from the horizon, $0 < \varepsilon \ll r_+$, and L is the thickness of the brick wall. Then the constraint of the semiclassical quantum condition imposed on k is

$$\int\limits_{r_H+\varepsilon}^{r_H+\varepsilon+L} k\,dr = n\pi, \quad (12)$$

where n is a nonnegative integer, and the number of eigenstates with the energy smaller than E is given by

$$g(E) = \sum_p \sum_l (2l+1)n = \frac{1}{\pi}\sum_p \int\limits_s^{l_{max}} (2l+1)\,dl \times \int\limits_{r_H+\varepsilon}^{r_H+\varepsilon+L} dr \left(\frac{(r^2-D^2)^2E^2}{(r-r_+)^2(r-r_-)^2} + \frac{B(r,p)-(l-p)(l+p+1)}{(r-r_+)(r-r_-)}\right)^{1/2} = \frac{2}{3\pi}\sum_p \int\limits_{r_H+\varepsilon}^{r_H+\varepsilon+L} \frac{(r^2-D^2)^3 dr}{(r-r_+)^2(r-r_-)^2} \times \left[E^2 + \frac{(r-r_+)(r-r_-)}{(r^2-D^2)^2}\eta(r,p)\right]^{3/2}. \quad (13)$$

Here, l_{max} is determined by Eq. (11) and

$$\eta(r,p) = B(r,p) - s + p. \quad (14)$$

The free energy at the inverse Hawking temperature β_H is given by

$$-\beta_H F = \pm \sum_\alpha \ln\left(1 \pm \exp(-\beta_H E_\alpha)\right). \quad (15)$$

The «+» sign in Eq. (15) corresponds to the Fermi case and the «−» sign corresponds to the Bose case. Using Eq. (13) to determine the density of states, we obtain the free energy

$$F = \mp\frac{1}{\beta_H}\int\limits_0^\infty dE \frac{dg(E)}{dE}\ln\left(1 \pm \exp(-\beta_H E)\right) =$$

$$= \begin{cases} -\dfrac{2\omega\pi^3}{45\beta_H^4}\displaystyle\int\limits_{r_H+\varepsilon}^{r_H+\varepsilon+L}\frac{(r^2-D^2)^3 dr}{(r-r_+)^2(r-r_-)^2} - \frac{\pi}{6\beta_H^2}\sum_p \int\limits_{r_H+\varepsilon}^{r_H+\varepsilon+L}\frac{(r^2-D^2)\eta(r,p)\,dr}{(r-r_+)(r-r_-)}, & s = 1, 2, \\ -\dfrac{7\omega\pi^3}{180\beta_H^4}\displaystyle\int\limits_{r_H+\varepsilon}^{r_H+\varepsilon+L}\frac{(r^2-D^2)^3 dr}{(r-r_+)^2(r-r_-)^2} - \frac{\pi}{12\beta_H^2}\int\limits_{r_H+\varepsilon}^{r_H+\varepsilon+L}\frac{(r^2-D^2)\eta(r,p)\,dr}{(r-r_+)(r-r_-)}, & s = 1/2, \end{cases} \quad (16)$$

where $\omega = \sum_p 1$ ($\omega = 2$ for the gravitational and electromagnetic fields and $\omega = 1$ for the neutrino field). Using the formulas

$$S = \beta_H^2 \frac{\partial F}{\partial \beta_H}$$

and

$$U = \frac{\partial(\beta_H F)}{\partial \beta_H},$$

we can obtain the entropy and energy as

$$S=\begin{cases}\dfrac{8\omega\pi^3}{45\beta_H^3}\displaystyle\int\limits_{r_H+\varepsilon}^{r_H+\varepsilon+L}\dfrac{(r^2-D^2)^3dr}{(r-r_+)^2(r-r_-)^2}+\dfrac{\pi}{3\beta_H}\displaystyle\int\limits_{r_H+\varepsilon}^{r_H+\varepsilon+L}\dfrac{(r^2-D^2)^2\eta^*(r)\,dr}{(r-r_+)(r-r_-)}, & s=1,2,\\ \dfrac{7\omega\pi^3}{45\beta_H^3}\displaystyle\int\limits_{r_H+\varepsilon}^{r_H+\varepsilon+L}\dfrac{(r^2-D^2)^3dr}{(r-r_+)^2(r-r_-)^2}+\dfrac{\pi}{6\beta_H}\displaystyle\int\limits_{r_H+\varepsilon}^{r_H+\varepsilon+L}\dfrac{(r^2-D^2)^2\eta^*(r)\,dr}{(r-r_+)(r-r_-)}, & s=1/2,\end{cases}\qquad(17)$$

$$U=\begin{cases}\dfrac{2\omega\pi^3}{15\beta_H^4}\displaystyle\int\limits_{r_H+\varepsilon}^{r_H+\varepsilon+L}\dfrac{(r^2-D^2)^3dr}{(r-r_+)^2(r-r_-)^2}+\dfrac{\pi}{6\beta_H^2}\displaystyle\int\limits_{r_H+\varepsilon}^{r_H+\varepsilon+L}\dfrac{(r^2-D^2)^2\eta^*(r)\,dr}{(r-r_+)(r-r_-)}, & s=1,2,\\ \dfrac{7\omega\pi^3}{60\beta_H^4}\displaystyle\int\limits_{r_H+\varepsilon}^{r_H+\varepsilon+L}\dfrac{(r^2-D^2)^3dr}{(r-r_+)^2(r-r_-)^2}+\dfrac{\pi}{12\beta_H^2}\displaystyle\int\limits_{r_H+\varepsilon}^{r_H+\varepsilon+L}\dfrac{(r^2-D^2)^2\eta^*(r)\,dr}{(r-r_+)(r-r_-)}, & s=1/2,\end{cases}\qquad(18)$$

where

$$\eta^*(r)=\begin{cases}\dfrac{1}{r^2-D^2}\displaystyle\sum_p\eta(r,p)=\dfrac{16s^2r^2-2(4s^2+3)(r_++r_-)r-2r_+r_-}{(r^2-D^2)^2}+\\ +\dfrac{2(r-r_+)(r-r_-)[(4s^2+7)r^2+(2s^2+1)D^2]}{(r^2-D^2)^3}-\dfrac{2s}{r^2-D^2}, & s=1,2,\\ \dfrac{1}{r^2-D^2}\eta(r,p)=\dfrac{B(r,p)-s+p}{r^2-D^2}, & s=1/2.\end{cases}\qquad(19)$$

On the other hand, the total entropy and energy are

$$S=\int\limits_{r_H+\varepsilon}^{r_H+\varepsilon+L}\sigma(r)\frac{A(r)\,dr}{\sqrt{g_{00}}}=\int\limits_{r_H+\varepsilon}^{r_H+\varepsilon+L}\sigma(r)\frac{4\pi(r^2-D^2)\,dr}{\sqrt{g_{00}}},\qquad(20)$$

$$U=\int\limits_{r_H+\varepsilon}^{r_H+\varepsilon+L}\rho(r)A(r)\,dr=\int\limits_{r_H+\varepsilon}^{r_H+\varepsilon+L}\rho(r)4\pi(r^2-D^2)\,dr,\qquad(21)$$

where we have taken a spherical shell as the volume element. The factor

$$\frac{1}{\sqrt{g_{00}}}=\sqrt{\frac{r^2-D^2}{(r-r_+)(r-r_-)}}$$

does not appear in the integral for the total energy of the thermal excitations [18]. Comparing Eqs. (20) and (21) with Eqs. (17) and (18), we obtain the entropy density and energy density

$$\sigma(r)=\begin{cases}\dfrac{2\omega\pi^2}{45}T^3(r)+\dfrac{1}{12}\eta^*(r)T(r), & s=1,2,\\ \dfrac{7\omega\pi^2}{180}T^3(r)+\dfrac{1}{24}\eta^*(r)T(r), & s=1/2,\end{cases}\qquad(22)$$

$$\rho(r)=\begin{cases}\dfrac{\omega\pi^2}{30}T^4(r)+\dfrac{1}{24}\eta^*(r)T^2(r), & s=1,2,\\ \dfrac{7\omega\pi^2}{240}T^4(r)+\dfrac{1}{48}\eta^*(r)T^2(r), & s=1/2,\end{cases}\qquad(23)$$

where $T(r)$ is the local temperature determined by the Tolman relation [17]

$$T(r)=\frac{1}{\beta_H\sqrt{g_{00}}}=\frac{1}{\beta_H}\sqrt{\frac{r^2-D^2}{(r-r_+)(r-r_-)}}\,.\qquad(24)$$

The pressure is given by [25]

$$P(r)=\sigma(r)T(r)-\rho(r).\qquad(25)$$

Substituting Eqs. (22) and (23) in Eq. (25), we obtain the pressure

$$P(r) = \\ = \begin{cases} \dfrac{\omega\pi^2}{90} T^4(r) + \dfrac{1}{24}\eta^*(r)T^2(r), & s = 1, 2, \\ \dfrac{7\omega\pi^2}{720} T^4(r) + \dfrac{1}{48}\eta^*(r)T^2(r), & s = 1/2. \end{cases} \quad (26)$$

The equation of state is

$$\rho(r) - 3P(r) = \begin{cases} -\dfrac{1}{12}\eta^*(r)T^2(r), & s = 1, 2, \\ -\dfrac{1}{24}\eta^*(r)T^2(r), & s = 1/2. \end{cases} \quad (27)$$

In summary, we have evaluated the entropy density, energy density, and pressure for the perfect relativistic gases of massless particles with spins $s = 1/2$, 1, and 2 in the vicinity of the static Gibbons–Maeda dilaton black hole by the WKB approximation, which are given by Eqs. (22), (23), and (26), respectively. Our results show that any one of these quantities for a spin field includes an additive spin-dependent term. These additional terms cannot be neglected at sufficiently low temperature, for example, in the vicinity of a near-extremal black hole, and lead to the equations of state (27) being also spin-dependent. The usual result for any spin field that the entropy density, energy density, and pressure take the same forms as in a flat space–time holds only for the leading term in powers of the temperature. Of course, when r is sufficiently large, the space–time becomes a Minkowski one, the spin-dependent terms decrease as $1/r^2$ or more rapidly and can be neglected, and therefore the result is consistent with that in Minkowski space–time.

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