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# KERR–GAUSS–BONNET BLACK HOLES: EXACT ANALYTIC SOLUTION

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Gauss–Bonnet gravity provides one of the most promising frameworks to study curvature corrections to the Einstein action in supersymmetric string theories, while avoiding ghosts and keeping second-order field equations. Although Schwarzschild-type solutions for Gauss–Bonnet black holes have been known for a long time the Kerr–Gauss–Bonnet metric was missing. In this paper, a five dimensional Gauss–Bonnet solution is analytically derived for spinning black holes and briefly outlined.

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## 1. INTRODUCTION

In any attempt to perturbatively quantize gravity as a field theory, higher-derivative interactions must be included in the action. Such terms also arise in the effective low-energy action of string theories. Furthermore, higher-derivative gravity theories are intrinsically attractive because in many cases they display features of renormalizability and asymptotic freedom. Among such approaches, Lovelock gravity [1] is especially interesting because the resulting equations of motion contain no more than second derivatives of the metric, include the self interaction of gravitation, and are free of ghosts in the expansion around flat space. The four-derivative Gauss–Bonnet term is most probably the dominant correction to the Einstein–Hilbert action [2] when considering the dimensionally extended Euler densities used in the Lovelock Lagrangian, which straightforwardly generalizes the Einstein approach in $(4+n)$ dimensions. The action is therefore given by

$$S_{GB} = \frac{1}{16\pi G}\int d^D x\sqrt{-g}\Big[-2\Lambda + R + \\ + \alpha(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2)\Big], \quad (1)$$

where $\alpha$ is a coupling constant of dimension $(\text{length})^2$ and $G$ is the $D$-dimensional Newton constant defined as $G = 1/M_*^{D-2}$ in terms of the fundamental Planck scale $M_*$. Gauss–Bonnet gravity was shown to exhibit a very rich phenomenology in cosmology (see, e.g., [3] and the references therein), high-energy physics (see, e.g., [4] and the references therein) and black-hole theory (see, e.g., [5] and the references therein). It also provides interesting solutions to the dark energy problem [6], offers a promising framework for inflation [7, 8], allows a useful modification of the Randall–Sundrum model [9], and, of course, resolves most divergences associated with the endpoint of the Hawking evaporation process [10].

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Either in $D$ dimensions or in 4 dimensions with a dilaton coupling (required to make the Gauss–Bonnet term dynamical), Gauss–Bonnet black holes and their rich thermodynamical properties [11] have only been studied in the nonspinning (i.e., Schwarzschild-like) case. Although some general features can be derived in this framework, it remains largely unrealistic because both astrophysical black holes and microscopic black holes possibly formed at colliders [12–14] are expected to be rotating (i.e., be Kerr-like). Of course, the latter — which should be copiously produced at the Large Hadron Collider if the Planck scale is in the TeV range as predicted by some large-extra-dimension models [15] — are especially interesting for the Gauss–Bonnet gravity because they could be observed in the high-curvature region of general relativity and allow a direct measurement of the related coupling constant [4]. The range of impact parameters corresponding to the formation of a nonrotating black hole being of zero measure, the Schwarzschild or Schwarzschild–Gauss–Bonnet solutions are mostly irrelevant. This is also of experimental importance because only a few quanta should be emitted by those light black holes, evading the Gibbons [16] and Page [17] arguments usually adduced to neglect the angular momentum of primordial black holes.

It should be underlined that $D$-dimensional spinning black hole solutions are anyway very important within different theoretical frameworks (e.g., in conservation law studies) [18]. Thanks to perturbation theory several attempts were made [19] to derive the solution. In what follows, we focus on an analytical approach.

## 2. 5*D*-SOLUTION

To investigate the detailed properties of black holes in the Lovelock gravity, it is mandatory to derive the general solution, i.e., the metric for the spinning case. In contrast to the numerical attempts that were presented in Ref. [20] for degenerate angular momenta, we focus on the exact solution in 5 dimensions.

The Einstein equations in Gauss–Bonnet gravity with a cosmological constant $\Lambda$ are given by

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \\ = \Lambda g_{\mu\nu} + \alpha\left[\frac{1}{2}g_{\mu\nu}\left(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2\right) - \right. \\ \left. - 2RR_{\mu\nu} + 4R_{\mu\gamma}R^{\gamma}_{\nu} + 4R_{\gamma\delta}R^{\gamma\delta}_{\mu\nu} - 2R_{\mu\gamma\delta\lambda}R^{\gamma\delta\lambda}_{\nu}\right], \quad (2)$$

and the 5-dimensional metric of the spherically symmetric Kerr–Schild type can be written as

$$ds^2 = dt^2 - dr^2 - (r^2 + a^2)\sin^2\theta d\phi_1^2 - \\ - (r^2 + b^2)\cos^2\theta d\phi_2^2 - \rho^2 d\theta^2 - \\ - 2dr\left(a\sin^2\theta d\phi_1 + b\cos^2\theta d\phi_2\right) - \\ - \beta\left(dt - dr - a\sin^2\theta d\phi_1 - b\cos^2\theta d\phi_2\right)^2, \quad (3)$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta + b^2\sin^2\theta$$

and $\beta = \beta(r,\theta)$ is the unknown function.

The $\theta\theta$ component of the Einstein equations is

$$A\beta'' + B\beta'^2 + C\beta' + D\beta + E = 0, \quad (4)$$

where

$$A = r\rho^2(4\alpha\beta - \rho^2),\\ B = 4\alpha r\rho^2,\\ C = 2\left[4\alpha\beta(\rho^2 - r^2) - \rho^2(\rho^2 + r^2)\right],\\ D = 2r(2r^2 - 3\rho^2),\\ E = 2r\Lambda\rho^4.$$

Equation (4) can be split into two relations respectively involving only $\beta$ and $z \equiv \beta\beta'$ as independent unknown functions. It is then possible to introduce a new function $f(r,c)$, where

$$c = a^2\cos^2\theta + b^2\sin^2\theta,$$

such that the equations are equivalent to the system

$$\beta'' + 2\left(\frac{\rho^2+r^2}{r\rho^2}\beta' - \frac{2r^2-3\rho^2}{\rho^4}\beta - \Lambda\right) - \frac{f(r,c)}{r\rho^4} = 0, \quad (5)$$

$$z' + 2\frac{\rho^2 - r^2}{r\rho^2}z - \frac{1}{2}\frac{f(r,c)}{\alpha r\rho^2} = 0. \quad (6)$$

With the new function $p(r,c)$ introduced via the transformation

$$f(r,c) = \frac{\rho^4}{r}\frac{\partial p(r,c)}{\partial r}, \quad (7)$$

the second equation can be solved (with $p_r \equiv \equiv \partial p(r,c)/\partial r$), leading to

$$z = \frac{1}{2}\frac{(\int p_r dr + 2C_{21}\alpha)(r^2 + c^2)}{\alpha r^2} = (\beta\beta'), \quad (8)$$

where $C_{ij}$ are constants of integration in the $i$th equation. This equation can be integrated to obtain

$$\beta^2 = \frac{1}{\alpha}\int\left(p\frac{r^2+c^2}{r^2}\right)dr + 2C_{21}\frac{r^2-c^2}{r} + C_{20}. \quad (9)$$

The first equation results in

$$\beta = \left(C_{12}r - C_{11}(r^2-c^2) - r\int\frac{(p_r-2r^2\Lambda)(r^2-c^2)}{r}dr + (r^2+c^2)\int(p_r+2\Lambda r^2)\,dr\right)\frac{1}{r(r^2+c^2)}, \quad (10)$$

where a simple integration by parts

$$\int p_r\frac{r^2-c^2}{r}dr = p\frac{r^2-c^2}{r} - \int p\frac{r^2+c^2}{r^2}dr \quad (11)$$

yields

$$\beta r(r^2+c^2) = C_{12}r + C_{11}(r^2-c^2) + r\int\left(p\frac{r^2+c^2}{r^2}\right)dr + \frac{\Lambda r^3}{6}(r^2+c^2). \quad (12)$$

With the same integral combination

$$Q = \int\left(p\frac{r^2+c^2}{r^2}\right)dr, \quad (13)$$

the system leads to the quadratic equation

$$\alpha\beta^2 - (r^2+c^2)\beta + \left(C_{32} + C_{31}\frac{r^2-c^2}{r} + \frac{\Lambda r^2}{6}(r^2+2c^2)\right) = 0, \quad (14)$$

where $C_{3i}$ are new integration constants obtained from a combination of $C_{2i}$ and $C_{1i}$.

Taking the asymptotic forms at infinity into account (and therefore finding the values of the integration constants, $M$ being the Arnowitt–Deser–Misner (ADM) mass), we obtain

$$\beta = \frac{\rho^2 \pm \sqrt{\rho^4 - 4\alpha M - \frac{2}{3}\alpha\Lambda r^2(2\rho^2 - r^2)}}{2\alpha}, \quad (15)$$

where the "−" branch should be chosen so as to recover the usual Kerr solution in the limit $\alpha \to 0$. When $\alpha \to 0$, we recover the pure Kerr case [21]. In case of a vanishing rotation ($a = b = 0$), the obtained solution corresponds to the one suggested in Ref. [22]. When used in metric (3), this leads to the exact Kerr–Gauss–Bonnet–(anti)de Sitter solutions of Einstein equations. Because only the $\theta\theta$ component of the field equations was used to derive this result, the compatibility with the other components was carefully checked. Although the equations are far too intricate to allow analytic investigations, numerical results show that they are indeed satisfied.

## 3. TRANSFORMATION TO THE BOYER–LINGUIST FORM

To obtain the value of the horizon radius $r_h$, it is necessary to transform metric (3) back to the Boyer–Linguist form with

$$dt' = A\,dt + B\,dr + C d\theta,$$

$$d\phi_1' = D\,d\phi_1 + E\,dr + F\,d\theta,$$

$$d\phi_2' = D\,d\phi_2 + H\,dr + F\,d\theta.$$

Taking into account that the processes relevant for thermodynamical investigations occur in the vicinity of the horizon, $M/\rho^2$ can be considered a small parameter and used for a Taylor expansion of $\beta$ as

$$\beta \approx \frac{M}{\rho^2} + \frac{8M^2\alpha}{\rho^6}.$$

As a necessary condition, the Boyer–Linguist parameterization imposes vanishing coefficients for nondiagonal components except for $dt\,d\phi_1$ and $dt\,d\phi_2$. This leads to a system of light equations with light variables. The solutions are explicitly the Boyer–Linguist parameterization of the Kerr–Gauss–Bonnet metric.

Solving those equations (without substituting the direct expression of $\rho(r,\theta)$), we obtain that

i) all the coefficients before the components $d\theta\,dx$ (where $x$ is an arbitrary coordinate) vanish automatically, as in the classical Kerr case;

ii) the coefficients $A$ and $D$ can be set equal to 1 to recover the classical case;

iii) other coefficients are:

$$B = B_1/B_2,$$

$$E = E_1/E_2,$$

$$H = H_1/H_2,$$

where

$$B_1 = -\rho^6 a(r^2+b^2),$$

$$B_2 = r^4\rho^6 + 8M^2\alpha a^2\cos^2\theta r^2 - M\rho^4 a^2\cos^2\theta r^2 + \\ + \rho^6 b^2 r^2 + \rho^6 b^2 a^2 - M\rho^4 r^4 + r^2\rho^6 a^2 + 8M^2\alpha r^4 - \\ - M\rho^4 b^2 r^2 + 8M^2\alpha b^2 r^2 + M\rho^4 b^2\cos^2\theta r^2 - \\ - 8M^2\alpha b^2\cos^2\theta r^2,$$

$$E_1 = -(a^2+r^2)\rho^6 b,$$

$$E_2 = r^4\rho^6 + 8M^2\alpha a^2\cos^2\theta r^2 - M\rho^4 a^2\cos^2\theta r^2 + \\ + \rho^6 b^2 r^2 + \rho^6 b^2 a^2 - M\rho^4 r^4 + r^2\rho^6 a^2 + 8M^2\alpha r^4 - \\ - M\rho^4 b^2 r^2 + 8M^2\alpha b^2 r^2 + M\rho^4 b^2\cos^2\theta r^2 - \\ - 8M^2\alpha b^2\cos^2\theta r^2,$$

$$H_1 = Mr^2(8M\alpha a^2\cos^2\theta - \rho^4 a^2\cos^2\theta - \rho^4 r^2 + \\ + 8M\alpha r^2 - \rho^4 b^2 + 8M\alpha b^2 + \rho^4 b^2\cos^2\theta - 8M\alpha b^2\cos^2\theta),$$

$$H_2 = r^4\rho^6 + 8M^2\alpha a^2\cos^2\theta r^2 - M\rho^4 a^2\cos^2\theta r^2 + \\ + \rho^6 b^2 r^2 + \rho^6 b^2 a^2 - M\rho^4 r^4 + r^2\rho^6 a^2 + 8M^2\alpha r^4 - \\ - M\rho^4 b^2 r^2 + 8M^2\alpha b^2 r^2 + M\rho^4 b^2\cos^2\theta r^2 - \\ - 8M^2\alpha b^2\cos^2\theta r^2.$$

After substituting these coefficients in metric (3) and some rearrangements, the metric becomes

$$ds^2 = dt^2 - (r^2+a^2)\cos^2\theta\, d\phi_1^2 - (r^2+b^2)\sin^2\theta\, d\phi_2^2 - \rho^2 d\theta^2 - \left(\frac{M}{\rho^2} + \frac{8M^2\alpha}{\rho^6}\right) \times \\ \times \left(dt + a\sin^2\theta\, d\phi_1 + b\cos^2\theta\, d\phi_2\right)^2 - \frac{\Phi}{\rho^6\left((r^2+a^2)(r^2+b^2) - r^2\rho^2\left(\frac{M}{\rho^2} + \frac{8M^2\alpha}{\rho^6}\right)\right)}\, dr^2, \quad (16)$$

where $\Phi$ is a coefficient whose value is irrelevant because this investigation requires only the denominator of the last term (the $g_{11}$ component of the metric), which is

$$\rho^6\left((r^2+a^2)(r^2+b^2) - r^2\rho^2\beta\right). \quad (17)$$

## 4. THERMODYNAMICAL PROPERTIES

To investigate the black hole topology, we must study singular points of the metric component $g_{11}$, i.e., study the zeros of expression (17):

$$(r^2+a^2)(r^2+b^2) - r^2\rho^2\beta = 0. \quad (18)$$

This is an 8th-order equation for $\rho$ when $\beta$ is Taylor expanded in the lowest order in $\alpha$. As shown in Ref. [23], the cosmological constant can change the temperature. In what follows, we restrict our study to the $\Lambda = 0$ case. Using the value of $\beta$ in (15), we obtain

$$M = \frac{M^*}{4\alpha r^4\rho^4}, \quad (19)$$

where

$$M^* = r_+^4\rho^8 - 4\alpha(r_+^2+a^2)^2(r_+^2+b^2)^2 + \\ + 4\alpha r_+^2\rho^4(r_+^2+a^2)(r_+^2+b^2) + r_+^8\rho^4$$

and $r_+$ is the horizon radius. As $\alpha \to 0$, this leads to the usual Kerr case.

We emphasize that the anglular variable $\theta$ is included in expression (19) (as $\rho^2 = r_+^2 + a^2\cos^2\theta + b^2\sin^2\theta$), which indicates a good choice of coordinates. To remove this dependence, we must set $\theta = \pi/4$. This allows computing the temperature, which requires the surface gravity given by

$$\kappa^2 = -\frac{1}{4}g^{tt}g^{ij}(\partial_i g_{tt})(\partial_j g_{tt})\Big|_{r=r_+}. \quad (20)$$

In the considered case, this leads to

$$\kappa = -\frac{1}{4}(1+\beta)\left[g^{rr}(\partial_r\beta)^2 + g^{\theta\theta}(\partial_\theta\beta)^2\right]\Big|_{r=r_+}. \quad (21)$$

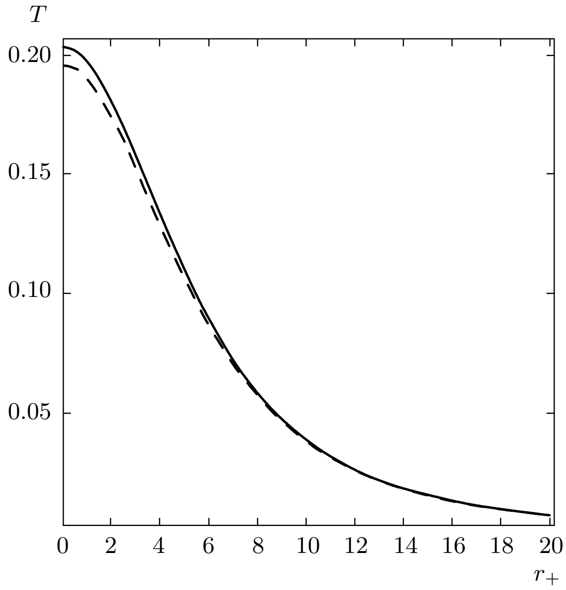
After substituting all the values, this formula becomes

$$\kappa = -\frac{1}{4}(1+\beta)\left(\frac{\kappa_1}{\kappa_2} - \frac{\left(\frac{\partial}{\partial\theta}\beta\right)^2}{\rho^2}\right), \quad (22)$$

where

$$\kappa_1 = (\beta r^2\cos^2\theta(a^2-b^2) - (r^2+a^2)(r^2+b^2) + \\ + \beta r^2(r^2+b^2))\left(\frac{\partial}{\partial r}\beta\right)^2,$$

$$\kappa_2 = -\cos^2\theta(a^2-b^2) + (r^2+b^2).$$




Black-hole temperature $T$ ($y$ axis, relative Plank values) versus the black-hole size $r_+$ ($x$ axis, relative Plank values) in the pure Kerr case (lower line) and the Kerr–Gauss–Bonnet case (upper line)

The black hole temperature $T$ can be easily computed as $T = \kappa/2\pi$. The pure Kerr $5D$-formula, as given in [24], leads to

$$T = \frac{r_+^2 \Delta'}{4\pi(r_+^2 + a^2)(r_+^2 + b^2)}, \tag{23}$$

where

$$\Delta = \frac{(r_+^2 + a^2)(r_+^2 + b^2)}{r_+^2}.$$

The figure displays the pure Kerr temperature and the Kerr–Gauss–Bonnet temperature. As expected, both values become very close for large masses. They differer by about 5 % in the limit of very small masses for $\alpha = 1$ in Planck units.

## 5. DISCUSSION AND CONCLUSIONS

If, as is suggested by geometrical arguments and by low-energy effective superstring theories, the Gauss–Bonnet gravity is a realistic path toward the full quantum theory of gravity, then Kerr–Gauss–Bonnet black holes are probably among the most important objects to understand the physical basis of our World. This article has established the solution of Einstein equations in the 5-dimensional Gauss–Bonnet theory. This allows investigating into the details of the physics of "realistic" spinning black holes, from both a pure theoretical and a phenomenological (in the framework of low-Planck-scale models) standpoint.

Some improvements and developments can be foreseen. First, it should be very welcome to obtain the same kind of solutions for any number of dimensions. Unfortunately, the method introduced in this article is not easy to generalize and a specific study is required in each case. Then, it would be interesting to compute the greybody factors for those black holes. Following the techniques in [25], it is possible (although not straightforward) to obtain a numerical solution as soon as the metric is known, at least in the $\Lambda = 0$ case. The Kerr–Gauss–Bonnet–(anti)de Sitter situation is more intricate because the metric is nowhere flat, requiring a more detailed investigation, as suggested in Ref. [26].

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