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BKT PHASE IN SYSTEMS OF SPINLESS STRONGLY INTERACTING ONE-DIMENSIONAL FERMIONS

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We present the ground-state wave functions for a system of spinless one-dimensional fermions in the limit of an infinitely strong interaction and demonstrate explicitly that the system symmetry is lower than the original symmetry of the Hamiltonian. As a result, the system in this limit undergoes a second-order phase transition into a phase with finite density of chiral pairs. The phase transforms continuously into a Berezinskii–Kosterlitz–Thouless (BKT) phase if the interaction in the model decreases. Therefore, just the BKT phase is realized in nature. The temperature of the smearing phase transition is calculated.

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1. INTRODUCTION

For a long time, one-dimensional fermion systems were a subject of intensive studies only in theoretical physics. Tomonaga [1] and Luttinger [2] demonstrated in their pioneering papers that the longwave excitations of such a system (under rather general conditions) can be expressed in terms of noninteracting bosons. These degrees of freedom were made explicit in the elegant method of bosonization proposed by Mattis and Leeb [3]. The recent interest in this field is mainly due to the development of submicron techniques, which allowed producing very pure quantum wires. In such wires, only few levels (or sometimes even one) corresponding to the quantization of electrons in perpendicular directions are occupied. Hence, the systems under discussion are accessible by experiment today (see, e.g., [4–7]).

The bosonization technique allows calculating all n-point correlation functions for systems of interacting fermions in one dimension. However, these correlators give only indirect information about the ground state of the system, which requires further interpretation. The correlation functions of the Luttinger model reveal a number of anomalies of the fermion system (see [8–10]): they have oscillating contributions with wave vectors equal to $2p_f$ or $4p_f$, which decay very slowly with distance. In the literature, these contributions were interpreted as follows: the oscillations with the Fermi momentum p_f doubled were related to the Peierls instability (related to the charge density wave [9, 11]) and the oscillations with the $4p_f$ frequency were interpreted as a marginal Wigner crystal [12]. Although the correlators of chiral complexes obey a power law (see [13]), it is commonly believed that the system under discussion is a kind of normal liquid because quantum fluctuations destroy any order parameter. As a result, a phase with a long-range order is impossible even in the zero-temperature region [9]. The common point of view was formulated as "Luttinger liquid is a normal (not symmetry-broken) metallic phase" [14] with a gapless boson spectrum. However, if we speak about quantum fluctuations, two more points should be taken into account.

1. In low dimensions, some of the systems appear to be in the Berezinskii–Kosterlitz–Thouless phase (BKT) [15, 16]. In this phase, the order parameter density tends to zero in an infinite system, but a long-range order exists because correlation functions decay as some power of the distance. This means that the correlation is present in a whole specimen. As a result, in

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its macroscopic properties, the system are quite similar to a system with broken symmetry. A BKT phase can form if the gapless excitations, which should be present in the system after spontaneous symmetry breaking due to the Goldstone theorem [17, 18], do not interact. At nonzero temperatures ($\Theta \neq 0$), this phase can occur in two-dimensional systems; at $\Theta = 0$, a BKT phase is possible in one dimension. (The crossover temperature and its relation to the smearing phase transition temperature Θ_c is considered below; see Sec. 3.)

2. The Goldstone theorem itself can be broken in one-dimensional models with the Adler–Schwinger anomaly. For that to happen, the interaction has to be strong enough. As an illustration one can consider the massless Schwinger model [19]. In the Coulomb gauge, this model is a particular case of the Luttinger model with the potential given by a linear function of the distance between electrons. This leads to the Goldstone theorem violation: all excitations have a gap in spite of spontaneous breaking of the chiral symmetry. For this reason, excitations cannot suppress the long-range order at temperatures below the gap. As a result, a second-order phase transition occurs in one dimension even at finite temperature [20].

This makes the statement that the Luttinger liquid at $\Theta = 0$ is in an unbroken phase doubtful.

To clarify this question, we calculate the wave functions of the ground states in the Tomonaga–Luttinger model directly in the fermion representation. Although the details of these wave functions could depend on the interacting potential, all possible ground states qualitatively reveal the same phenomenon.

In one dimension, the Fermi surface reduces to two isolated points in phase space ($p = \pm p_f$). Transitions between these two points can be neglected. This is a good approximation, at least if the potential is a decreasing function of the momentum transfer. As a result, the number of electrons near each point (left and right particles) must be conserved and the system acquires a complementary (chiral) symmetry. This symmetry, as we see in what follows, breaks down spontaneously in the model.

It is common knowledge that the electron distribution function in one dimension changes drastically even for the systems with a weak electron–electron interaction. It is of the order $1/2$ near the Fermi level [21]. This means that a hole is located near each electron. Naturally, they attract each other and form a kind of bound state consisting of a right electron and a left hole ($R\bar{L}$-pair). This is quite similar to the formation of a Cooper pair in a superconductor, but the quantum numbers of the bound state are different: instead of a nonzero electric charge, exciton-like (neutral) pairs with nonzero chirality occur[1)].

Of course, this fact itself is not enough to speak about a new correlated phase. Using the explicit ground-state wave function constructed in this paper, we verify that long-range order is indeed present in the system. As a result, the Luttinger liquid undergoes a phase transition at low temperatures.

We see in what follows that in the limit of the infinitely strong interaction and at zero temperature, the system is in a phase with broken chiral symmetry and nonvanishing order parameter density. The properties of the Luttinger liquid in this limit are quite analogous to the properties of the massless Schwinger model, where the spontaneous breakdown of the chiral symmetry is well known. This can be expected because the interaction in the Schwinger model is also infinite (it increases with distance). On the other hand, in contrast to the Schwinger model, the spectrum of the Luttinger model remains gapless.

If the interaction in the Luttinger model is considered finite, as it is realized in nature, the order parameter density vanishes (in the infinite system). In this case, the system appears to be in a BKT phase with correlators decaying as some power of the distance. But the properties of this phase are rather close to those of a system with a nonzero order parameter. The BKT system transforms smoothly into the phase with a nonzero order parameter as the coupling constant increases.

The main point of this paper is that the existence of a symmetry-broken phase underlies the existence of anomalies in the correlation functions. We believe that the usual interpretation of the anomalies as an instability of the Peierls type is misleading. There is a clear-cut distinction between the chiral phase and the Peierls one. Indeed, the phase transition to the Peierls phase is a second-order phase transition in the phonon system, while the chiral symmetry of electrons is broken explicitly, in the Hamiltonian. On the contrary, the chiral phase originates from the spontaneous symmetry breaking in the electron system. To manifest the breakdown of the chiral symmetry in the Luttinger liquid, we exactly calculate the ground-state wave function for the model in the fermion representation and demonstrate explicitly that its symmetry is lower than the original symmetry of the Hamiltonian. (This is the definition of spontaneous symmetry breaking.)

As regards, possible observations of a condensate

[1)] To draw an analogy between an exciton-like pair and a bound state, we should take into account that we have a correlation between the filled states rather than a real bound pair.

we note that a charge-neutral condensate cannot reveal itself in experiments associated with charge transfer. But it contributes to the effects involving energy currents and should not transfer heat. Hence, we can think about thermal anomalies related to the condensate. In fact, we keep in mind the effect similar to the thermomechanical effect in superfluid helium. (The temperature decreases with an increase in the superfluid mass [22].) We plan to discuss this problem elsewhere.

This paper is organized as follows. In Sec. 2, we introduce the Hamiltonian, the definitions of left and right particles, and so on. We present our results and discussion in Sec. 3. We relegate the derivation of the results to Sec. 4 because the calculation is rather cumbersome. In Sec. 5, we show the symmetry breaking in a different way, namely, from the well-known ground-state wave function in the boson representation [23]. Justifiably, the boson representation may be considered a nonobvious way to see the symmetry breaking in a fermion system. However, we perform the calculation in order to compare our approach with a common method. We conclude the paper with three appendices that contain some mathematical details related to the calculation.

2. NOTATION AND GENERAL EQUATIONS

We begin with the usual separation of left and right particles in the electron wave functions $\hat{\Psi}(x)$ [24]:

$$\hat{\Psi}(x) = \exp(ip_f x)\,\hat{\Psi}_R(x) + \exp(-ip_f x)\,\hat{\Psi}_L(x). \quad (1)$$

It is implied here that the wave functions $\hat{\Psi}_{R,L}(x)$ vary over distances much longer than $1/p_f$. We also restrict ourself to the Tomonaga–Luttinger model [24]. For simplicity, we consider only an electrically neutral system, where a positive charge of ions is distributed homogeneously along the channel.

We pass to the electron–hole representation for the right (left) particles:

$$\hat{\Psi}_{R,L}(x) = \int_0^\infty \frac{dp}{2\pi}\left(\exp(\pm ipx)\,\hat{a}_{R,L}(p) + \exp(\mp ipx)\,\hat{b}^\dagger_{R,L}(p)\right) = \hat{a}_{R,L}(x) + \hat{b}^\dagger_{R,L}(x), \quad (2)$$

where $\hat{a}^\dagger(\hat{a})$ and $\hat{b}^\dagger(\hat{b})$ are creation (annihilation) operators for electrons and holes. The Hamiltonian of interacting spinless electrons in one dimension can be written for a neutral system in terms of the electron density operator

$$\varrho(x) = \varrho_R(x) + \varrho_L(x)$$

as

$$H = \int dx \times \left[\hat{\Psi}^\dagger_R(x)\, v_f\,(-i\partial_x)\,\hat{\Psi}_R(x) + \hat{\Psi}^\dagger_L(x)\, v_f\, i\partial_x \hat{\Psi}_L(x)\right] + \int dx\, dy\, \varrho(x)\, V(x-y)\, \varrho(y), \quad (3)$$

where v_f is the Fermi velocity.

The form of $V(x-y)$ depends on the relation between the usual $3D$ screening radius R_D (we consider the case of Debye screening for simplicity) and the transverse size of the channel d. Indeed, we must take into account that the electrons are one-dimensional only for distances $|x-y|$ much larger than d. Therefore, if $R_D \ll d$, we can use a point-like interaction $V_0\delta(x-y)$[2]. This is the case for metals[3]. In the opposite case $R_D \gg d$ (a semiconductor), an ordinary Coulomb potential must be used. In what follows, we restrict ourself to the simplest case, a point-like interaction. Thus,

$$V(p) = V_0. \quad (4)$$

The Hamiltonian of the Luttinger model in Eq. (3) represented in terms of electrons and holes is completely defined without any additional regularization of the electron operators. In particular, the commutator of the R- and L-densities in this representation reproduces the well-known Schwinger anomaly [19]:

$$[\varrho_{R,L}(x), \varrho_{R,L}(y)] = \pm\frac{i}{2\pi}\frac{\partial}{\partial x}\delta(x-y). \quad (5)$$

[2] We note that this case also includes backscattering of electrons with the transition $L \to R$ and back:

$$\int dx \hat{\Psi}^\dagger_R(x)\Psi_L(x)\hat{\Psi}^\dagger_L(x)\Psi_R(x).$$

By anticommuting the $\hat{\Psi}$-operators, we can reduce this term to the term without backscattering [9]; for this model, therefore, the Tomonaga–Luttinger Hamiltonian describes the entire electron–electron interaction.

[3] To obtain a real parameter, we should use the standard expression for the Debye screening radius $R_D = 1/\sqrt{4\pi e^2 \partial n/\partial\mu}$ with the concentration $n = p_f/\pi^2 d^2$. We took into account here that there is only one state for electrons in directions perpendicular to the channel. Thus, $R_D \approx (d/2)\sqrt{\pi p_f a_b}$, where $a_b = (me^2)^{-1}$ is the Bohr radius. Hence, the condition $R_D \ll d$ is equivalent to $p_f a_b \ll 1$. The last parameter depends only on the effective mass of the electron. If it is of the order of the free electron mass, then $a_b \approx 0.5 \cdot 10^{-8}$ cm and we obtain $p_f a_b \ll 1$ for the concentrations typical for metals. In other words, we then deal with a short-range interaction.

These relations are the starting point of the bosonization technique. Equations (5) are usually derived by regularizing the product of Ψ operators by point splitting [25]. This is not necessary, however, in the electron–hole representation [20] because the creation and annihilation operators for R, L electrons ($\hat{a}^{\pm}_{R,L}(x)$) and holes ($\hat{b}^{\pm}_{R,L}(x)$) are nonlocal in the coordinate space as their anticommutators are:

$$\{\hat{a}^{\dagger}_{R}(x),\hat{a}_{R}(x_1)\} = \{\hat{b}^{\dagger}_{R}(x),\hat{b}_{R}(x_1)\} = \\ = \frac{1}{2\pi i}\,\frac{1}{x-x_1-i\delta}, \quad (6)$$

$$\{\hat{a}^{\dagger}_{L}(x),\hat{a}_{L}(x_1)\} = \{\hat{b}^{\dagger}_{L}(x),\hat{b}_{L}(x_1)\} = \\ = \frac{1}{2\pi i}\,\frac{1}{x_1-x-i\delta} \quad (7)$$

(in momentum space, these anticommutators are δ-functions). Using these anticommutators for the densities of right and left electrons $\varrho_{R,L}(x)$ of form (75), we immediately reproduce the Schwinger anomaly. This means that, being formulated in the electron–hole representation, our theory is completely defined without any further redefinition of the density operators.

Besides, Hamiltonian (3) is invariant under the chiral transformations

$$\Psi_R(x) = e^{i\alpha_c}\Psi_R, \quad \Psi_L(x) = e^{-i\alpha_c}\Psi_L, \quad (8)$$

where α_c is a constant parameter of the transformation. This invariance leads to conservation of the chiral charge (the difference of the numbers of right and left electrons). However, we see below that the ground state of the model is constructed such that the symmetry is spontaneously broken.

3. THE APPROACH, RESULTS, AND DISCUSSION

The standard approach to many-particle systems is based on Green's functions. The one-particle Green's function gives the information about the spectrum of excitations; the many-particle Green's functions allow calculating different correlation and response functions. Of course, the Green's functions give some information about the wave functions of the states, but this information is indirect.

In principle, the wave functions of stationary states (and of the ground state in particular) can be obtained by solving the corresponding Schrödinger equation directly. However, for systems with an infinite number of the degrees of freedom, this equation is too complicated. A more practical approach can be based on the evolution operator [26]

$$S(T) = \sum_{m,n} |n\rangle\langle n|\exp -iHT|m\rangle\langle m|, \quad (9)$$

where $|n\rangle$ are the exact wave functions of the Hamiltonian H in the second-quantized representation and T is the time of observation. The evolution operator determines the evolution of an arbitrary initial wave function ($\langle m|$) from the instant $t=0$ to final states $|n\rangle$ at $t=T$. (We assume from now on that the Schrödinger representation for operators with time-dependent wavefunctions is used.)

Formula (9) suggests the general method to obtain wave functions. We first calculate the evolution operator and represent it as a sum of time-dependent exponentials. The coefficients in front of these exponentials are products of exact wave functions and their complex conjugates. To extract the ground-state wave function, we must take the limit $T\to\infty$ (with an infinitesimal imaginary part added to the energy). Passing to the Euclidean time ($T\to -i/\Theta$), we see that the evolution operator determines the density matrix for the equilibrium system at nonzero temperature (see end of this section).

The advantage of this method is that the evolution operator can be written explicitly as a functional integral with definite boundary conditions (see Eq. (22)). The expression is nontrivial because it allows rewriting the wave function in the second-quantized representation as a functional integral with unusual boundary conditions at $t=0$ and $t=T$. This is possible because creation and annihilation operators in Eq. (9) anticommute once they correspond to different instances of time (see Appendix A for the details). That is, they should be considered Grassmann variables. (An analogous representation for the Feynman Green's function describes the vacuum–vacuum transitions. In this case, therefore, Grassmann variables obey zero boundary conditions. This distinction is extremely essential.) The functional integral is rather simple for the Luttinger model with Hamiltonian (3) and can be calculated exactly. This allows constructing wave functions of all states in the model and, in particular, the ground state in terms of the electron and hole operators. This suffices to demonstrate the symmetry breaking. (See Sec. 5 for further comparison of the approach and the bosonization one.)

We keep the size of the system finite. This is important not only for regularizing infrared divergences in the system but mostly because the characteristic tem-

peratures of the problem depend on the system size L. Therefore, we have to discuss the concept of phase transition in finite systems. In fact, we identify a parameter that allows applying the concept of phase transition formulated in the thermodynamic limit ($L \to \infty$, with the electron density finite) to a real finite system. We see in what follows that our system is in a certain sense large enough and we can speak about a smearing second-order phase transition for the problem.

Usually, the critical temperature is defined as a point where thermodynamic quantities have a singularity. Of course, this is the case only in an infinite system because all singularities smear out if the size of the system is finite. The same is true for the coherence length, which cannot be larger than the system size.

In this paper, we adopt the point of view suggested by Landau in order to describe second-order phase transitions [27]. He introduced the order parameter as the main quantity for the description of phase transitions related to the spontaneous symmetry breaking. By definition, the order parameter is zero in the high-symmetry phase (with the same symmetry as the Hamiltonian) and nonzero in a phase with broken symmetry. In fact, this is only one of the possible definitions of the broken (unbroken) phase, but we use it below because it is convenient for us. In the case of chiral symmetry breaking in Eq. (8), the following quantity can serve as the order parameter:

$$\Delta = \int dx \langle \Omega | \hat{a}_R^\dagger \left(x + \delta x/2\right) \times \times \hat{b}_L^\dagger \left(x - \delta x/2\right) |\Omega\rangle|_{\delta x \to 0}. \quad (10)$$

This quantity is not invariant under transformations in Eq. (8) and should be zero if the chiral symmetry remains unbroken. We note that we use a macroscopic order parameter (the integral over the entire system). In the broken phase, this quantity is proportional to the volume of the system.

The BKT phase represents the intermediate case where

$$\Delta \sim L^{\alpha_T}, \quad 0 < \alpha_T < 1, \quad (11)$$

and hence Δ is still infinite in the thermodynamic limit, while the density of the order parameter Δ/L vanishes. We note that Δ appears to be nonzero even at $\Theta > \Theta_c$ due to fluctuations of the broken phase in the higher-symmetry phase. It is important that Δ does not increase with L in this case.

Intensive thermodynamic quantities remain smooth for a finite-size system even at the phase transition point. However, an essential circumstance is that they depend explicitly on the system size and tend to infinity (or acquire a jump) as $L \to \infty$.

Usually, one proves that the system is in the BKT phase by investigating the behavior of the four-fermion correlator that does not break the chiral invariance (below, in Sec. 4.1, we consider such a correlator—the probability to find an $R\bar{L}$-pair at a large distance r from an $L\bar{R}$ pair). If such a correlator decreases sufficiently slowly with the distance, the system is in the BKT phase. The limit case where the correlator remains constant at large distances corresponds to a nonzero density of the order parameter and the ordinary broken symmetry. In fact, this definition of the BKT phase is equivalent to our definition given above, but the definition in (10) and (11) is more convenient for us.

In one dimension, the BKT phase can exist only at $\Theta = 0$ or, to be more precise, for temperatures that tend to zero as $L \to \infty$. There is no need in a microscopic theory in order to estimate the characteristic temperature Θ_c at which the system changes one type of behavior for another. To obtain an estimate, we can use the general phenomenology applicable to all BKT systems (see, e.g., [24]). We assume that the chiral symmetry is indeed spontaneously broken in the Luttinger model (of course, this can be proved only in a microscopic theory). According to the Goldstone theorem, the chiral phase α_c (the phase of the $\Psi_R^+\Psi_L$ operator) becomes a massless boson field. In the long-range limit, only fluctuations of this field are relevant and its effective Lagrangian reduces to (Euclidean time $\tau = it$ is used because we want to consider nonzero temperatures below)

$$S_{eff}[\alpha_c] = \frac{V^2}{2} \int d\tau \, d^m x \left[(\partial_t \alpha_c)^2 + (w \partial_x \alpha_c)^2 \right], \quad (12)$$

where V and w are phenomenological constants (calculable in a microscopic theory) and m is number of spatial dimensions.

To decide if the system is in the BKT phase, it suffices to consider the behavior at large distances $|x - y|$ of the chirality-conserving correlator:

$$F(x - y) = \langle \Psi_R^+(x) \Psi_L(x) \Psi_L^+(y) \Psi_R(y) \rangle. \quad (13)$$

We can neglect fluctuations of the modulus of the operator $\Psi_R^+(x)\Psi_L(x)$ (as well as higher derivatives in the effective action for the chiral phase). Correlator (13) then reduces to

$$F(x - y) = \text{const} \int \mathcal{D}\alpha_c \exp(-S[\alpha_c]) \times \times \exp(2i\alpha_c(x)) \exp(-2i\alpha_c(y)). \quad (14)$$

Calculating this integral at $\Theta = 0$, we obtain

$$F(x-y) \sim \exp\left[2iV^{-2}\int \frac{d\omega d^m k}{(2\pi)^{m+1}} \times \right.$$
$$\left. \times \frac{\sin^2\left[(\mathbf{k}\cdot(\mathbf{x}-\mathbf{y}))/2\right]}{\omega^2 - w^2\mathbf{k}^2}\right]. \quad (15)$$

For $m = 1$, the two-dimensional integral (one space and one time dimension) in the exponent diverges logarithmically, and hence

$$F(x-y) \sim \frac{V^2}{(k_{max}|x-y|)^{1/2\pi V^2 w}}. \quad (16)$$

This proves the existence of a BKT phase at $\Theta = 0$.

If the temperature is nonzero, the integral over ω in Eq. (15) should be replaced by a sum over discrete values $\omega_n = 2\pi n\Theta$ (with integer n). At high temperatures, only the term with $n = 0$ survives at large distances and we are left with a one-dimensional integral with respect to k, which leads to the correlator exponentially decreasing with distance:

$$F(x-y) \sim V^2 \exp\left(-\frac{\Theta}{2\pi V^2 w^2}|x-y|\right). \quad (17)$$

Clearly, this correlator describes the unbroken phase.

The power-like behavior of the correlator in (16) is valid in the region

$$|x-y| < w/\Theta.$$

For

$$\Theta < \Theta_c \equiv w/L,$$

this takes place for the entire sample, i.e., the system is in a broken phase. The temperature Θ_c is the temperature of the phase smear transition.

In this estimate, we can recognize the excitation energy with the smallest momentum possible in a finite-size system. In the Luttinger model, this energy is equal to $\omega_{min} = 2\pi v_f^c/L$, with the renormalized Fermi velocity $v_f^c = v_f\sqrt{1 + V_0/\pi v_f}$ [9]. Therefore, we see that if the spectrum of excitations is gapless (as in the Luttinger model), then the phase transition temperature is inversely proportional to the sample length. This result can be obtained in the microscopic theory as well.

At $\Theta \gg \Theta_c$, the integral in Eq. (17) diverges logarithmically for $m = 2$. Therefore, Θ_c can also be considered a crossover point where the critical dimension of the system changes from 1 to 2.

Turning to the microscopic theory, we discuss the simplest case: the short-range potential in the limit of an infinitely strong interaction

$$\frac{\pi v_f}{V_0} \ll 1. \quad (18)$$

In the leading order in this parameter, the evolution operator appears to be very simple and the ground-state wave function can be represented in a closed form. In the temperature region

$$\Theta_{chiral} = \frac{2\pi v_f}{L} \ll \Theta \ll \Theta_c = \omega_{min}, \quad (19)$$

the ground-state wave function is of the form[4)]

$$|\Omega\rangle_\theta = \sqrt{Z_0}\exp\left[\int dx\, \exp(i\theta)\,\hat{a}_R^\dagger(x)\,\hat{b}_L^\dagger(x) + \right.$$
$$\left. + \int dy\, \exp(-i\theta)\,\hat{a}_L^\dagger(y)\,\hat{b}_R^\dagger(y)\right]|F\rangle, \quad (20)$$

where $|F\rangle$ is the filled Fermi sphere and Z_0 is the normalization coefficient. There is an infinite set of degenerate ground states labeled by the continuous parameter θ, which has the meaning of the order parameter phase.

The symmetry breaking can be seen immediately because wave function (20) is not invariant under chiral transformation (8). This is the definition of spontaneous symmetry breaking. Besides, it can be verified directly that $\Delta \propto L$[5)]. This means that a second-order phase transition occurs at a higher temperature in this limit.

Wave function (20) is a mixture of states with different chiralities. (We assign chirality +1 to a right electron and a left hole and −1 to their counterparts. Therefore, bosons in Eq. (20) are neutral in terms of electric charge but have the nonzero chiralities ±2). As a result, the order parameter Δ (see Eq. (10)) does not vanish because such a ground state implies that

[4)] Strictly speaking, at nonzero temperatures, the system is described not by the wave function but by the density matrix. But we are interested in the low-temperature region, where the probability to find the system in an excited state is small. In the leading approximation, we can describe such a system by the wave function.

[5)] To calculate this, we use the identity

$$\left[\hat{a}(p), \exp\left(\int \frac{dp'}{2\pi}\hat{a}^\dagger(p')\,\hat{b}^\dagger(p')\right)\right] =$$
$$= \hat{b}^\dagger(p)\exp\left(\int \frac{dp'}{2\pi}\hat{a}^\dagger(p')\,\hat{b}^\dagger(p')\right)$$

and determine the anticommutator with the same momenta in a regular way: $\{\hat{a}^\dagger(p_n), \hat{a}(p_m)\} = L\delta_{n,m}$.

the states with different chiralities are all degenerate in energy and transitions between states with different chiralities in Eq. (10) exist. The degeneracy permits constructing a symmetry-breaking wave function, although the Hamiltonian has no symmetry-breaking terms. This is a typical situation for a system with a condensate: adding one pair to the condensate does not cost any energy. But this degeneracy is possible only if the size of the system is large enough; we see in what follows that it should be $L \gg L_{min} \sim 2\pi v_f/\Theta$. In the opposite case $L \ll L_{min}$, the ground state has a fixed chirality (equal to zero; see the discussion below Eq. (58)) and the order parameter Δ vanishes, i.e., there is no spontaneous symmetry breaking. In fact, the first inequality is the parameter that allows the concept of phase transition, which has been formulated in the thermodynamic limit, to be applied to finite real systems. If the states with different chiralities are degenerate, the sample can be considered infinite with regard to symmetry breaking. (See the end of Sec. 5 for a further discussion of these statements. There, we recalculate Eq. (20) from the boson representation.)

These considerations put a lower bound on the temperature region where a chiral phase can exist: $\Theta \gg \Theta_{chiral}$, with Θ_{chiral} being the degeneration temperature. It is the characteristic energy difference between states with varied chirality.

We estimate the density of chiral pairs in the ground state. Wave function (20) implies that all electrons are bound into pairs. Hence, the density of $R\bar{L}$ coincides with the density of R-electrons:

$$N_R(p) = {}_\theta\langle\Omega|\hat{a}_R^\dagger(p)\,\hat{a}_R(p)\,|\Omega\rangle_\theta = L/2 \qquad (21)$$

(see footnote 5). This quantity reflects a well-known fact: the distribution function of electrons is of the order $1/2$ near the Fermi surface [21]. If the interaction is infinitely strong, all electrons and all holes are bound into exciton-like pairs. As a result, we obtain the value in (21), which is the maximal possible.

In the model under consideration, $N_R(p)$ is momentum independent and the total number of pairs N_R diverges at large p. (This is the defect of the point-like electron–electron interaction.) The sum over all states should be restricted either by p_f or, at $p_f d \gg 1$, by the inverse size of the channel because electrons cannot be considered one-dimensional at larger p (see footnote 3). In the case, therefore,

$$N_R \sim \frac{L}{4\pi d}.$$

Hence, the number of pairs N_R, in the case $p_f d \gg 1$, is only a small fraction of the total number of electrons $(Lp_f/2\pi)$. This does not mean, of course, that the Luttinger liquid behaves like a normal one in this case. The response of the system to slowly varying external fields is completely determined by the electrons near the Fermi surface, which are all paired. This situation resembles superfluid helium, where (even at zero temperature) the density of the condensate is only a few percent of the total one. Nevertheless, the whole mass of helium is superfluid [28] at $\Theta = 0$.

We proceed with the region of high temperatures: $\Theta \gg \Theta_c = \omega_{min}$. In this region, macroscopic order parameter (10) is proportional not to the volume of the system but to some characteristic length $\zeta(\Theta) = v_f^c/(\Theta - \Theta_c)$ (see the end of the next section) and the density of the order parameter Δ/L vanishes (as ζ/L) in the limit $L \to \infty$, as it should. Hence, the temperature Θ_c indeed has the meaning of the temperature of a smearing phase transition from the symmetric phase to the phase with broken chiral symmetry.

The length $\zeta(\Theta)$ plays the role of coherence length in our system. At lengths less than $\zeta(\Theta)$, the wave function of the system coincides with coherent exponent (20). But at larger distances, the order disappears.

The macroscopic order parameter Δ can be nonzero even in the symmetric phase due to fluctuations of the broken to the unbroken phase. What matters is the behavior of Δ with the size of the system. If Δ does not increase with L, it follows that $\Delta \propto \zeta$ and $\zeta \ll L$, and we deal with an unbroken phase; if Δ increases with L, a long-range order appears. (This is just the region $\Theta \sim \Theta_c$, where $\zeta \sim v_f^c/\,\Theta_c \sim L$.) This condition can also be considered the definition of the smearing phase transition temperature for a finite-size system.

On the other hand, it is obvious from such a definition that the phase transition temperature in finite-size systems can be defined only up to $1/L$ corrections and the phase transition is smooth within the $1/L$ region near the phase transition temperature. In the Luttinger model, where the temperature Θ_c itself is of the order $1/L$, we can define Θ_c only up to a factor of the order of unity. This is the price we have to pay for considering a phase transition of a large but finite-size system. The above discussion should make it clear that this transition is smeared over the temperature region about Θ_c. But there is still a clear distinction between the case with a correlation length of the order of the system size (broken phase) and the case where $\zeta \ll L$ (unbroken phase). We note that Θ_c is not so small for real systems. Indeed, if we take $v_f \sim 10^7$ cm/sec and $L \sim 10^{-4}$ cm, then $\Theta_c \approx 1^\circ\,\mathrm{K} \cdot v_f^c/v_f$.

As was already pointed out in the Introduction, the

case of the infinitely strong interaction is very special. We see in what follows that if the interaction is finite, then the macroscopic order parameter Δ increases with the system size, but more slowly than L (at $\Theta \ll \Theta_c$). In the case of a short-range potential (see Sec. 4.1), Δ behaves as some power of L. This corresponds literally to the definition of the BKT phase. If we considered a potential of the Coulomb type, Δ would depend on L in a more complicated way, but would nevertheless increase with the size L. Physically, this case is quite similar to the usual BKT one. We note that in the formal limit $e^2/\pi v_f \to \infty$, we have the condensate of independent chiral pairs with wave function (20).

To summarize, the Luttinger model at $\Theta < \Theta_c$ is always in the BKT phase with broken chiral symmetry. At $\Theta \sim \Theta_c$, it undergoes a phase transition, which in the limit of the infinitely strong interaction turns into the smearing second-order phase transition with a finite density of pairs.

4. GROUND STATE OF THE TOMONAGA–LUTTINGER MODEL

Evolution operator (9) of a quantum system can be represented as a functional integral with definite boundary conditions (cf. [26]). This representation is usually derived for boson systems, and we therefore give the derivation for fermions in Appendix A.

The theory with an arbitrary electron–electron interaction can be reduced to a theory in an external field by means of the Hubbard–Stratonovich transformation [29] (see Eq. (33) below). Integrating over the external field is required in order to return to the original 4-fermion interaction. Therefore, we first consider the evolution operator for one-dimensional electrons placed into an external field $\Phi(x,t)$. It is given by

$$\hat{S}\left(\Phi\right) = \int\limits_{\left(\overline{\Psi},\Psi\right)} \mathcal{D}\Psi\mathcal{D}\overline{\Psi} \exp \mathcal{S}\left(\overline{\Psi},\Psi\right), \tag{22}$$

where $\overline{\Psi}$ and Ψ are the electron fields (Grassmann variables) and $\mathcal{S}$ is the action:

$$\mathcal{S} = i\int\limits_0^T dt \int dx \overline{\Psi}_R\left(x,t\right)\left[i\partial_t - v_f i\partial_x + \Phi\left(x,t\right)\right] \times \\ \times \Psi_R\left(x,t\right) + \left(R, v_f \leftrightarrow L, -v_f\right). \tag{23}$$

Integration over $\overline{\Psi}, \Psi$ in Eq. (22) is performed with given boundary conditions at $t = 0$ and $t = T$.

At $t \to +0$,

$$\Psi_{R,L}\left(x,t\right) = \hat{a}_{R,L}\left(x\right) + \\ + \text{arbitrary negative-frequency part},$$

$$\overline{\Psi}_{R,L}\left(x,t\right) = \hat{b}_{R,L}\left(x\right) + \\ + \text{arbitrary negative-frequency part}.$$

At $t \to T - 0$,

$$\Psi_{R,L}\left(x,t\right) = \hat{b}^{\dagger}_{R,L}\left(x\right) + \\ + \text{arbitrary positive-frequency part},$$

$$\overline{\Psi}_{R,L}\left(x,t\right) = \hat{a}^{\dagger}_{R,L}\left(x\right) + \\ + \text{arbitrary positive-frequency part}. \tag{24}$$

The creation operators of electrons and holes $\hat{a}^+$ and $\hat{b}^+$ are the variables entering the wave functions of the states in the sum in Eq. (9). The annihilation operators $\hat{a}$ and $\hat{b}$ enter the conjugate wave functions. They anticommute, $\{\hat{a},\hat{a}^+\} = \{\hat{b},\hat{b}^+\} = 0$, once they belong to different instances of time if the evolution operator is calculated (see Appendix A for the details). Therefore, we can regard them as Grassmann variables in calculating the functional integral.

It is possible to explicitly separate the dependence on the creation and annihilation operators for the evolution operator in a given external field determined by the functional integration region in (22). We introduce new integration variables

$$\begin{aligned} \Psi_{R,L} &= \Psi^0_{R,L} + \chi_{R,L}, \\ \overline{\Psi}_{R,L} &= \overline{\Psi}^0_{R,L} + \overline{\chi}_{R,L}. \end{aligned} \tag{25}$$

The saddle-point fields $\Psi^0_{R,L}$ are supposed to obey the Schrödinger equation in the external field $\Phi(x,t)$ with given boundary conditions (24). The "quantum" fields $\chi_{R,L}(x,t)$ are arbitrary but obey zero boundary conditions: $\chi_{R,L}\left(0\right) = \chi_{R,L}\left(T\right) = 0$.

The solutions $\Psi^0_{R,L}$ can be represented in terms of the Feynman Green's function $G_{R,L}$ in a finite time, defined as follows. It is a solution of the Schrödinger equation

$$\left[i\partial_t \pm v_f i\partial_x + \Phi\left(x,t\right)\right] G_{R,L}\left(x,t;x_1,t_1\right) = \\ = i\delta^{(2)}\left(x - x_1, t - t_1\right) \tag{26}$$

with the following boundary conditions: at $t \to +0$, the Green's function $G_R(x,t,x_1,t_1)$ coincides with the

Green's function of free fermions in the lower halfplane of the complex variable x (being arbitrary in the upper halfplane). At $t \to T-0$, it coincides with the free Green's function in the upper halfplane. For the Green's function of left electrons $G_L(x,t,x_1,t_1)$, the upper and lower halfplanes are to be exchanged.

The free Feynman Green's function is given by [8]

$$G^0_{R,L}(x,t;x_1,t_1) = \\ = \frac{1}{2\pi i}\left[v_f(t-t_1) \mp (x-x_1) - i\delta\,\mathrm{sign}(t-t_1)\right]^{-1}. \quad (27)$$

In one dimension, Schrödinger equation (26) can be solved for an arbitrary external field $\Phi(x,t)$:

$$G_{R,L}(x,t;x_1,t_1) = G^0_{R,L}(x,t;x_1,t_1) \times \\ \times \exp\left[i\int\limits_0^T dt' \int dy\,\Phi(y,t')\left(G^0_{R,L}(x,t;y,t') - \right.\right. \\ \left.\left. - G^0_{R,L}(x_1,t_1;y,t')\right)\right]. \quad (28)$$

Now it is easy to verify that the saddle-point fields $\Psi^0_{R,L}$ can be expressed in terms of these Green's functions as

$$\Psi^0_{R,L}(x,t) = \int dx' \left[G_{R,L}(x,t;x',0)\,\hat a_{R,L}(x') - \right. \\ \left. - G_{R,L}(x,t;x',T)\,\hat b^\dagger_{R,L}(x')\right], \\ \overline{\Psi^0}_{R,L}(x,t) = -\int dx' \left[G_{R,L}(x',0;x,t) \times \right. \\ \left. \times \hat b_{R,L}(x') - G_{R,L}(x',T;x,t)\,\hat a^\dagger_{R,L}(x')\right]. \quad (29)$$

To verify that these fields obey the required boundary conditions, we note that $\hat a_R(x)$ and $\hat b_R(x)$ are regular in the upper halfplane (see Eq. (2)). Therefore, the positive-frequency part of $G_R(x,t,x_1,t_1)$ at $t \to +0$ is determined by the pole contribution at $x' = x + i\delta$ and is equal to $\hat a_R(x)$, as it should. The second term in Eq. (29) yields a negative-frequency part, which is arbitrary. Similarly, we verify the boundary condition at $t \to T-0$. Inside the time interval $(0,T)$, the saddle-point fields satisfy the Schrödinger equation, as can be seen from Eq. (26) for the Green's functions.

The contribution of the saddle-point field to the action is

$$\mathcal{S}_0 = \sum_{i=R,L}\int dx\,dx' \left[\hat b_i(x')\,G_i(x',0;x,\epsilon)\,\hat a_i(x) + \right. \\ + \hat a^\dagger_i(x')\,G_i(x',T;x,T-\epsilon)\,\hat b^\dagger_i(x) - \\ - \hat a^\dagger_i(x')\,G_i(x',T;x,0)\,\hat a_i(x) - \\ \left. - \hat b_i(x')\,G_i(x',0;x,T)\,\hat b^\dagger_i(x)\right]. \quad (30)$$

We here take Eq. (26) into account. Since the saddle-point fields obey the Schrödinger equation, there is no term linear in the quantum field χ in the action.

The dependence of the evolution operator in the external field on the creation and annihilation fermion operators is completely determined by Eq. (30). The integral over quantum fluctuations produces the determinant of the Schrödinger operator in the external field Φ:

$$\ln\left[\mathrm{Det}\,\Phi(T)\right] = -\frac{1}{4\pi}\int\limits_0^T dt\,dt_1 \times \\ \times \int\limits_{-\infty}^{\infty}\frac{dp}{2\pi}\Phi(-p,t)\,\Phi(p,t_1)\,|p| \times \\ \times \exp\left[-i|p|v_f|t-t_1|\right]. \quad (31)$$

(It is calculated in Appendix B. In fact, we introduced an ultraviolet cut-off there.) The complete expression for the evolution operator in the external field has the form

$$\hat S(\Phi) = \exp\left(\mathcal{S}_0 + \ln[\,\mathrm{Det}\,\Phi(T)\,]\right)|F\rangle\langle F|, \quad (32)$$

Now we can express the evolution operator for the system of interacting fermions in terms of this operator. We use the well-known identity [29]

$$\exp\left[-\frac{i}{2}\int\limits_0^T dt\int\limits_{-\infty}^{\infty}\frac{dp}{2\pi}V(p)\,\varrho(p,t)\,\varrho(-p,t)\right] = \frac{1}{\mathcal{N}} \times \\ \times \int \mathcal{D}\Phi \exp\left[\frac{i}{2}\int\limits_0^T dt\int\limits_{-\infty}^{\infty}\frac{dp}{2\pi}\Phi(p,t)\,\Phi(-p,t)\,V^{-1}(p) - \right. \\ - \frac{i}{2}\int\limits_0^T dt\int\limits_{-\infty}^{\infty}\frac{dp}{2\pi}\left(\varrho(p,t)\,\Phi(-p,t) + \right. \\ \left.\left. + \varrho(-p,t)\,\Phi(p,t)\right)\right]. \quad (33)$$

The normalization coefficient $\mathcal{N}$ is

$$\mathcal{N}=\int \mathcal{D}\Phi \exp\left[\frac{i}{2}\int\limits_0^T dt\,\times\right.$$
$$\left.\times\int\limits_{-\infty}^{\infty}\frac{dp}{2\pi}\Phi\left(p,t\right)\Phi\left(-p,t\right)V^{-1}\left(p\right)\right]. \quad (34)$$

To prove Eq. (33), it suffices to shift the integration variable Φ to $\Phi - V\varrho$ in the integral

$$\int \mathcal{D}\Phi \exp\left[\frac{i}{2}\mathrm{Tr}\left(\Phi^{\dagger}\Phi V^{-1}\right)\right].$$

Applying identity (33) to the functional integral that determines the evolution operator for the Tomonaga–Luttinger model, we express it in terms of the evolution operator in the external field at the price of an additional functional integration over the scalar field $\Phi(x,t)$:

$$\hat{S}_{e\text{-}e}=\frac{1}{\mathcal{N}}\int \mathcal{D}\Phi \exp\left[\frac{i}{2}\int\limits_0^T dt\,\times\right.$$
$$\left.\times\int\limits_{-\infty}^{\infty}\frac{dp}{2\pi}\Phi\left(p,t\right)\Phi\left(-p,t\right)V^{-1}\left(p\right)\right]\hat{S}\left(\Phi\right). \quad (35)$$

Expression (35) is explicit: while it is not possible to perform the final integration in $\Phi(x,t)$ in closed form, it is easy to obtain an arbitrary term of the evolution operator by expanding it in the creation and annihilation operators. This suffices for the calculation of the evolution operator.

Indeed, we expand the evolution operator in powers of the external field S_0^n. The arbitrary term of the expansion contains a number of Green's functions in the external field (28), which are exponentials linear in the external field. Together with action (35) and determinant (34), we obtain a Gaussian-type integral in $\Phi(x,t)$, which can be easily performed. The result of the integration depends of the electron–hole configuration considered. It is specified by the concrete term of the expansion.

We introduce the following notation for the coordinates entering the electron–hole creation and annihilation operators.

1. We let x denote the coordinates of the right particles and y the coordinates of the left particles.

2. We put a tilde on coordinates related to annihilation operators (initial state) and leave coordinates of creation operators (final state) without a tilde.

3. We put primes on coordinates related to holes.

It is convenient to proceed in the exponents of Green's functions (28) to momentum space using the expression for the free Feynman Green's functions:

$$G^0_{R,L}\left(p,t,t_1\right)=\theta_{\pm p}\theta\left(t-t_1\right)\times$$
$$\times\exp\left[\mp ipv_f\left(t-t_1\right)\right]-\theta_{\mp p}\theta\left(t_1-t\right)\times$$
$$\times\exp\left[\mp ipv_f\left(t-t_1\right)\right]. \quad (36)$$

Collecting all terms in the exponents arising from Green's function (28), we obtain the contribution to the action linear in the external field Φ,

$$\mathcal{S}_c=i\int\limits_0^T dt\int\limits_{-\infty}^{\infty}\frac{dp}{2\pi}\Phi\left(-p,t\right)\mathcal{R}_c\left(p,t\right), \quad (37)$$

where the "current" $\mathcal{R}_c$ depends on the chosen configuration, i.e., on the concrete term in the expansion of S_0^n. It depends on the electron (hole) operators directly and as a result on their coordinates. (See Eq. (50) below. A possible configuration can be seen in explicit form there.) It is equal to

$$\mathcal{R}_c\left(p,t\right)=\mathcal{R}_i\left(p\right)\exp\left(-i|p|v_f t\right)+$$
$$+\mathcal{R}_f\left(p\right)\exp\left(-i|p|v_f\left(T-t\right)\right), \quad (38)$$

and

$$\mathcal{R}_f\left(p\right)=\sum_{x\ldots;x'\ldots;y\ldots;y'\ldots}\theta\left(p\right)\left[\exp\left(ipx\right)-\right.$$
$$\left.-\exp\left(ipx'\right)\right]+\theta\left(-p\right)\left[\exp\left(ipy\right)-\exp\left(ipy'\right)\right],$$
$$\mathcal{R}_i\left(p\right)=\sum_{\tilde{x}\ldots;\tilde{x}'\ldots;\tilde{y}\ldots;\tilde{y}'\ldots}\theta\left(-p\right)\left[\exp\left(ip\tilde{x}\right)-\right. \quad (39)$$
$$\left.-\exp\left(ip\tilde{x}'\right)\right]+\theta\left(p\right)\left[\exp\left(ip\tilde{y}\right)-\exp\left(ip\tilde{y}'\right)\right]$$

for the initial (annihilation operators) and final (creation operators) configurations respectively. Coordinates $x,\ldots,y,\ldots$ in Eq. (39) are the coordinates of annihilation and creation operators for the configuration in which we are interested. Finally, we obtain the functional integral

$$\int \mathcal{D}\Phi \exp\left[\frac{i}{2}\int_0^T dt\,dt_1 \times \right.$$
$$\times \int_{-\infty}^{\infty}\frac{dp}{2\pi}\Phi(p,t)\,\Phi(-p,t_1)\,V^{-1}(p)\,\delta(t-t_1) -$$
$$-\frac{1}{4\pi}\int_0^T dtdt_1\int_{-\infty}^{\infty}\frac{dp}{2\pi}\Phi(-p,t)\,\Phi(p,t_1)\,|p|\times$$
$$\times\exp\left[-i|p|v_f|t-t_1|\right]+$$
$$\left.+\,i\int_0^T dt\int_{-\infty}^{\infty}\frac{dp}{2\pi}\Phi(-p,t)\,\mathcal{R}_c(p,t)\right], \quad (40)$$

where the first term is the action in Eq. (35), the second term is the quantum determinant, and the third term comes from the Green's functions in Eq. (37).

The integral in Eq. (40) is Gaussian: it can be calculated by standard methods, by finding the saddle-point field Φ_0 and shifting the integration variables as $\Phi \to \Phi - \Phi_0$. The integral with respect to the fluctuations $\Phi - \Phi_0$ yields a shift of the ground-state energy due to the electron interaction and the normalization coefficient of the ground-state wave function. We calculate this integral in Appendix C. The operator structure of the evolution operator is completely determined by the terms that appear as a result of substituting the saddle-point Φ_0 in Eq. (40). We write them as an "effective action":

$$\mathcal{S}_{eff} = \frac{i}{2}\int_{-\infty}^{\infty}\frac{dp}{2\pi}\int_0^T dt\,\Phi_0(p,t)\,\mathcal{R}_c(p,t). \quad (41)$$

The saddle-point field $\Phi_0(x,t)$ satisfies the integral equation

$$\frac{i}{V_0}\Phi_0(p,t) - \frac{1}{2\pi}\int_0^T dt_1\,\Phi_0(p,t_1)\,|p|\times$$
$$\times\exp\left[-i|p|v_f|t-t_1|\right] = -i\mathcal{R}_c(p,t), \quad (42)$$

which can be reduced to the following differential equation (to see this, it suffices to differentiate both sides of Eq. (42) with respect to time):

$$\partial_t^2\Phi_0(p,t) + \omega_p^2\Phi_0(p,t) = 0, \quad (43)$$

where

$$\omega_p = |p|v_f\sqrt{1+\frac{V_0}{\pi v_f}}. \quad (44)$$

The boundary conditions for this equation follow from the original integral equation (42):

$$\begin{aligned}&\partial_t\Phi_0(p,0) - i|p|v_f\Phi_0(p,0) = 2i|p|v_fV_0\mathcal{R}_i(p),\\ &\partial_t\Phi_0(p,T) + i|p|v_f\Phi_0(p,T) =\\ &\qquad = -2i|p|v_fV_0\mathcal{R}_f(p).\end{aligned} \quad (45)$$

In the derivation of Eq. (43), we have used the fact that our system is electrically neutral, and hence

$$\mathcal{R}_f(p=0,t) = \mathcal{R}_i(p=0,t) = 0.$$

The solution of the differential equation for the saddle-point field (Eq. (43)) gives

$$\Phi_0(p,t) = \frac{-2|p|v_fV_0}{(\omega_p+|p|v_f)(1-\xi_p^2)}\times$$
$$\times\{\mathcal{R}_i[\exp(-i\omega_pt)+\xi_p\exp(-i\omega_p(T-t))]+$$
$$+\,\mathcal{R}_f[\xi_p\exp(-i\omega_pt)+\exp(-i\omega_p(T-t))]\}, \quad (46)$$

where

$$\xi_p = \frac{1-\sqrt{1+V_0/\pi v_f}}{1+\sqrt{1+V_0/\pi v_f}}\exp(-i\omega_pT).$$

Substituting the saddle-point field in the effective action (41), we finally obtain

$$\mathcal{S}_{eff} = -\frac{1}{L}\sum_{p\neq0}\frac{V_0}{1+\sqrt{1+V_0/\pi v_f}}\frac{1}{1-\xi_p^2}\times$$
$$\times[[\mathcal{R}_f(-p)\,\mathcal{R}_f(p)+\mathcal{R}_i(-p)\,\mathcal{R}_i(p)]\,F_2(p)+$$
$$+\,2F_1(p)\,\mathcal{R}_f(-p)\,\mathcal{R}_i(p)], \quad (47)$$

where we introduce the functions

$$\begin{aligned}F_1(p) &= \frac{\exp(-i|p|v_fT)-\exp(-i\omega_pT)}{\omega_p-|p|v_f}+\\ &+\xi_p\frac{1-\exp(-i(\omega_p+|p|v_f)T)}{\omega_p+|p|v_f},\\ F_2(p) &= \frac{1-\exp(-i(\omega_p+|p|v_f)T)}{\omega_p+|p|v_f}+\\ &+\xi_p\frac{\exp(-i|p|v_fT)-\exp(-i\omega_pT)}{\omega_p-|p|v_f}.\end{aligned} \quad (48)$$

In expression (47) for the effective action, we return to a sum over the particle momenta $p_n = 2\pi n/L$ in accordance with the ordinary rule[6)]:

$$\int_{-\infty}^{\infty}\frac{dp}{2\pi} \longrightarrow \frac{1}{L}\sum_p.$$

[6)] This corresponds to periodic boundary conditions for the Ψ and $\overline{\Psi}$ fields at the boundaries of the sample. (See also the end of Appendix D.)

This allows qualifying different infrared divergences that appear in the effective action. We note that there is no term with $p = 0$ in these sums. This fact is related to the gauge invariance of the system: constant (in space) fields $\Phi(t)$ correspond to a pure gauge electric potential and should not contribute.

We proceed with the ground-state wave function in the Tomonaga–Luttinger model. As mentioned above, in order to separate the ground state, we have to take the limit $T \to \infty$. (This corresponds to the case of zero temperature.) We can omit oscillating exponentials in this limit. As a result, we are left with only the function F_2, which becomes

$$F_2(p) \sim \left[|p| v_f \left(1 + \sqrt{1 + \frac{V_0}{\pi v_f}}\right)\right]^{-1}.$$

The effective action factores into contributions of initial and final states:

$$\mathcal{S}_{eff} = -\frac{1}{L}\sum_{p\neq 0} \frac{V_0}{|p| v_f \left[1 + \sqrt{1 + \frac{V_0}{\pi v_f}}\right]^2} \times$$

$$\times \left[\mathcal{R}_f(-p)\,\mathcal{R}_f(p) + \mathcal{R}_i(-p)\,\mathcal{R}_i(p)\right] =$$

$$= \mathcal{S}^f_{eff} + \mathcal{S}^i_{eff}. \quad (49)$$

In addition to S_{eff}, we have to calculate preexponential factors that arise from the free Feynman Green's functions. Only the Green's functions with equal-time arguments survive as $T \to \infty$. As a result, we see that the whole expression for the evolution operator for large T factores into the product of the ground-state wave function $|\Omega\rangle$ and its complex conjugate. The final expression for the wave function is of the form

$$|\Omega\rangle = \sum_{n=0}^{\infty} \frac{1}{n!}\left[\int \frac{dx\,dx'}{2\pi i}\,\frac{\hat a^\dagger_R(x)\,\hat b^\dagger_R(x')}{x' - x - i\delta} + \right.$$

$$\left. + \int \frac{dy\,dy'}{2\pi i}\,\frac{\hat a^\dagger_L(y)\,\hat b^\dagger_L(y')}{y - y' - i\delta}\right]^n \times$$

$$\times \exp \mathcal{S}^f_{eff}(x, x', \ldots, y, y', \ldots)\,|F\rangle. \quad (50)$$

We verify that the wave function of noninteracting fermions ($V = 0$) is $|F\rangle$. The general term in the sum in Eq. (50) is a product of factors:

$$\int \frac{dx\,dx'}{2\pi i}\,\frac{\hat a^\dagger_R(x)\,\hat b^\dagger_R(x')}{x' - x - i\delta}|F\rangle.$$

We note now that all singularities of the operator $\hat b^\dagger_R(x')$ are those in the upper halfplane (see Definition 2) and the pole of the Green's function. We can close the contour of x' in the lower halfplane and prove that the corresponding integral vanishes. The only term that survives is the one with $n = 0$, and hence $|\Omega\rangle = |F\rangle$, as it should be for noninteracting fermions.

A nontrivial answer for the wave function appears only owing to singularities of the effective action. It is clear from the general structure of the action (which is the product $R_f(p)R_f(-p)$) that the wave function contains only terms where both R- and L-particles are present. All terms with only R (or only L) electrons or holes vanish. The simplest possible contribution to the ground-state wave function $|\Omega\rangle$ (see Eq. (50)) is

$$\int \frac{dx\,dx'}{2\pi i}\,\frac{dy\,dy'}{2\pi i}\,\frac{\hat a^\dagger_R(x)\,\hat b^\dagger_R(x')}{x' - x - i\delta}\,\frac{\hat a^\dagger_L(y)\,\hat b^\dagger_L(y')}{y - y' - i\delta} \times$$

$$\times \exp \mathcal{S}^f_{eff}(x, x', y, y'). \quad (51)$$

The effective action $\mathcal{S}_{eff}$ for this term is given by

$$\mathcal{S}^f_{eff}(x, x', y, y') = -\frac{2\alpha}{L} \times$$

$$\times \sum_{p_n > 0} \frac{1}{p_n}\left\{\exp[ip_n(x - y + i\delta)] + \right.$$

$$+ \exp[ip_n(x' - y' + i\delta)] - \exp[ip_n(x' - y + i\delta)] -$$

$$\left. - \exp[ip_n(x - y' + i\delta)]\right\}, \quad (52)$$

where

$$\alpha = \frac{V_0}{v_f\left[1 + \sqrt{1 + V_0/\pi v_f}\right]^2}. \quad (53)$$

The sums in Eq. (52) can be easily calculated. We obtain

$$\mathcal{S}^f_{eff}(x, x', y, y') =$$

$$= \frac{\alpha}{\pi}\ln\frac{(x - y + i\delta)(x' - y' + i\delta)}{(x' - y + i\delta)(x - y' + i\delta)}. \quad (54)$$

According to the charge conservation law, the number of electrons has to be equal to the number of holes, and therefore the number of exponentials with the opposite sign in Eq. (52) is the same. As a result, the action $\mathcal{S}_{eff}$ does not diverge and singularities in the integrand in Eq. (51) are removed by zeroes of the action or by the integrations over x' and y'. Therefore, divergences in the wave function do not exist even for short-range interactions.

Expression (51) describes the simplest possible complex in the vacuum of the interacting fermions. This

complex has all quantum numbers equal to zero. In fact, it describes electron–electron scattering (in the cross channel). Correspondingly, all coordinates x, x', y, y' are close to each other. In general, this complex does not break any continuous symmetry.

But in the Tomonaga–Luttinger model, a special situation arises. The leading contribution to term (54) comes from the region $x'-y, x-y' \to 0$ (of the order of the transverse size of the channel), but $x-y$ and $x'-y'$ can be arbitrarily large. In other words, the complex decays into $R\bar{L}$- and $\bar{R}L$-pairs. As we see in what follows, such a wave function leads to a spontaneous chiral symmetry breaking.

We first consider the strong-interaction limit:

$$\frac{V_0}{\pi v_f} \gg 1. \tag{55}$$

In this limit, $\alpha/\pi \to 1$. It can be seen that for $\alpha/\pi = 1$, the poles ($x = x'$ and $y = y'$) corresponding to free fermions are canceled completely by the fermion–fermion interaction (described by $\exp S_{eff}$) with the effective action in Eq. (54). Instead, we obtain new poles at the points $x' = y - i\delta$ and $y' = x + i\delta$. Recalling that $\hat{b}^\dagger_R(x')$ is analytic in the lower and $\hat{b}^\dagger_L(y')$ in the upper halfplane, we can integrate further over x' and y'. As a result, we obtain the following contribution to the ground-state wave function:

$$\int dx\, \hat{a}^\dagger_R(x)\, \hat{b}^\dagger_L(x) \int dy\, \hat{a}^\dagger_L(y)\, \hat{b}^\dagger_R(y). \tag{56}$$

Thus, the 4-particle complex decays into 2 noninteracting "bosons". They are neutral in the electric charge but have a nonzero chirality ± 2.

It can be verified that no other connected complexes appear in the limit of strong interaction. For example, we consider charged complexes. The four-fermion contribution is exhausted by Eq. (56), and hence we have to consider a 6-fermion complex:

$$\frac{\hat{a}^\dagger_R(x)\,\hat{b}^\dagger_R(x')}{x'-x-i\delta}\,\frac{\hat{a}^\dagger_R(x_1)\,\hat{b}^\dagger_R(x'_1)}{x'_1-x_1-i\delta}\,\frac{\hat{a}^\dagger_L(y)\,\hat{b}^\dagger_L(y')}{y-y'-i\delta}\times$$

$$\times\frac{(x-y+i\delta)\,(x_1-y+i\delta)\,(x'-y'+i\delta)\,(x'_1-y'+i\delta)}{(x-y'+i\delta)\,(x_1-y'+i\delta)\,(x'-y+i\delta)\,(x'_1-y+i\delta)}. \tag{57}$$

This complex, indeed, decays into 2 fermions as $x_1 \to y' \to x$ and $x' \to y \to x'_1$ (the relative distance $x - x'$ is supposed to be large). These fermions are of the form $\hat{a}^\dagger_R(x)\,\hat{a}^\dagger_R(x)\,\hat{b}^\dagger_L(x)$ and $\hat{a}^\dagger_L(x')\,\hat{b}^\dagger_R(x')\,\hat{b}^\dagger_R(x')$. Hence, this contribution is zero owing to the Pauli principle. We can also consider more complicated configurations that could produce charged connected complexes and verify that they do not appear in the ground-state wave function.

The Pauli principle allows one more complex that describes scattering of chiral pairs:

$$\hat{a}^\dagger_R(x)\,\hat{b}^\dagger_L(x)\,\hat{a}^\dagger_L(x)\,\hat{b}^\dagger_R(x).$$

The corresponding contribution can be extracted from the connected part of the general expression (51). The integral over x' and y' is easily calculated and we obtain

$$\int dx\, dy\, \hat{a}^\dagger_R(x)\,\hat{b}^\dagger_L(x)\,\hat{a}^\dagger_L(y)\,\hat{b}^\dagger_R(y)\,\Phi(x-y),$$

where

$$\Phi(x-y) = \frac{-i\delta}{y-x-2i\delta}\left(2 + \frac{i\delta}{y-x-2i\delta}\right).$$

The function $\Phi(x-y)$ is finite at any x and y (even at $x = y$) and therefore its contribution to the integral vanishes in the limit $\delta \to +0$. In other words, in the limit of an infinitely strong interaction, the chiral pairs do not interact. This interaction appears, however, in the next approximations in the inverse coupling constant (see Sec. 4.1).

To obtain the complete expression for the ground-state wave function, we have to consider complexes with 8, 12, ... particles and separate the connected parts of these complexes. This is not necessary, however, because, according to a general theorem [30], the complete wave function is the exponent of the connected complexes[7)] and we have proved that the only connected complexes are the chiral pairs in Eq. (56). On the other hand, the total chirality C of $|\Omega\rangle$ must be zero and only terms with $C = 0$ can occur in the expansion of $|\Omega\rangle$. To take this into account, we introduce the projector $P_{C=0}$ onto the state with chirality zero. Then the wave function can be written as

$$|\Omega\rangle = \sqrt{Z_0}\hat{P}_{C=0}\exp\left[\int dx\,\hat{a}^\dagger_R(x)\,\hat{b}^\dagger_L(x) + \right.$$

$$\left. + \int dy\,\hat{a}^\dagger_L(y)\,\hat{b}^\dagger_R(y)\right]|F\rangle. \tag{58}$$

The normalization coefficient Z_0 is calculated in Appendix C. We have already discussed the wave function in Sec. 3. Function (58) corresponds to an unbroken symmetry phase in spite of the presence of an

[7)] This theorem is in fact a purely combinatorial statement. In field theory, it is mostly applied to Green's functions. In statistical physics, it is known as the first Mayer's theorem (see, e.g., [31]).

infinite number of chiral pairs with zero momentum. If the chiral symmetry is broken, the states with different chiralities should be degenerate in energy. This is not the case if the system size is finite: the energy of the state with $C = 0$ is still minimal and the order parameter Δ is zero for $\Theta = 0$.

Wave function (58) corresponds to the state with the minimal possible energy. Hence, it is the wave function of the system at $\Theta = 0$. To discuss the nonzero temperature region, we can proceed to the Euclidean time ($T \to -i/\Theta$) for equilibrium systems. We have seen that the action contains two types of exponentials (see the term with $\mathcal{R}_f(p)\mathcal{R}_i(-p)$ in Eq. (48)): $\exp(-v_f p/\Theta)$ and $\exp(-v_f^c p/\Theta)$ (with a renormalized Fermi velocity). Corrections of the second type correspond to excitations, and we omit them. But there are no excitations with the energy $v_f p$ (this can be seen, e.g., by the method of bosonization). In fact, these exponentials describe the change of the ground state with temperature (see footnote 4). (We discussed the meaning of the crossover temperature $2\pi v_f/L$ in Sec. 3 in detail.) Obviously, for $\Theta \gg 2\pi v_f/L$, this exponential factor is not small but the preexponential factor, i.e., the Green's function with the imaginary time differences about $1/\Theta$, gives the smallness. This is compensated by the action, because it is proportional to $\ln(1/\Theta)$. This is the case for the temperature region under consideration. In the opposite case $\Theta \ll 2\pi v_f/L$, the Green's function (27) is inapplicable. We can use Eq. (36), but it is impossible to transform the sums over p_n to integrals in order to obtain Eq. (27). As a result, the Green's function is proportional to a small exponential factor. It cannot be compensated by a logarithmic divergence from the action and the whole term with $\exp(-v_f p/\Theta)$ is small. Therefore, ground-state wave function (58) is valid if

$$\Theta \ll \Theta_{chiral} = \frac{2\pi v_f}{L}. \tag{59}$$

Of course, we assume that the number of states is large, i.e., $p_f L \gg 1$. This allows passing from sums to integrals in the expressions independent of Θ (or T).

In the region of higher temperatures, $\Theta_{chiral} \ll \Theta$, Eq. (27) for the Green's function is applicable. In this case, after similar algebraic transformations, the effective action $\mathcal{S}_{eff}$ in Eq. (47) for an infinitely strong interaction can be rewritten as

$$\begin{aligned}\mathcal{S}_{eff} = &-\frac{\pi}{L}\sum_{n\neq 0}\frac{1}{|p_n|}\times\\ &\times \operatorname{th}\frac{|p_n|v_f^c}{2\Theta}\left[\mathcal{R}_f(-p)\mathcal{R}_f(p)+\mathcal{R}_i(-p)\mathcal{R}_i(p)\right]-\\ &-\frac{2\pi}{L}\sum_{n\neq 0}\frac{1}{|p_n|}\frac{\exp\dfrac{-|p_n|v_f}{\Theta}-\exp\dfrac{-|p_n|v_f^c}{\Theta}}{1+\exp\dfrac{-|p_n|v_f^c}{\Theta}}\times\\ &\times\mathcal{R}_f(p)\mathcal{R}_i(-p),\end{aligned} \tag{60}$$

where $v_f^c = v_f\sqrt{1+V_0/\pi v_f}$. If

$$\Theta_{chiral} \ll \Theta \ll 2\pi v_f^c/L \tag{61}$$

then Eq. (60) can be transformed to[8)]

$$\begin{aligned}\mathcal{S}_{eff} = &-\frac{\pi}{L}\sum_{n\neq 0}\frac{1}{|p_n|}\left[\mathcal{R}_f(-p)\times\right.\\ &\left.\times\,\mathcal{R}_f(p)+\mathcal{R}_i(-p)\mathcal{R}_i(p)\right]-\\ &-\frac{2\pi}{L}\sum_{n\neq 0}\frac{1}{|p_n|}\exp\frac{-|p_n|v_f}{\Theta}\mathcal{R}_f(-p)\mathcal{R}_i(p).\end{aligned} \tag{62}$$

Hence, in the temperature region of interest, we should take another 4-fermion contribution to the ground state into account:

$$\begin{aligned}&\int\frac{dx\,d\tilde{x}}{2\pi i}\,\frac{dy'd\tilde{y}'}{2\pi i}\,\frac{\hat{a}_R^\dagger(x)\,\hat{a}_R(\tilde{x})}{\tilde{x}-x+v_fT-i\delta}\times\\ &\times\frac{\hat{b}_L^\dagger(y')\,\hat{b}_L(\tilde{y}')}{\tilde{y}'-y'-v_fT+i\delta}\exp\mathcal{S}_{eff}^f(x,\tilde{x},y',\tilde{y}').\end{aligned} \tag{63}$$

(At lower temperatures, this contribution is exponentially small. Here, we work with real time T until the end and proceed to Euclidean time only at the last step.) The action for this configuration is

$$\ln\frac{(\tilde{y}'-y'-v_fT+i\delta)(x-\tilde{x}-v_fT+i\delta)}{(x-y'+i\delta)(\tilde{y}'-\tilde{x}+i\delta)}. \tag{64}$$

Thus, we have a similar result: a pair $\hat{a}_R^\dagger(x)\,\hat{b}_L^\dagger(x)$ in $|\Omega\rangle$ and $\hat{a}_R(\tilde{x})\,\hat{b}_L(\tilde{x})$ in $\langle\Omega|$. However, the existence of an extra pair implies that the total chirality C of the state is nonzero. Hence, states with any C exist. Their energies differ by values of the order of $2\pi v_f/L$. For temperatures (61), these states can be considered degenerate. Then a state with a fixed chirality is unstable relative to an infinitesimal interaction that mixes

[8)] The terms in the equation for $\mathcal{S}_{eff}$ with the factor $\exp\left(-|p_n|v_f^c/\Theta\right)$ describe an excited state. We do not discuss them because such effects can be calculated more easily by using the bosonization technique.

right and left particles (e.g., infinitesimal back scattering). Similarly to the theory of superconductivity, the real ground state of the system is a mixture of states with different chiralities, but with a fixed chiral phase θ. Therefore, we have derived the wave function corresponding to Eq. (20) discussed in Sec. 3. An alternative (in the boson representation) form of the ground-state wave function is given in Sec. 5.

To prove that

$$\Theta_c = \omega(2\pi/L) = 2\pi v_f^c/L \tag{65}$$

is the phase transition temperature, we must consider the higher-temperature region $\Theta \gg \Theta_c$. The logarithmic contribution to the action $\mathcal{S}_{eff}$ arises from $n \gg n_{min} \sim L\Theta/2\pi v_f^c \gg 1$. At smaller n, the logarithmic divergence does not occur. Hence,

$$\mathcal{S}_{eff} = -\frac{\pi}{L}\sum_{|n|>n_{min}}\frac{1}{|p_n|}\times \\ \times\left[\mathcal{R}_f(-p)\,\mathcal{R}_f(p)+\mathcal{R}_i(-p)\,\mathcal{R}_i(p)\right]- \\ -\frac{2\pi}{L}\sum_{|n|>n_{min}}\frac{1}{|p_n|}\exp\frac{-|p_n|v_f}{\Theta}\mathcal{R}_f(-p)\,\mathcal{R}_i(p). \tag{66}$$

This is to be compared with Eq. (60). The sums in Eq. (66) are calculated in Appendix D. As a result, the logarithms in Eq. (54) are to be replaced by

$$\ln\frac{(\Delta x+i\delta)}{(\Delta x'+i\delta)}\to\int\limits_{-(\Delta x'+i\delta)}^{-(\Delta x+i\delta)}\frac{dz}{z}\exp\left(-\frac{iz}{\zeta(\Theta)}\right), \tag{67}$$

where

$$\zeta(\Theta) = v_f^c/(\Theta-\Theta_c) \tag{68}$$

is the coherence length.

The right-hand side of Eq. (67) can be expressed in terms of the integral exponential function with imaginary argument. To prove that $\zeta(\Theta)$ is the coherence length, we note that at lengths $\Delta x \ll \zeta(\Theta)$, the right-hand side of Eq. (67) tends to $\ln((\Delta x+i\delta)/(\Delta x'+i\delta))$, i.e., the system is characterized by the wave function in Eq. (20), with the exception of the normalization coefficient. (Indeed, it is then possible to repeat the calculations in the previous section with all connected complexes separated by distances shorter than $\zeta(\Theta)$.) Thus, in a region of a sample smaller than ζ, a coherent state exists. In the opposite case (distances between pairs $\Delta x = |x-y|$ larger than $\zeta(\Theta)$), the integrand begins to oscillate and the divergence does not occur. As a result, we have small corrections to the action approximately given by

$$\exp\left(-\frac{i\Delta x}{\zeta(\Theta)}\right)\frac{\zeta(\Theta)}{\Delta x}.$$

Hence, the 4-fermion contribution in (63) leads to the term

$$\int dx\,dy\left(\frac{\zeta(\Theta)}{|x-y|}\right)^2\hat{a}_R^\dagger(x)\,\hat{b}_L^\dagger(x)\times \\ \times|F\rangle\langle F|\hat{b}_L(y)\,\hat{a}_R(y) \tag{69}$$

in the evolution operator. Thus, at distances $|x-y| \gg \zeta$, we have configurations with free bosons. Consequently, the state is noncoherent at this scale. Therefore, the long-range order does not exist at lengths larger than $\zeta(\Theta)$. We can also verify this directly. For this, we calculate the contribution of state (69) to the order parameter density correlator $\langle|\hat{a}_R^\dagger(y_1)\,\hat{b}_L^\dagger(y_1)\,\hat{b}_L(x_1)\,\hat{a}_R(x_1)|\rangle$ in the region $|x_1-y_1| \sim L \gg \zeta$. After the integration over x_1 and y_1, we have the contribution of this state to Δ^2:

$$\int\limits_{-L/2}^{L/2}dx\frac{\zeta^2(\Theta)}{L}\sim\zeta^2(\Theta).$$

Because Δ is independent of L, we have the normal phase (see Sec. 3) with low-symmetry phase fluctuations. This means that Θ_c is indeed the phase transition temperature and ζ is the coherence length. Besides, we have a more obvious definition of Θ_c:

$$\zeta(\Theta_c)\sim L.$$

In this case, the entire system can be described by the broken-symmetry wave function in Eq. (20). Hence, the low-symmetry phase should be regarded as realized if $\Theta<\Theta_c$. The above discussion must make it clear that this transition is smeared over the temperature region about Θ_c, as it should be for a finite-size sample.

4.1. Berezinskii–Kosterlitz–Thouless phase

We prove that the BKT phase [16] is likely to form in the Tomonaga–Luttinger model if corrections to the action due to $\pi v_f/V_0$ are taken into account.

We begin with the case of zero temperature and again consider the 4-fermion contribution, Eq. (51), to the ground wave function

$$\int\frac{dx\,dx'}{2\pi i}\,\frac{dy\,dy'}{2\pi i}\,\frac{\hat{a}_R^\dagger(x)\,\hat{b}_R^\dagger(x')}{x'-x-i\delta}\,\frac{\hat{a}_L^\dagger(y)\,\hat{b}_L^\dagger(y')}{y-y'-i\delta}\times \\ \times\left[\frac{(x-y+i\delta)\,(x'-y'+i\delta)}{(x'-y+i\delta)\,(x-y'+i\delta)}\right]^{\alpha_0}|F\rangle, \tag{70}$$

where $\alpha_0 \equiv \alpha/\pi$. For simplicity, we consider α_0 close to unity. We consider the configuration with two connected chiral complexes separated by a distance R large compared to the transverse size of the channel d: $x'-y$, $x-y' \sim d \to 0$ and $|x-x'| \sim R$, $|y-y'| \sim R \to \infty$. The contribution in which we are interested is determined by two cuts, $y' = x + i\delta$ and $x' = y + i\delta$, and is proportional to

$$\left(1 - e^{2\pi i\alpha_0}\right) \int\limits_{-\infty}^{x} \frac{dy'}{2\pi i} \hat{b}_L^\dagger\,(y')\, \frac{1}{(y'-x)^{\alpha_0}}\, \frac{(x'-y')^{\alpha_0}}{(y'-y)}.$$

The last factor in the integrand is of the order of $1/R^{1-\alpha_0}$. Distances inside the pair $y'-x$, $x'-y$ are of the order of d. The contribution of the distant chiral pairs to integral (70) is

$$\int dx\, dy \hat{a}_R^\dagger\,(x)\, \hat{b}_L^\dagger\,(x)\, \hat{a}_L^\dagger\,(y)\, \hat{b}_R^\dagger\,(y) \times \\ \times \left(\frac{d}{|x-y|}\right)^{2(1-\alpha_0)} |F\rangle. \quad (71)$$

In temperature region (19), we can also consider contributions of the states with $C \neq 0$ to the ground state. The simplest contribution comes again from Eq. (63) and has the form[9)]

$$\int dx\, dy \left(\frac{d}{|x-y|}\right)^{2(1-\alpha_0)} \times \\ \times \hat{a}_R^\dagger\,(x)\, \hat{b}_L^\dagger\,(x)\, |F\rangle\langle F| \hat{b}_L\,(y)\, \hat{a}_R\,(y)\,. \quad (72)$$

As can be seen from Eqs. (71) and (72), the probability to find chiral pairs at the distance R is

$$P(R) = |\Phi(R)|^2 \sim 1/R^{2(1-\alpha_0)}.$$

This probability decreases with R but much more slowly than in the theory without interaction. The average distance between correlated pairs

$$\langle R \rangle = \int\limits_0^L dR\, R\, P(R) \sim L^{2\alpha_0}$$

diverges as $L \to \infty$.

[9)] The bosonization technique allows calculating the 4-particle correlator exactly (i.e., with pair scattering). As a result, we have the well-known exponent

$$\alpha_T = 1 - 1/\sqrt{1 + \frac{V_0}{\pi v_f}}.$$

It coincides with α_0 only in the strong interaction limit. This distinction arises from the fact that the term in which we are interested involves only the direct interaction between two pairs.

It is instructive to consider the same quantities in the theory with noninteracting electrons. There, the probability to find a chiral pair is

$$P_{free} = \left(\frac{d}{|x-y|}\right)^2$$

(see Eq. (69)). As we have seen, this results in the independence of Δ from L. The other limit case is the system with a nonzero density of the order parameter. There, the probability to find a chiral pair is independent of the distance R and $\Delta \propto L$. The probability under discussion has an intermediate behavior. As a result, Δ increases with L, but the power exponent is smaller than unity. Both these properties can be considered a definition of the BKT phase. Moreover, the existence of a macroscopic, i.e., increasing with volume, number of bosons in the ground state is a sufficient condition for a long-range order itself. This is the case although their density tends to zero in the thermodynamic limit because each matrix element is proportional to the square root of the boson number in the state.

In the BKT system at $\alpha_0 < 1$, the temperature Θ_c of the phase transition to the unbroken phase is of the same order as in the limit of the infinitely strong interaction. Indeed, our estimate of Θ_c in the previous section was based on the logarithmic divergence of the action. This divergence also exists for $\alpha_0 < 1$ and hence our expressions for Θ_c and the correlation length ζ are valid in this case. We note that the upper-temperature boundary of the chiral phase coincides with the temperature region where power-law correlators exist, as it should. Indeed, it is well-known that for a finite temperature, the correlators decrease exponentially at distances longer than v_f^c/Θ [11]. Because we do not wish that the exponential asymptotic be reached within the size of the sample, it has to be smaller than v_f^c/Θ, or $\Theta \ll v_f^c/L \sim \Theta_c$.

The wave function for the BKT phase does not have the simple form in Eq. (20) because the interaction of chiral pairs is nonzero. Also chiral complexes with more than two particles are present in the ground-state wave function. However, properties of this phase are quite similar to properties of the phase with broken symmetry that appears in the limit of infinitely strong interaction.

5. BOSON REPRESENTATION AND SYMMETRY BREAKING

We consider the relation between the ground-state wave function in the boson representation and in our

representation. It is convenient to use the first in the form given in [23] for the Luttinger model (as it should, it coincides in the limit of infinitely strong interaction $\operatorname{th}\theta \to 1$ with the Schwinger model [32]):

$$|GS_0\rangle = \\ = N \exp\left(-\frac{1}{L}\sum_{n>0} \operatorname{th}\theta \hat{C}_L^\dagger(p_n)\,\hat{C}_R^\dagger(p_n)\right)|F\rangle, \quad (73)$$

where $\hat{C}_R^\dagger$ is the boson creation operator determined by the density of the right electrons; the subscript L denotes left bosons:

$$\hat{C}_{R,L}(p) = \sqrt{\frac{2\pi}{p}}\int dx \exp(\mp ipx)\varrho_{R,L}(x), \quad (74)$$

all $p > 0$, and

$$\varrho_{R,L}(x) = \hat{a}_{R,L}^\dagger(x)\,\hat{a}_{R,L}(x) - \hat{b}_{R,L}^\dagger(x)\,\hat{b}_{R,L}(x) + \\ + \hat{a}_{R,L}^\dagger(x)\,\hat{b}_{R,L}^\dagger(x) + \hat{b}_{R,L}(x)\,\hat{a}_{R,L}(x). \quad (75)$$

The parameters $\operatorname{sh}\theta$ and $\operatorname{ch}\theta$ allow passing to the new diagonalized fields $\hat{C}(\pm p)$. Wave function (73) has the lowest possible energy and satisfies the relation $\hat{C}(\pm p)|GS_0\rangle = 0$. We claim that the wave function $|GS_0\rangle$ coincides with our ground-state wave function with zero chirality (see Eq. (58)). We used the fermion representation in the paper because we need to see the symmetry breaking in the electron system. Rewriting $|GS_0\rangle$ in terms of electron operators directly with the help of Eq. (75) is a rather involved and nonuniversal procedure. In the boson approach, it is preferable to know results in advance because the boson representation is the most veiled way to see a symmetry breaking. The problems begin with noncommuting terms in Eq. (75). It is extremely difficult to find complexes decaying into chiral pairs and to prove the absence of the neutral ones in all orders. (We note that the form of the chiral complexes depends on a problem. For example, it changes drastically with increasing the electron component number, see [33], and the calculations become much more bulky in the case.) In the fermion treatment in this paper, the operator structure of the wave function is determined by the contribution of the saddle-point field to the action, S_0 (Eq. (30)). The operators are here considered anticommuting (see Appendix A for the details). This allows formulating the calculation rule for the action for a given electron operator configuration in explicit form (Eq. (37)).

In addition, the pairing effect cannot be obtained in any order of the perturbation theory (i.e., in the expansion of (73) in powers of $\hat{C}_L^\dagger\hat{C}_R^\dagger$). It is a nonperturbative effect. However, if the complexes are known from the outset, it is possible to rewrite $|GS_0\rangle$ in the fermion representation. Indeed, we know that the simplest complex decaying into chiral pairs for a zero chirality state is $\hat{a}_R^\dagger\hat{b}_L^\dagger\hat{a}_L^\dagger\hat{b}_R^\dagger$. We extract the relevant term from the entire state $|GS_0\rangle$:

$$N\exp\left(-\frac{1}{L}\sum_{n>0}\hat{C}_L^\dagger(p_n)\,\hat{C}_R^\dagger(p_n)\right)|F\rangle = \\ = N\Big[1+\operatorname{Tr}\Big(F^{(2)}(x,x',y,y')\,\hat{a}_R^\dagger(x)\,\hat{b}_L^\dagger(y')\,\hat{a}_L^\dagger(y)\times \\ \times\,\hat{b}_R^\dagger(x')\Big)+\ldots\Big]|F\rangle. \quad (76)$$

To calculate the coefficient $F^{(2)}$, we have to project the entire state on $\langle F|\hat{a}_R\hat{b}_L\hat{a}_L\,\hat{b}_R$, and hence

$$F^{(2)}(\tilde{x},\tilde{x}',\tilde{y},\tilde{y}') = \langle F|\hat{a}_R(\tilde{x})\,\hat{b}_L(\tilde{y}')\,\hat{a}_L(\tilde{y})\,\hat{b}_R(\tilde{x}')\times \\ \times\exp\left(-\frac{1}{L}\sum_{n>0}\hat{C}_L^\dagger(p_n)\,\hat{C}_R^\dagger(p_n)\right)|F\rangle. \quad (77)$$

The further train of thought is obvious. After bosonization, the matrix element is to be rewritten as a functional integral using the well-known relation

$$\langle F|\mathcal{R}\left(\hat{C}\right)\mathcal{R}'\left(\hat{C}^\dagger\right)|F\rangle = \\ = \int \mathcal{D}C\mathcal{D}\overline{C}\mathcal{R}(C)\,\mathcal{R}'\left(\overline{C}\right)\exp\left(-\operatorname{Tr}\overline{C}C\right),$$

where $\mathcal{R}$ and $\mathcal{R}'$ are arbitrary functions and C is a complex Bose field. Wave function (73) depends on the left and right electron densities, and it is therefore convenient to use the bosonization scheme involving left and right fields, $\hat{C}_L$ and $\hat{C}_R$ (see [32]), rather than a scheme with the total density and momentum canonically conjugate to it:

$$\Psi_{R,L}^\dagger(x) = \exp\left(A_{R,L}^\dagger(x)\right)\frac{\sigma_{R,L}^\dagger}{\sqrt{L}}\times \\ \times\exp\left(-A_{R,L}(x)\right). \quad (78)$$

Here

$$A_{R,L}^\dagger(x) = \frac{1}{L}\sum_{n>0}\exp(\mp ipx)\sqrt{\frac{2\pi}{p}}\hat{C}_{R,L}^\dagger,$$

and σ is the operator with a set of characteristics determined by $\Psi_{R,L}$. For example, from the anticommutator of the electron operators, we have $\sigma_{R,L}^\dagger\sigma_{R,L} = 1$ and $\{\sigma_{R,L},\sigma_{L,R}\} = 0$. Also, σ and $\sigma^\dagger$ should commute with $\hat{C}_{L,R}$ and $\langle F|\sigma_{R,L}^\dagger|F\rangle = 0$. It follows that

$$\int dx\,\Psi_{R,L}^\dagger(x)\,|F\rangle = L^{-1/2}\sigma_{R,L}^\dagger|F\rangle,$$

that is, $\sigma^{\dagger}_{R,L}$ coincides with the ladder operator of Haldane [23]. The scheme in (78) is identical to the standard one for condensed matter physics (see, e.g., [11]) but is more convenient in our problem.

To extract the electron or hole parts from (78), we can use the identities such as

$$\hat{a}_L\left(\tilde{y}\right) = \frac{1}{2\pi i}\int dx \frac{\Psi_L\left(x\right)}{\left(\tilde{y}-x-i0\right)}.$$

As a result, we obtain

$$F^{(2)}\left(\tilde{x},\tilde{x}',\tilde{y},\tilde{y}'\right) = \\ = \frac{1}{\left(2\pi i\right)^2}\frac{1}{\left(\tilde{x}-\tilde{y}'+i0\right)\left(\tilde{x}'-\tilde{y}+i0\right)}. \quad (79)$$

This implies that the second term in Eq. (76) is equal to

$$\int d\tilde{x}\, d\tilde{y}\, \hat{a}^{\dagger}_R\left(\tilde{x}\right)\hat{b}^{\dagger}_L\left(\tilde{x}\right)\hat{b}^{\dagger}_R\left(\tilde{y}\right)\hat{a}^{\dagger}_L\left(\tilde{y}\right),$$

in accordance with our previous result.

To obtain the chiral wave function, we prove that the state $|GS_1\rangle = \sigma_L\sigma^{\dagger}_R|GS_0\rangle$ is the state with an additional chiral pair. In the same way, we have

$$F^{(1)}\left(\tilde{x},\tilde{y}'\right) = \frac{1}{2\pi i}\frac{1}{\tilde{x}-\tilde{y}'+i0},$$

and hence the one-pair state is

$$\int d\tilde{x}\,\hat{a}^{\dagger}_R\left(\tilde{x}\right)\hat{b}^{\dagger}_L\left(\tilde{x}\right)|F\rangle$$

and the entire state $|GS_1\rangle$ is not invariant under chiral transformations. (It is highly essential that the electron and the hole are at the same spatial point. In principle, the form of $|GS_1\rangle$ might suggest that their positions are uncorrelated.) Of course, the energy difference between the state and $|GS_0\rangle$ is $4\pi v_f/L$. At the same time, in order to have a nonzero order parameter, the states with different chiralities have to be degenerate in energy. This would allow constructing a wave function giving a nonzero order parameter, Eq. (10), although the Hamiltonian has no symmetry-breaking term. To obtain the degeneracy, the thermodynamic limit $L\to\infty$ is typically used. This treatment is forbidden for us because $\Theta_c\to 0$ in the limit as well. At the same time, in the temperature region $\Theta \gg \Theta_{chiral} = 2\pi v_f/L$, we can consider these states degenerate too (cf. the discussion in Sec. 3 between Eqs. (20) and (21)). Therefore, the ground-state wave function with an arbitrary chirality and fixed phase can be constructed as

$$|\theta\rangle = \sum_{-\infty}^{\infty}\exp\left(in\theta\right)|GS_n\rangle, \quad (80)$$

where

$$|GS_n\rangle = \left(\sigma_L\sigma^{\dagger}_R\right)^n|GS_0\rangle$$

for $n>0$ and

$$|GS_n\rangle = \left(\sigma_R\sigma^{\dagger}_L\right)^n|GS_0\rangle$$

for $n<0$. (Indeed, $\hat{C}\left(\pm p\right)|\theta\rangle = 0$ and in order to check chirality of the state, any n-pair amplitude can be calculated in the same way.) Equation (80) is an alternative (in the boson representation) form of our symmetry-breaking ground-state wave function, Eq. (20).

It is not surprising that the boson representation is a nonobvious way to see a symmetry breaking in a fermion system. To recalculate the wave function, we should know the result in advance. We believe that the exclusive use of the boson representation in an analytic calculation is the reason why the fact of symmetry breaking has been unknown so far.

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APPENDIX A

Evolution operator for fermion systems

In this appendix, we derive the representation for the evolution operator of fermions in the external field as a functional integral with definite boundary conditions.

In the Schrödinger representation, the evolution operator $S(T)$ is

$$S[T] = T\exp(-i\int\limits_0^T H\,dt)|F\rangle\langle F|,$$

where H is a fermion Hamiltonian in the external field, which is bilinear in the fermion fields. As we have seen, the general problem with the electron–electron interaction can be reduced to this problem at the price of integration over the external field. For simplicity, we begin with the model with an empty ground state $|0\rangle$ rather than the Fermi one. (This allows writing the equations in a more compact form.) We also omit the spatial arguments.

We divide the time interval T into N infinitesimal pieces $\delta = T/N$ (with the point $i=N$ corresponding to

$t = 0$ and $i = 1$ to $t = T$) and introduce the sum over the complete set of quantum mechanical states $|k\rangle\langle k|$ at the intermediate points:

$$S[T] = \sum_{n_i} |k_N\rangle\langle k_N| \left(1 - i\delta H\right) |k_{N-1}\rangle \ldots \\ \ldots \langle k_2| \left(1 - i\delta H\right) |k_1\rangle\langle k_1|. \quad \text{(A.1)}$$

For any complete set of wave functions in the second-quantized representation, we have

$$\sum_n |k_i(n)\rangle\langle k_i(n)| = \int \mathcal{D}\xi_i \mathcal{D}\xi_i^\dagger \exp\left(-\mathrm{Tr}\, \xi_i^\dagger \xi_i\right) \times \\ \times \exp\left(\mathrm{Tr}\, \xi_i \hat{a}^\dagger\right) |0\rangle\langle 0| \exp\left(-\mathrm{Tr}\, \xi_i^\dagger \hat{a}\right) \quad \text{(A.2)}$$

(the index n corresponds to the set of all quantum numbers). The Grassmann variables ξ are defined in the usual way:

$$\int d\xi_i(n) = 0, \quad \int d\xi_i(n)\xi_i(n) = 1,$$

$$\left[\xi_i^\dagger(n), \xi_i(n)\right]_+ = 0, \quad \mathcal{D}\xi_i = \prod_n d\xi_i(n).$$

Equation (A.2) can be proved by direct comparison of the left- and right-hand sides. We use this representation to rewrite the sum over states as a functional integral.

At each point i, we obtain the following matrix element of the Hamiltonian:

$$\exp\left(-\mathrm{Tr}\, \xi_i^\dagger \xi_i\right) \langle 0| \exp\left(-\mathrm{Tr}\, \xi_i^\dagger \hat{a}\right) \times \\ \times \left(1 - i\delta H\left(\hat{a}^\dagger, \hat{a}\right)\right) \exp\left(-\mathrm{Tr}\, \xi_{i+1}^\dagger \xi_{i+1}\right) \times \\ \times \exp\left(\mathrm{Tr}\, \xi_{i+1} \hat{a}^\dagger\right) |0\rangle. \quad \text{(A.3)}$$

To calculate this matrix element, we move all creation operators to the right. For the Hamiltonian H depending linearly on $\hat{a}$ and $\hat{a}^+$, e.g., for the Hamiltonian in the external field, the result is

$$\exp\left(\mathrm{Tr}\, \xi_i^\dagger \left(\xi_{i+1} - \xi_i\right) + i\delta \mathrm{Tr}\, H\left(\xi_i^\dagger, \xi_{i+1}\right)\right).$$

Thus, the result of the calculation is that the creation and annihilation operators in the Hamiltonian are substituted by the Grassmann variables ξ and ξ^+.

The product over all intermediate points as $N \to \infty$ tends to

$$\exp\left(-\int_0^T dt\, \overline{\Psi}\,(t)\left[\partial_t + i\mathcal{H}\right]\Psi\,(t)\right) = \exp\left(i\int_0^T dt\, \mathcal{L}\right),$$

$$\mathcal{L} = \overline{\Psi}\left[i\partial_t - \mathcal{H}\right]\Psi,$$

where $\mathcal{L}$ is the Lagrangian of the system. This expression should be integrated over $\Psi, \bar{\Psi}$ at all intermediate points in time. The boundary points are special, however. The creation operators entering $|k_N\rangle$ and the annihilation operators entering $\langle k_1|$ are not contracted. They are variables on which the evolution operator depends.

We integrate over all intermediate variables and consider the answer as a function of the Grassmann variable ξ_1^+ (and $\xi_N^\dagger$). This function can be only linear: $A + B\xi_1$. Then the last integration in ξ_1 and ξ_1^+ gives

$$\int \mathcal{D}\xi_1 \mathcal{D}\xi_1^\dagger \exp\left(-\mathrm{Tr}\, \xi_1^\dagger \xi_1\right) \times \\ \times \exp\left(-\mathrm{Tr}\, \xi_1^\dagger \hat{a}\right)\left(A_1 + \mathrm{Tr}\, B_1 \xi_1\right) = A_1 + \mathrm{Tr}\, B_1 a.$$

Thus, we see that the variable ξ_1 should be substituted by an annihilation operator. Integrating over ξ_N^+, we conclude that ξ_N^+ is substituted by a creation operator.

Finally, we can formulate the following recipe: to calculate the evolution operator, we integrate $\exp\left(i\int_0^T \mathcal{L}\right)$ over $\Psi, \bar{\Psi}$ at all intermediate points. At $t = 0$, Ψ is fixed to $\hat{a}$, and at $t = T$, $\bar{\Psi}$ is fixed to $\hat{a}^+$. The values of $\bar{\Psi}$ at $t = 0$ and Ψ at $t = T$ remain arbitrary. As a result, the operators $\hat{a}$ and $\hat{a}^+$ are defined at different times. Therefore, they are to be regarded here as anticommuting.

If the ground state of our system is a filled Fermi sphere, we have to introduce two types of creation and annihilation operators $\hat{a}^\pm$ and $\hat{b}^\pm$ corresponding to electrons and holes. Then we can apply the above derivation in this case as well. We should introduce negative (Ψ^-) and positive (Ψ^+) frequency parts of Ψ variables and double the number of the ξ variables.

APPENDIX B

Calculation of Det Φ

We calculate the functional integral over the fields χ and $\bar{\chi}$. They obey zero initial conditions:

$$\mathrm{Det}\,\Phi = \int \mathcal{D}\overline{\chi}\,\mathcal{D}\chi \times \\ \times \exp\left(i\int_0^T dt \int dx \overline{\chi}\left(i\partial_t - \mathcal{H}_{ext}\left(x\right)\right)\chi\right), \quad \text{(B.1)}$$

where $\mathcal{H}_{ext} = \mathcal{H}_0\left(x\right) + \Phi\left(x,t\right)$. In the ordinary case, Det Φ can be calculated in the usual way using the identity

$$\ln\left[\mathrm{Det}\,\Phi\right] = \mathrm{Tr}\,\ln\Phi.$$

After differentiation over λ, the right-hand side of this identity is represented as

$$\mathrm{Tr}\left[-i\int\limits_0^1 d\lambda\left(-\partial_t - i\mathcal{H}_0(x) - i\lambda\Phi(x)\right)^{-1}\Phi(x)\right],$$

where the inverse operator is the Green's function with the same arguments. The result is usually independent of the order of the arguments. However, in the theory with the Adler–Schwinger anomaly, the sequence of time and spatial arguments is essential. The simplest way is to make spatial arguments equal first. In this case, the result contradicts the gauge invariance of the theory. In paper [20], a procedure free of this difficulty was suggested. The problem does not exist in the procedure because all calculations are done with nonequal variables until the end. It is based on the Heisenberg equation for the electron evolution operator $\hat{S}(\Phi) = \mathrm{Det}\,\Phi\exp\mathcal{S}_0(T)$ in the external field (without a direct electron–electron interaction). This claim guarantees the conservation of the electron number and, as a result, the theory is gauge invariant. The dependence of $\mathrm{Det}\,\Phi$ on the sequence time and spatial arguments considered above implies the existence of an ultraviolet divergence in the theory. In fact, we have regularized it in the usual way for the theory with the Adler–Schwinger anomaly: we required a gauge-invariant result (see [19]).

In the Heisenberg representation, we have

$$i\frac{\partial\hat{S}}{\partial T} = \left[H_{ext}, \hat{S}\right],$$

where H_{ext} is the noninteracting electron Hamiltonian (the external field is dependent on the time T), and the action $\mathcal{S}_0$ is defined by Eq. (30). We note that all creation operators are defined at the instant T and the annihilation operators at $t = 0$, and therefore, in the commutator $[H_{ext}, \mathcal{S}_0(T)]$, only the terms with creation operators do not commute with $\mathcal{S}_0$.

We can rewrite the last equation as

$$i\frac{\partial\ln\mathrm{Det}\,\Phi}{\partial T} = \exp(-\mathcal{S}_0)\left[H_{ext}, \exp\mathcal{S}_0\right] - i\frac{\partial\mathcal{S}_0}{\partial T}. \quad \text{(B.2)}$$

To calculate the commutator in this equation, we can use the well-known identity

$$\left[\hat{a}(x), \exp\left(\int dx' K(x')\,\hat{a}^\dagger(x')\right)\right] =$$
$$= \int dx_1\Delta(x_1 - x')\,K(x_1)\exp\left(\int dx' K(x')\,\hat{a}^\dagger(x')\right),$$

which can be proved by expanding the exponentials. (Here, K is an operator anticommuting with $\hat{a}$ and $\Delta(x_1 - x')$ is the anticommutator $\{\hat{a}(x), \hat{a}^\dagger(x')\}$ defined in Eq. (7).) The left-hand side of Eq. (B.2) is a c-number; this means that all operators in the right-hand side of this equation have to vanish. The c-number parts arise only from the following commutators:

$$\int dx\,\Phi(x)\left[\hat{b}(x)\,\hat{a}(x),\right.$$
$$\left.\exp\left(\int dy\,dy'\hat{a}^\dagger(y')\,G(y'T, yT-\varepsilon)\,\hat{b}^\dagger(y)\right)\right].$$

As a result, we have

$$i\frac{\partial\ln\mathrm{Det}\,\Phi}{\partial T} = \int\frac{dx\,dy\,dy'}{(2\pi i)^2}\Phi(x,T)\times$$
$$\times\left[\frac{G_R(y'T, yT-\varepsilon)}{(y'-x-i\delta)(y-x-i\delta)} + \right.$$
$$\left. + \frac{G_L(y'T, yT-\varepsilon)}{(x-y'-i\delta)(x-y-i\delta)}\right]. \quad \text{(B.3)}$$

This representation is general. To rewrite the right-hand side of this equation in our case, we recall that only the region $y \to y' \to x$ is essential in the first term. However, at the point $y \to y'$, the argument of the exponential in Green's function (28) vanishes. This means that the contribution is determined by the pre-exponential pole and only the first and the second terms of the expansion of the exponential can give nonvanishing contributions. All singularities in the integrand with respect to y in the function coming from the first term are in the same halfplane. We can close the contour in the other halfplane and prove that this integral vanishes. In the next order in Φ, only the part with a singularity in the lower halfplane of y gives a nonvanishing term. After the integration over y', we have (in the momentum-space representation)

$$\frac{-i}{2\pi}\int\limits_0^T dt_1\int\limits_0^\infty\frac{dp}{2\pi}p\Phi_{-p}(T)\,\Phi_p(t_1)\exp\left(-ipv_f(T-t_1)\right).$$

The L electrons give the same result but with the opposite sign of p in the region $p < 0$. After the integration of Eq. (B.3) and symmetrization, we obtain Eq. (31). We note that Eq. (31) is gauge invariant: the fields depending only on time do not contribute to Eq. (31).

APPENDIX C

Normalization coefficient and energy shift

We have seen that the matrix element in Eq. (9) can be expressed as a Gaussian-type functional integral. It

gives the normalization coefficient and the ground-state energy shift. Indeed, we can expand the exact wave function with respect to the free-electron functions. In the limit $T \to \infty$, only the matrix element between the lowest energy level survives. It can be represented as

$$Z = \exp(-i\Delta E T)|\langle\Omega|F\rangle|^2,$$

where ΔE is the ground-state energy shift. Comparing Z with the definition of the normalization coefficient Z_0 in Eq. (20), we can see that it is equal to the overlap probability of the ground states of the free and interacting electrons, $|\langle\Omega|F\rangle|^2$. The normalization coefficient should be calculated for a finite-size system, because it is exponentially small with the volume.

On the other hand, the matrix element in which we are interested is

$$Z = \frac{1}{\mathcal{N}} \int \mathcal{D}\Phi \exp\left[\frac{i}{2}\int_0^T dt\, dt_1 \times \right.$$
$$\times \int_{-\infty}^{\infty} \frac{dp}{2\pi} \Phi(p,t)\, \Phi(-p,t_1)\, V^{-1}(p)\, \delta(t-t_1) -$$
$$- \frac{1}{4\pi}\int_0^T dt\, dt_1 \int_{-\infty}^{\infty} \frac{dp}{2\pi} \Phi(-p,t)\, \Phi(p,t_1)\, |p| \times$$
$$\left. \times \exp\left[-i|p|v_f|t-t_1|\right]\right], \quad \text{(C.1)}$$

where $1/\mathcal{N}$ is normalization coefficient (34).

It is convenient to transform the integral operator. A more effective procedure is the transformation of integral operator (C.1) to a differential form. For this, we note the identity

$$\frac{1}{-2i|p|v_f}\left(\frac{\partial^2}{\partial t^2} + p^2 v_f^2\right)\int_0^T dt_1 \Phi_p(t_1) \times$$
$$\times \exp(-i|p|v_f|t-t_1|) = \Phi_p(t). \quad \text{(C.2)}$$

Thus, symbolically,

$$\exp(-i|p|v_f|t-t_1|) = \frac{-2i|p|v_f}{\left(\dfrac{\partial^2}{\partial t^2} + p^2 v_f^2\right)}\delta(t-t_1),$$

and the kernel in Eq. (C.1) is equal to

$$\frac{i}{2V_0}\frac{\dfrac{\partial^2}{\partial t^2} + \omega_p^2}{\dfrac{\partial^2}{\partial t^2} + p^2 v_f^2}\delta(t-t_1). \quad \text{(C.3)}$$

As a result, we have

$$Z = \frac{1}{\mathcal{N}}\int \mathcal{D}\Phi \exp\left[\frac{i}{2}\int_0^T dt \int_{-\infty}^{\infty}\frac{dp}{2\pi} V^{-1}(p) \times \right.$$
$$\left. \times\ \Phi(-p,t)\frac{\dfrac{\partial^2}{\partial t^2} + \omega_p^2}{\dfrac{\partial^2}{\partial t^2} + p^2 v_f^2}\Phi(p,t)\right]. \quad \text{(C.4)}$$

Taking into account that the normalization coefficient $\mathcal{N}$ cancels $\sqrt{\mathrm{Det}\,(i/2V_p)}$ that arises from the differential kernel definition, we obtain

$$Z^{-2} = \mathrm{Det}\,\frac{i\left(\dfrac{\partial^2}{\partial t^2} + \omega_p^2\right)}{i\left(\dfrac{\partial^2}{\partial t^2} + p^2 v_f^2\right)} = \frac{D}{D_0}.$$

To define the differential operator, we should have two initial conditions. In exactly the same way as in the derivation of the equation for the saddle-point field, we obtain the conditions

$$\begin{aligned} &\partial_t \Phi(p,0) - i|p|v_f \Phi(p,0) = 0, \\ &\partial_t \Phi(p,T) + i|p|v_f \Phi(p,T) = 0. \end{aligned} \quad \text{(C.5)}$$

Usually, determinants are calculated with zero boundary conditions

$$\Phi_p(0) = \Phi_p(T) = 0.$$

To reduce our problem to the problem with zero boundary conditions, we introduce

$$\Phi_p(t) = \overline{\Phi}_p(t) + \phi(t).$$

The field $\overline{\Phi}_p(t)$ is assumed to obey the equation $\Delta_t \overline{\Phi}_p(t) = 0$ with the initial conditions in Eq. (C.5). (As usual, $\Delta_t = \partial^2/\partial t^2 + \omega_p^2$.) The field $\phi(t)$ is arbitrary but with zero boundary conditions. The solution $\overline{\Phi}_p(t)$ can be expressed as

$$\overline{\Phi}_p(t) = \Phi_p(0)\frac{\sin(\omega_p(T-t))}{\sin(\omega_p T)} + \Phi_p(T)\frac{\sin(\omega_p t)}{\sin(\omega_p T)}.$$

(The constants $\Phi_p(0)$ and $\Phi_p(T)$ are arbitrary.) This means that the determinant is given by

$$D^{-1/2} = \int_{-\infty}^{\infty} d\Phi_p(0)\, d\Phi_p(T) \times$$
$$\times \exp\left[i\left(\Phi_p(T)\,\partial_t\phi_p(T) - \Phi_p(0)\,\partial_t\phi_p(0)\right)\right] \times$$
$$\times \int \mathcal{D}\phi_p(t) \exp\left(\phi_{-p}(t)\,\Delta_t\phi_p(t)\right). \quad \text{(C.6)}$$

The integral over ϕ_p can be calculated in the usual way,

$$C\left(p=0\right)\frac{\omega_p}{\sin(\omega_p T)},$$

where $C\left(p=0\right)$ is the $p=0$ contribution. It cancels in the final expression. With the identity

$$\begin{aligned}\Phi_p\left(T\right)\partial_t\phi_p\left(T\right)-\Phi_p\left(0\right)\partial_t\phi_p\left(0\right)=\\ =-\left(\Phi_p\left(0\right)^2+\Phi_p\left(T\right)^2\right)\left(|p|v_f-i\omega_p\operatorname{ctg}(\omega_p T)\right)+\\ +\frac{2i\omega_p}{\sin(\omega_p T)}\Phi_p\left(0\right)\Phi_p\left(T\right),\end{aligned}$$

we have

$$Z^{-2}=\prod_{p\neq 0}\frac{\sin(\omega_p T)}{\sin\left(v_f|p|T\right)}\times \\ \times\frac{p^2v_f^2-2i\omega_p|p|v_f\operatorname{ctg}(\omega_p T)+\omega_p^2}{2|p|v_f\omega_p\left(1-i\operatorname{ctg}(\omega_p T)\right)}. \quad \text{(C.7)}$$

If the temperature is nonzero, we should substitute T by $1/\Theta$. We note that Eq. (C.7) is valid even at the temperatures $\Theta\ll\Theta_{chiral}$, because only the Green's functions with equal-time arguments were used. In this temperature region, Z can be expressed as

$$Z=\prod_{p\neq 0}\exp\left(-\frac{\omega_p-|p|v_f}{2\Theta}\right)\frac{4\sqrt{|p|v_f\omega_p}}{\omega_p+|p|v_f}. \quad \text{(C.8)}$$

It is convenient to rewrite this equation in the form

$$Z=\exp\left[-\frac{L}{\Theta}\int\limits_0^\infty\frac{dp}{2\pi}\left(\omega_p-pv_f\right)+\right. \\ \left.+\frac{1}{2}\sum_{p\neq 0}\ln\frac{4\sqrt{|p|v_f\omega_p}}{\omega_p+|p|v_f}\right], \quad \text{(C.9)}$$

which shows the energy shift (the first term in the exponent) and the normalization coefficient (the second term) explicitly. The sums in this equation diverge because of the gapless spectrum. They have to be cut off at $p_{max}\sim 1/d$. To take the preexponential factor into account, we should calculate the next correction after the Riemann sum. As a result, for a short-range potential, we have

$$\Delta E\sim\frac{L}{4\pi d}\frac{v_f}{d}\sqrt{\frac{V_0}{\pi v_f}}$$

for the energy shift and

$$Z_0=4\sqrt{\frac{\pi v_f}{V_0}}\exp\left(-\frac{L}{4\pi d}\ln\frac{V_0}{\pi v_f}\right)$$

for the normalization coefficient.

APPENDIX D

Calculation of Sums

All the sums in the equation for the action can be calculated by differentiating $S\left(\alpha\right)$ with respect to the parameter α:

$$S\left(\alpha\right)=-\frac{2\pi}{L}\sum_{n_{min}}^{\infty}\frac{1}{p_n}\exp\left[\frac{2\pi in\alpha}{L}\left(x+i\delta\right)\right],$$

(α ranges within $(1,i\infty)$). After summation of the geometric series, we can rewrite it as

$$S\left(1,x\right)=\int\limits_1^{1-y_0}\frac{dy}{y}\left(1-y\right)^{n_{min}-1},$$

where

$$y_0\left(x\right)=\exp\left(2\pi i/L\right)\left(x+i\delta\right).$$

It is understood that $x\ll L$ here. This result can apply at $x\sim L$ as an order-of-magnitude estimate only. The final expression appearing in the action is

$$S\left(1,x\right)-S\left(1,y\right)=\int\limits_{y+i\delta}^{x+i\delta}\frac{dz}{z}\exp\left(-\frac{iz}{\zeta}\right), \quad \text{(D.1)}$$

where $\zeta=L/2\pi\left(n_{min}-1\right)^{-1}$. If $n_{min}\sim L\Theta/2\pi v_f^c$, then ζ is equal to the coherence length in Eq. (68).

We consider the influence of the boundary conditions on the action. In principle, any of them can be rewritten as $p_n=2\pi\left(n+\delta n\right)/L, |\delta n|<1/2$. In this case, at $\Theta=0$, the action is determined by the sum

$$S'\left(\alpha\right)=-\frac{2\pi}{L}\sum_{1}^{\infty}\frac{1}{p_n}\exp\left[\frac{2\pi i\left(n+\delta n\right)\alpha}{L}\left(x+i\delta\right)\right].$$

In the same way, we obtain

$$S'\left(1,x\right)-S'\left(1,y\right)=\int\limits_{y+i\delta}^{x+i\delta}\frac{dz}{z}\exp\left(-\frac{2\pi iz\delta n}{L}\right).$$

The result is that up to $|x-y|\sim L$ at $\Theta=0$, the action is independent of the boundary conditions. For $\Theta\gg\Theta_c$, we should cut off the sum at some $n=n_{min}$. As a result, ζ is substituted by $L/2\pi\left(n_{min}+\delta n-1\right)^{-1}$ in Eq. (D.1). This suggests the replacement of Θ_c by $\left(1-\delta n\right)\Theta_c$[10]. However, the transition temperature can be defined only up to a factor of the order of unity. Therefore, we should not take this into account.

[10] This means that Θ_c is determined by the excitation energy with the smallest momentum.

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