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# ISING MODELS ON THE $2 \times 2 \times \infty$ LATTICES

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Exact analytic solutions are presented for two $2\times2\times\infty$ Ising étagères. The first model has a simple cubic lattice with fully anisotropic interactions. The second model consists of two different types of linear chains and includes noncrossing diagonal bonds on the side faces of the $2 \times 2 \times \infty$ parallelepiped. In both cases, the solutions are expressed through square radicals and obtained by using the obvious symmetry of the Hamiltonians, $\mathbf{Z}_2 \times \mathbf{C}_{2v}$, and the hidden algebraic $\lambda\lambda$ symmetry of the transfer matrix secular equations. The solution found for the second model is used to analyze the behavior of specific heat in a frustrated many-chain system.

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## 1. INTRODUCTION

Models of interacting Ising chains play an important role in many fields of physics (see, e. g., [1–5]). Allowing an accurate mathematical description, they, on the one hand, find numerous applications in the interpretation of various collective phenomena in one-dimensional and pseudo-one-dimensional systems. On the other hand, the coupled Ising chains appearing as clusters allow greatly improving the precision of calculated characteristics of two- and three-dimensional materials in the framework of general approaches such as the mean-field theory or the renormalization-group method. Moreover, the exact solutions quite often serve as heuristic examples, and are also the good tests for debugging of complicated computer code.

The problem of the Ising model on a $2 \times 2 \times \infty$ lattice with a simple cubic cell and under the condition of equality of the interactions in both transverse directions was actually solved in the famous paper of Onsager [6]. This paper is dedicated to the two-dimensional Ising model, and as an intermediate result, the expression for the largest eigenvalue $\lambda_{max}$ (and consequently for the free energy) of the transfer matrix of the model on a $n \times \infty$ cylinder was obtained:

$$\lambda_{max} = [2\,\mathrm{sh}(2K_1)]^{n/2} \times \exp\left[\frac{1}{2}(\gamma_1 + \gamma_3 + \ldots + \gamma_{2n-1})\right], \quad (1)$$

where $\gamma_k$ ($k = 1, 3, \ldots, 2n-1$) are the positive solutions of the equation

$$\mathrm{ch}(\gamma_k) = \mathrm{cth}(2K_1)\,\mathrm{ch}(2K_2) - \cos\left(\frac{\pi k}{n}\right)\frac{\mathrm{sh}(2K_2)}{\mathrm{sh}(2K_1)}. \quad (2)$$

Here,

$$K_1 = \frac{1}{2}\beta J_1, \quad K_2 = \frac{1}{2}\beta J_2, \quad \beta = \frac{1}{k_B T}.$$

With the number of chains $n$ equal to 4, formulas (1) and (2) lead to the solution of the above $2\times2\times\infty$ Ising system.

In this paper, simple analytic solutions are obtained for two other $2 \times 2 \times \infty$ Ising lattices. One lattice has the cell in the form of a rectangular parallelepiped in which the interactions are different along all three spatial directions. The cells of the second lattice are parallelepipeds with a rhombic base. Although the interactions in the base plane are equal, uncrossing diagonal couplings on the outside may be available and, moreover, the intrachain interactions in the given model must be equal only for chains situated in the $2\times2\times\infty$ system opposite each other. For both models, the lattice symmetry $\mathbf{C}_{2v}$ together with the symmetry under the spin inversion $\mathbf{Z}_2$ permit reducing the original transfer matrices to a block-diagonal form with the

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maximal size of subblocks $5 \times 5$. As the subsequent analysis shows, the secular equations of subblocks are virtually reciprocal, which allows reducing the solution of these equations to a chain of algebraic equations of at most second degree.

From the free energy of the second model in which the next-nearest-neighbor interactions are present, the expression for the specific heat is obtained and the peculiarities of its temperature behavior near the structure instability point are examined.

## 2. THE EIGENVALUES OF TRANSFER MATRICES

We consider the Ising models with the Hamiltonians

$$H_1 = -\frac{1}{2} \times \sum_i [J_x(\sigma_{1,i}\sigma_{4,i}+\sigma_{2,i}\sigma_{3,i})+J_y(\sigma_{1,i}\sigma_{2,i}+\sigma_{3,i}\sigma_{4,i})+ J_z(\sigma_{1,i}\sigma_{1,i+1} + \sigma_{2,i}\sigma_{2,i+1} + \sigma_{3,i}\sigma_{3,i+1} + \sigma_{4,i}\sigma_{4,i+1})] \quad (3)$$

and

$$H_2 = -\frac{1}{2}\sum_i [J_A(s_{1,i}s_{1,i+1} + s_{3,i}s_{3,i+1}) + J_B(s_{2,i}s_{2,i+1} + s_{4,i}s_{4,i+1}) + J_{AB}(s_{1,i} + s_{3,i})(s_{2,i} + s_{4,i}) + J'_{AB}(s_{2,i} + s_{4,i})(s_{1,i+1} + s_{3,i+1})]. \quad (4)$$

The topologies of the couplings represented by these Hamiltonians are illustrated in Figs. 1 and 2. The spin variables $\sigma_{l,i}$ and $s_{l,i}$ are located at the lattice sites and take the values $\pm 1$. Both lattices have the symmetry planes $\sigma_v$ and $\sigma'_v$. In the model given by Eq. (3), these planes pass through the middle of the opposite faces of an infinitely long $2\times2\times\infty$ parallelepiped having a rectangular cross section. In the model given by Eq. (4), the symmetry planes pass through the opposite edges of a $2\times2\times\infty$ parallelepiped, whose cross section is now a rhombus. We note that with $J_z = J_1$ and $J_x = J_y = J_2$ or with $J_A = J_B = J_1$, $J_{AB} = J_2$ and $J'_{AB} = 0$ (or, vise versa, with $J_{AB} = 0$ and $J'_{AB} = J_2$), we obtain a $2 \times 2 \times \infty$ model described by Onsager's formulas (1) and (2), in which we should of course put $n = 4$.

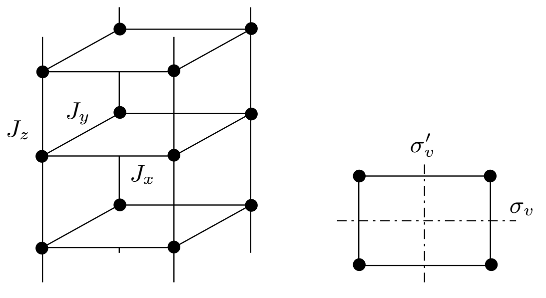


**Fig. 1.** Fully anisotropic simple cubic $2 \times 2 \times \infty$ Ising lattice given by Eq. (3) and its profile. To not overload the figure, the vertical symmetry planes $\sigma_v$ and $\sigma_v$ are shown only on the cross section of the lattice

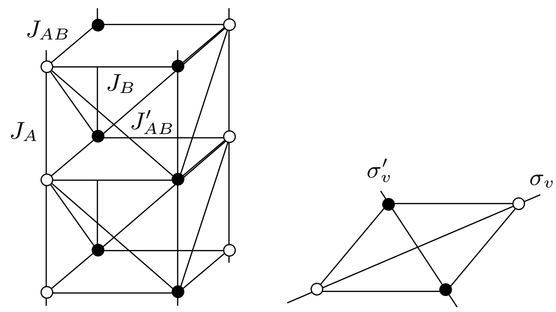


**Fig. 2.** Lattice $2 \times 2 \times \infty$ with a rhombic cross section and diagonal interactions (the model given by Eq. (4))

The principal task in calculating the statistical mechanical characteristics of models (3) and (4) is to solve the eigenvalue problem for the transfer matrices $\mathbf{V}_1$ and $\mathbf{V}_2$ with the elements

$$\langle \sigma_1, \sigma_2, \sigma_3, \sigma_4|\mathbf{V}_1|\sigma'_1, \sigma'_2, \sigma'_3, \sigma'_4\rangle = \exp\left[\frac{1}{2}K_x(\sigma_1\sigma_4 + \sigma_2\sigma_3 + \sigma'_1\sigma'_4 + \sigma'_2\sigma'_3) + \frac{1}{2}K_y(\sigma_1\sigma_2 + \sigma_3\sigma_4 + \sigma'_1\sigma'_2 + \sigma'_3\sigma'_4) + K_z(\sigma_1\sigma'_1 + \sigma_2\sigma'_2 + \sigma_3\sigma'_3 + \sigma_4\sigma'_4)\right] \quad (5)$$

and

$$\langle s_1, s_2, s_3, s_4|\mathbf{V}_2|s'_1, s'_2, s'_3, s'_4\rangle = \exp\left\{K_A(s_1s'_1 + s_3s'_3) + K_B(s_2s'_2 + s_4s'_4) + \frac{1}{2}K_{AB}[(s_1+s_3)(s_2+s_4)+(s'_1+s'_3)(s'_2+s'_4)] + K'_{AB}(s_2 + s_4)(s'_1 + s'_3)\right\}. \quad (6)$$

Here,

$$K_x = \frac{1}{2}\beta J_x, \quad K_y = \frac{1}{2}\beta J_y, \quad K_z = \frac{1}{2}\beta J_z,$$
$$K_A = \frac{1}{2}\beta J_A, \quad K_B = \frac{1}{2}\beta J_B,$$

$$K_{AB} = \frac{1}{2}\beta J_{AB}, \quad K'_{AB} = \frac{1}{2}\beta J'_{AB}.$$

We notice that the matrix $\mathbf{V}_1$ is symmetric and $\mathbf{V}_2$ is, generally speaking, not.

To solve the eigenvalue problem of transfer matrices (5) and (6), we use the invariance property of the appropriate Hamiltonians with respect to the transformations of the group $\mathbf{Z}_2 \times \mathbf{C}_{2v}$ where, as has already been mentioned,

$$\mathbf{Z}_2 = \{E, R\}$$

is the group of global reflections in the spin space ($E$ is the identity transformation and $R$ is the spin inversion operation) and

$$\mathbf{C}_{2v} = \{E, C_2, \sigma_v, \sigma'_v\}$$

is the point group generated by symmetry elements $\sigma_v$ and $\sigma'_v$ ($C_2 = \sigma_v \sigma'_v$ is a second-order symmetry axis).

### 2.1. Group-theoretical analysis

We construct representations of a group $\mathbf{Z}_2 \times \mathbf{C}_{2v}$ in the transfer-matrix spaces. For the first model, we set

$$R|\sigma_1, \sigma_2, \sigma_3, \sigma_4\rangle = |-\sigma_1, -\sigma_2, -\sigma_3, -\sigma_4\rangle, \qquad (7)$$

$$\sigma_v|\sigma_1, \sigma_2, \sigma_3, \sigma_4\rangle = |\sigma_2, \sigma_1, \sigma_4, \sigma_3\rangle, \qquad (8)$$

and

$$\sigma'_v|\sigma_1, \sigma_2, \sigma_3, \sigma_4\rangle = |\sigma_4, \sigma_3, \sigma_2, \sigma_1\rangle. \qquad (9)$$

The remaining elements of the group are the corresponding combinations of $R$, $\sigma_v$, and $\sigma'_v$ and their action on the vector $|\sigma_1, \sigma_2, \sigma_3, \sigma_4\rangle$ is easily found by using relation (7)–(9). Multiplying these equalities by conjugated vectors from the left and taking the orthonormality condition

$$\langle\sigma_1, \sigma_2, \sigma_3, \sigma_4|\sigma'_1, \sigma'_2, \sigma'_3, \sigma'_4\rangle = = \delta_{\sigma_1\sigma'_1}\delta_{\sigma_2\sigma'_2}\delta_{\sigma_3\sigma'_3}\delta_{\sigma_4\sigma'_4} \qquad (10)$$

into account, where $\delta_{\sigma\sigma'}$ is the Kronecker delta, we can calculate the matrix elements of the original representation $\Gamma_1$ of the group $\mathbf{Z}_2 \times \mathbf{C}_{2v}$ for the first model. It is not difficult to verify that all matrices obtained commute with $\mathbf{V}_1$.

For the second model, the inversion transformation in (7) preserves the analogous form and the reflections $\sigma_v$ and $\sigma'_v$ now act on a vector as

$$\sigma_v|s_1, s_2, s_3, s_4\rangle = |s_1, s_4, s_3, s_2\rangle \qquad (11)$$

and

$$\sigma'_v|s_1, s_2, s_3, s_4\rangle = |s_3, s_2, s_1, s_4\rangle. \qquad (12)$$

Again multiplying the equations of type (11) and (12) by conjugated vectors, we find a matrix set giving the representation $\Gamma_2$ of the group for the second model. These matrices commute with $\mathbf{V}_2$.

In the subsequent analysis, we first find the characters $\chi$ of the representations $\Gamma_1$ and $\Gamma_2$. For $\Gamma_1$, we obtain

$$\chi_1(E) = \langle\sigma_1, \sigma_2, \sigma_3, \sigma_4|E|\sigma_1, \sigma_2, \sigma_3, \sigma_4\rangle = = \langle\sigma_1, \sigma_2, \sigma_3, \sigma_4|\sigma_1, \sigma_2, \sigma_3, \sigma_4\rangle = = \delta_{\sigma_1\sigma_1}\delta_{\sigma_2\sigma_2}\delta_{\sigma_3\sigma_3}\delta_{\sigma_4\sigma_4} = 2^4, \qquad (13)$$

$$\chi_1(\sigma_v) = \langle\sigma_1, \sigma_2, \sigma_3, \sigma_4|\sigma_v|\sigma_1, \sigma_2, \sigma_3, \sigma_4\rangle = = \delta_{\sigma_1\sigma_2}\delta_{\sigma_2\sigma_1}\delta_{\sigma_3\sigma_4}\delta_{\sigma_4\sigma_3} = 2^2, \qquad (14)$$

and similarly for other group elements. By analogy, we can also calculate the characters for the representation $\Gamma_2$. The characters found, together with the known characters of the irreducible representations (IRs) $\Gamma^{(1)}, \ldots, \Gamma^{(8)}$ of $\mathbf{Z}_2 \times \mathbf{C}_{2v}$, are collected in the Table. Now, using the formula (see, e.g., [7])

$$a^{(\nu)} = \frac{1}{g}\sum_G \chi(G)\chi^{(\nu)^*}(G) \qquad (15)$$

(where $g$ is the order of a group, $\chi(G)$ is the character of an element $G$ in the considered representation, $\chi^{(\nu)}(G)$ is the character of the same element in the $\nu$th IR, and $a^{(\nu)}$ is the multiplicity with which the $\nu$th IR enters the original representation) we find

$$\Gamma_1 = 5\Gamma^{(1)} + 2(\Gamma^{(2)} + \Gamma^{(3)} + \Gamma^{(4)} + + \Gamma^{(5)}) + \Gamma^{(6)} + \Gamma^{(7)} + \Gamma^{(8)} \qquad (16)$$

and

$$\Gamma_2 = 5\Gamma^{(1)} + 4\Gamma^{(2)} + + 2(\Gamma^{(3)} + \Gamma^{(5)}) + \Gamma^{(6)} + \Gamma^{(7)} + \Gamma^{(8)}. \qquad (17)$$

This implies that by transitions using similar transformations to the new basis in which the representations $\Gamma_1$ and $\Gamma_2$ of the Abelian group $\mathbf{Z}_2 \times \mathbf{C}_{2v}$ are fully reducible, the matrices $\mathbf{V}_1$ and $\mathbf{V}_2$ take a block-diagonal form, where the first matrix $\mathbf{V}_1$ has one subblock of size $5 \times 5$, four $2 \times 2$ subblocks, and three $1 \times 1$ "subblocks", and the second matrix $\mathbf{V}_2$ consists of one $5 \times 5$ subblock, one $4 \times 4$ subblock, two $2 \times 2$ subblocks, and again three ready-made eigenvalues.

Character table of the group $\mathbf{Z}_2 \times \mathbf{C}_{2v}$

| $\mathbf{Z}_2 \times \mathbf{C}_{2v}$ | $E$ | $C_2$ | $\sigma_v$ | $\sigma'_v$ | $R$ | $RC_2$ | $R\sigma_v$ | $R\sigma'_v$ |
|---|---|---|---|---|---|---|---|---|
| $\Gamma^{(1)}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\Gamma^{(2)}$ | 1 | 1 | 1 | 1 | −1 | −1 | −1 | −1 |
| $\Gamma^{(3)}$ | 1 | −1 | −1 | 1 | −1 | 1 | 1 | −1 |
| $\Gamma^{(4)}$ | 1 | 1 | −1 | −1 | −1 | −1 | 1 | 1 |
| $\Gamma^{(5)}$ | 1 | −1 | 1 | −1 | −1 | 1 | −1 | 1 |
| $\Gamma^{(6)}$ | 1 | −1 | −1 | 1 | 1 | −1 | −1 | 1 |
| $\Gamma^{(7)}$ | 1 | 1 | −1 | −1 | 1 | 1 | −1 | −1 |
| $\Gamma^{(8)}$ | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 |
| $\Gamma_1$ | 16 | 4 | 4 | 4 | 0 | 4 | 4 | 4 |
| $\Gamma_2$ | 16 | 4 | 8 | 8 | 0 | 4 | 0 | 0 |

### 2.2. Basis vectors of irreducible representations

The next step is the quasi-diagonalization of transfer matrices in practice. For this, we first construct the basis vectors of IRs. In our case, this is easily done by acting with the projection operator

$$P^{(\nu)} = \sum_G \chi^{(\nu)^*}(G)G \tag{18}$$

(a normalizing coefficient is omitted) on the vectors of the original basis. Let

$$e_1 = |1,1,1,1\rangle, \quad e_2 = |1,1,1,-1\rangle, \quad \ldots \\ \ldots, \quad e_{16} = |-1,-1,-1,-1\rangle \tag{19}$$

be a basis in which the matrix $\mathbf{V}_1$ is defined according to Eq. (5). Applying operator (18) to vectors (19) successively and taking equalities (7)–(9) and the character Table into account, we obtain sets of basis vectors, which should be normalized. In particular, we have for the basis vectors of the identical IR

$$\begin{aligned} \psi_1^{(1)} &= \frac{e_1 + e_{16}}{\sqrt{2}}, \\ \psi_2^{(1)} &= \frac{e_2 + e_3 + e_5 + e_8 + e_9 + e_{12} + e_{14} + e_{15}}{2\sqrt{2}}, \\ \psi_3^{(1)} &= \frac{e_4 + e_{13}}{\sqrt{2}}, \quad \psi_4^{(1)} = \frac{e_6 + e_{11}}{\sqrt{2}}, \\ \psi_5^{(1)} &= \frac{e_7 + e_{10}}{\sqrt{2}}. \end{aligned} \tag{20}$$

For the next IR $\Gamma^{(2)}$, we obtain

$$\begin{aligned} \psi_1^{(2)} &= \frac{e_1 - e_{16}}{\sqrt{2}}, \\ \psi_2^{(2)} &= \frac{e_2 + e_3 + e_5 + e_9 - e_8 - e_{12} - e_{14} - e_{15}}{2\sqrt{2}}. \end{aligned} \tag{21}$$

In an analogous way, we find basis vectors for all other IRs in the original representation.

Knowing the basis functions, we directly calculate the matrix elements of subblocks using Eq. (5):

$$\left(\mathbf{V}_1^{(\nu)}\right)_{ij} = \psi_i^{(\nu)^+} \mathbf{V}_1 \psi_j^{(\nu)}, \tag{22}$$

and similarly for the second model.

### 2.3. Using the $\boldsymbol{\lambda\lambda}$ symmetry

Now the task is to solve the secular equations of subblocks. Calculating the determinant of the first equation

$$\det(\mathbf{V}_1^{(1)} - \lambda) = 0, \tag{23}$$

we obtain that it has the structure

$$\lambda^5 - a_1\lambda^4 + a_2\lambda^3 - \alpha a_2\lambda^2 + \alpha^3 a_1\lambda - \alpha^5 = 0, \tag{24}$$

where

$$a_1 = 2[1 + 4\,\mathrm{ch}(2K_x)\,\mathrm{ch}(2K_y)]\,\mathrm{ch}(4K_z) + 6, \tag{25}$$

$$a_2 = 32\,\mathrm{ch}(2K_x)\,\mathrm{ch}(2K_y)[\mathrm{ch}(4K_z)\,\mathrm{ch}^2(2K_z) - 1] + \\ + 8[1 + \mathrm{ch}(4K_x) + \mathrm{ch}(4K_y)]\,\mathrm{sh}^2(4K_z), \tag{26}$$

and

$$\alpha = 4\,\mathrm{sh}^2(2K_z). \tag{27}$$

According to Ref. [8], an algebraic equation like (24) is reciprocal. One root of Eq. (24) coincides obviously with $\alpha$. After its extraction, we obtain a quartic equation that is again reciprocal:

$$\lambda^4 - (a_1 - \alpha)\lambda^3 + [a_2 - \alpha(a_1 - \alpha)]\lambda^2 - \\ - \alpha^2(a_1 - \alpha)\lambda + \alpha^4 = 0. \quad (28)$$

By the substitution

$$r = \lambda + \frac{\alpha^2}{\lambda}, \quad (29)$$

Eq. (28) is reduced to the quadric resolvent

$$r^2 - (a_1 - \alpha)r + a_2 - \alpha(a_1 + \alpha) = 0. \quad (30)$$

First solving Eq. (30) and then quadric Eqs. (29) for each $r_i$, we find the eigenvalues of the subblock $\mathbf{V}_1^{(1)}$. The solution of secular equations of second-order subblocks causes no difficulties. As a result, we can obtain the complete set of eigenvalues of the transfer matrix $\mathbf{V}_1$. We note that all eigenvalues of $\mathbf{V}_1$ satisfy the $\lambda\lambda$ symmetry [9]: the eigenvalues $\lambda_i$ can be divided into pairs such that

$$\lambda_1\lambda_2 = \lambda_3\lambda_4 = \ldots \quad (31)$$

It is the $\lambda\lambda$ symmetry that reduces the Galois group and leads to the reciprocal property for the secular equations of the transfer matrix and its subblocks.

The largest eigenvalue of the transfer matrix always lies in the subblock of the identical IR (this follows from the Perron theorem [10]) and is in our case given by

$$\lambda_1 = \frac{1}{2}r_1 + \left(\frac{1}{4}r_1^2 - \alpha^2\right)^{1/2} \quad (32)$$

with

$$r_1 = \frac{1}{2}(a_1 - \alpha) + \left[\frac{1}{4}(a_1 + \alpha)^2 + \alpha^2 - a_2\right]^{1/2}. \quad (33)$$

We now return to the second model. The secular equation of the subblock corresponding to the identical IR is also reciprocal:

$$\lambda^5 - b_1\lambda^4 + b_2\lambda^3 - \gamma b_2\lambda^2 + \gamma^3 b_1\lambda - \gamma^5 = 0, \quad (34)$$

where

$$b_1 = 12\,\mathrm{ch}(2K_A)\,\mathrm{ch}(2K_B) + 2\exp[2(K_A + K_B)] \times \\ \times \mathrm{ch}[4(K_{AB} + K'_{AB})] + \\ + 2\exp[-2(K_A + K_B)]\,\mathrm{ch}[4(K_{AB} - K'_{AB})], \quad (35)$$

$$b_2 = 24\,\mathrm{ch}(2K_A)\,\mathrm{ch}(2K_B)\{\exp[2(K_A + K_B)] \times \\ \times \mathrm{ch}[4(K_{AB} + K'_{AB})] + \\ + \exp[-2(K_A + K_B)]\,\mathrm{ch}[4(K_{AB} - K'_{AB})]\} - \\ - 4\,[2 + \exp(4K_A) + \exp(4K_B)] \times \\ \times \mathrm{ch}[4(K_{AB} + K'_{AB})] - \\ - 4\,[2 + \exp(-4K_A) + \exp(-4K_B)] \times \\ \times \mathrm{ch}[4(K_{AB} - K'_{AB})] + \\ + 8\,\mathrm{ch}[4(K_A + K_B)] + 4[\mathrm{ch}(4K_A) + \mathrm{ch}(4K_B)] + \\ + 8\,\mathrm{sh}^2[2(K_A - K_B)] - \\ - 16[\mathrm{ch}(4K_{AB}) + \mathrm{ch}(4K'_{AB})], \quad (36)$$

and

$$\gamma = 4\,\mathrm{sh}(2K_A)\,\mathrm{sh}(2K_B). \quad (37)$$

This permits us to find the largest eigenvalue of the transfer matrix $\mathbf{V}_2$, which is most important in applications,

$$\Lambda = \frac{1}{2}h_1 + \left(\frac{1}{4}h_1^2 - \gamma^2\right)^{1/2}, \quad (38)$$

where

$$h_1 = \frac{1}{2}(b_1 - \gamma) + \left[\frac{1}{4}(b_1 + \gamma)^2 + \gamma^2 - b_2\right]^{1/2}. \quad (39)$$

The secular equation of the $4 \times 4$ subblock is also reciprocal. Therefore, it can be solved by square radicals. As a result, we also determine all eigenvalues of the transfer matrix $\mathbf{V}_2$. They again have the $\lambda\lambda$ symmetry property.

To conclude this section, we note the following. It is not possible to generalize Hamiltonians (3) and (4) and at the same time preserve the above reduction of the original problem: the attempts to include the additional single (external field), pair or multiparticle interactions in the Hamiltonians immutablely lead to the destruction of the obvious or hidden symmetries.

## 3. SPECIFIC HEAT OF THE FRUSTRATED CHAIN SYSTEM

Ising magnets with triangle lattices are an example of frustrated systems [11]. We illustrate the peculiarities of the specific heat behavior in such systems using the solution obtained for model (4).

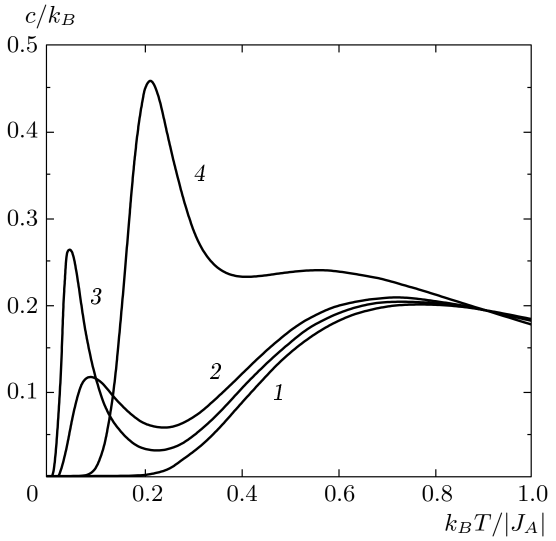


**Fig. 3.** Specific heat behavior in the frustrated systems: *1* — $J_A = J_B = J_{AB} = J'_{AB} = -1$; *2* — $J_A = -1$, $J_B = -0.8$, $J_{AB} = -0.9$, $J'_{AB} = -1$; *3* — $J_A = -1$, $J_B = -0.85$, $J_{AB} = -0.99$, $J'_{AB} = -0.95$; *4* — $J_A = -1$, $J_B = -1$, $J_{AB} = -0.9$, $J'_{AB} = -0.8$

The free energy per site of an infinitely long chain is given by

$$f(T) = -\frac{1}{4} k_B T \ln \Lambda, \qquad (40)$$

where $\Lambda$ is the largest eigenvalue of the transfer matrix $\mathbf{V}_2$. Taking the standard relations between the thermodynamical quantities into account, we then have from (40) the specific heat

$$c = \frac{k_B \beta^2}{4\Lambda} \left[ \frac{\partial^2 \Lambda}{\partial \beta^2} - \frac{1}{\Lambda} \left( \frac{\partial \Lambda}{\partial \beta} \right)^2 \right]. \qquad (41)$$

Substituting the expression for $\Lambda$ found in the previous section and performing the necessary differentiations, we arrive at an analytic formula for the specific heat of our system. From this formula, it follows that at high temperatures, the specific heat behaves as

$$\frac{c(T)}{k_B} \approx \frac{J_A^2 + J_B^2 + 2(J_{AB}^2 + {J'}_{AB}^2)}{8(k_B T)^2}, \quad T \to \infty. \qquad (42)$$

In the other limit, as $T \to 0$, the specific heat, a continued function of temperature, tends to zero and $c = 0$ at $T = 0$. This agrees with the Nernst theorem [12]. In an intermediate region ($0 < T < \infty$), the specific heat being a positive function, has one or more maxima in accordance with the Rolle theorem.

We have investigated the specific heat behavior as a function of temperature numerically using the analytic formula. For $k_B T/|J_A|$ in the region [0, 1], we calculated the specific heat for a frustrated system with

$$J_A = J_B = J_{AB} = J'_{AB} < 0$$

and for systems in which these negative exchange integrals are weakly disturbed (almost frustrated systems). The results are shown in Fig. 3.

The specific heat in the frustrated chain (curve *1* in Fig. 3) has one maximum. However, by any amount of disturbance of the absolute equality between the antiferromagnetic exchange constants, the second peak arises on the specific heat curve in the low-temperature region (curves *2*–*4*). The additional peak is very sensitive even to the smallest distortions of the equilateral triangle structure and hence the low-temperature maximum can serve as their indicator.

## 4. CONCLUSIONS

In this paper, we have obtained two different solutions of Ising model on the $2 \times 2 \times \infty$ lattices using the same obvious invariance, $\mathbf{Z}_2 \times \mathbf{C}_{2v}$, and the hidden $\lambda\lambda$ symmetry. But in one case, the symmetry planes $\sigma_v$ and $\sigma'_v$ of the group $\mathbf{C}_{2v}$ pass through the opposite faces of a $2 \times 2 \times \infty$ parallelepiped, and in other case, those symmetry planes pass through the opposite linear chains of a $2 \times 2 \times \infty$ system.

Both solutions found are unique in that they cannot be generalized with preservation of the combined symmetry, $\mathbf{Z}_2 \times \mathbf{C}_{2v}$ and $\lambda\lambda$.

We hope the presented solutions will be useful in statistical mechanics and in the theory of many-chain magnetic (and other) materials.

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