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SPONTANEOUS SYMMETRY BREAKING IN GENERAL RELATIVITY. VECTOR ORDER PARAMETER

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Gravitational properties of a hedgehog-type topological defect in two extra dimensions are considered in general relativity using a vector as the order parameter. All previous considerations were done using the order parameter in the form of a multiplet in the target space of scalar fields. The difference of these two approaches is analyzed and demonstrated in detail. Regular solutions of the Einstein equations are studied analytically and numerically. It is shown that the existence of a negative cosmological constant is sufficient for the spontaneous symmetry breaking of the initially flat bulk. Regular configurations have an increasing gravitational potential and are able to trap the matter on the brane. If the energy of spontaneous symmetry breaking is high, the gravitational potential has several minimum points. Spinless particles that are identical in the uniform bulk, being trapped at separate minima, acquire different masses and appear to the observer on the brane as different particles with integer spins.

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1. INTRODUCTION

The theories of brane world and multidimensional gravity are widely discussed in the literature. A natural physical concept is that a distinguished surface in the space–time manifold is a topological defect that appeared as a result of a phase transition with spontaneous symmetry breaking. The macroscopic theory of phase transitions allows considering the brane world concept self-consistently, even without the knowledge of the nature of the physical vacuum. The properties of topological defects (strings, monopoles, ...) are generally described with the aid of a multiplet of scalar fields forming a hedgehog configuration in extra dimensions (see [1] and the references therein). The scalar multiplet plays the role of the order parameter. The hedgehog configuration forms a vector proportional to a unit vector in the Euclidean target space of scalar fields. This model is self-consistent, but it is not the only generalization of a plane monopole to the curved space–time.

In a flat space–time, there is no difference between a vector and a hedgehog-type multiplet of scalar fields. On the contrary, in a curved space–time, scalar multiplets and vectors are transformed differently. In general relativity, the two approaches (based on a multiplet of scalar fields and a vector order parameter) therefore give different results which are worth comparing. Dealing with a vector order parameter seems to be more difficult, which is probably the reason why we could not find any papers considering phase transitions with a hedgehog-type vector order parameter in general relativity.

2. GENERAL FORMULAS

2.1. Lagrangian

The order parameter enters the Lagrangian via scalar bilinear combinations of its derivatives and via a scalar potential V allowing a spontaneous symmetry breaking. If ϕ_I is a vector order parameter, then V should be a function of the scalar

$$\phi^K \phi_K = g^{IK} \phi_I \phi_K,$$

and a bilinear combination of the derivatives is a tensor

$$S_{IKLM} = \phi_{I;K} \phi_{L;M}. \qquad (1)$$

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The index "$;K$" is used as usual for covariant derivatives. There are three ways to simplify S_{IKLM} into scalars, and the most general form of the scalar S formed via contractions of S_{IKLM} is

$$S = A\left(\phi^{K}_{;K}\right)^2 + B\phi^{L}_{;K}\phi^{;K}_{L} + C\phi^{M}_{;K}\phi^{K}_{;M}, \tag{2}$$

where A, B, and C are arbitrary constants. Different topological defects can be classified by these parameters. In a curved space–time, the scalar S depends not only on the derivatives of the order parameter but also on the derivatives of the metric tensor. This is the principal difference between a vector and a multiplet of scalar fields.

The general form of the Lagrangian determining gravitational properties of topological defects with a vector order parameter is

$$L\left(\phi_I, g^{IK}, \frac{\partial g_{IK}}{\partial x^L}\right) = L_g + L_d, \tag{3}$$

where

$$L_g = \frac{R}{2\kappa^2} \tag{4}$$

is the Lagrangian of the gravitational field, R is the scalar curvature of space–time, κ^2 is the (multidimensional) gravitational constant, and

$$L_d = A\left(\phi^{K}_{;K}\right)^2 + B\phi^{I}_{;K}\phi^{;K}_{I} + C\phi^{I}_{;K}\phi^{K}_{;I} - V\left(\phi^K\phi_K\right) \tag{5}$$

is the Lagrangian of a topological defect. The covariant derivative

$$\phi_{P;M} = \frac{\partial \phi_P}{\partial x^M} - \frac{1}{2}g^{LA}\left(\frac{\partial g_{AM}}{\partial x^P} + \frac{\partial g_{AP}}{\partial x^M} - \frac{\partial g_{MP}}{\partial x^A}\right)\phi_L \tag{6}$$

and raising of indices $\phi^K = g^{IK}\phi_I$ involve g^{IK} and $\partial g_{IK}/\partial x^L$, and it is therefore convenient to express the Lagrangian as a function of ϕ_I, g^{IK}, and $\partial g_{IK}/\partial x^L$.

2.2. Energy–momentum tensor

Varying the Lagrangian L_d in (5) with respect to δg^{IK} and having in mind that

$$\delta g_{IK} = -g_{KM}g_{IN}\delta g^{NM}, \tag{7}$$

we obtain the energy–momentum tensor[1)]

$$T_{IK} = \frac{2}{\sqrt{-g}}\left[\frac{\partial\sqrt{-g}\,L_d}{\partial g^{IK}} + g_{QK}g_{PI}\frac{\partial}{\partial x^L}\left(\sqrt{-g}\frac{\partial L_d}{\partial\dfrac{\partial g_{PQ}}{\partial x^L}}\right)\right]. \tag{8}$$

In the case of the vector order parameter, the potential

$$V\left(\phi^K\phi_K\right) = V\left(g^{IK}\phi_I\phi_K\right)$$

also undergoes a variation as g^{IK} is varied.

We proceed with the derivations taking the specific properties of particular topological defects into account.

3. GLOBAL STRING IN EXTRA DIMENSIONS

In our previous papers with Bronnikov (see [1] and the references therein), we considered global monopoles and strings as topological defects with the order parameter in the form of a hedge-hog-type multiplet of scalar fields in some flat target space. The aim of this paper is to describe these defects using a vector order parameter and compare the results.

3.1. Metric

The direction of the vector specifies one coordinate, and in the simplest case, the system is uniform and isotropic with respect to all the other coordinates. In [1], we presented the detailed properties of global strings in two extra dimensions. In what follows, we therefore consider a topological defect in a space–time with two extra dimensions. The order parameter is a space-like vector ($g^{IK}\phi_I\phi_K < 0$) directed normally from the brane hypersurface and depending on the only one specific coordinate, the distance from the brane. The whole ($D = d_0 + 2$)-dimensional space–time has the structure $\mathrm{M}^{d_0}\times \mathrm{R}^1\times \Phi^1$ and the metric

$$ds^2 = e^{2\gamma(l)}\eta_{\mu\nu}dx^\mu dx^\nu - \left(dl^2 + e^{2\beta(l)}d\varphi^2\right), \tag{9}$$

where $\eta_{\mu\nu} = \mathrm{diag}\,(1, -1, \ldots, -1)$ is the d_0-dimensional Minkowski brane metric ($d_0 > 1$) and φ is the angular cylindrical coordinate in extra dimensions; γ and β are functions of the distinguished extra-dimensional coordinate l, the distance from the center (i.e., from the brane), and $e^{\beta(l)} = r(l)$ is the circular radius. Greek indices $\mu, \nu, \ldots$ correspond to the d_0-dimensional space–time on the brane, and $I, K, \ldots$ to all the $D = d_0 + 2$

1) It differs from (94.4) in [2] because the Lagrangian is considered there as a function of g^{IK} and $\partial g^{IK}/\partial x^L$. Here and below, $\sqrt{-g}$ stands for $\sqrt{(-1)^{D-1}g}$.

coordinates. The metric tensor g_{IK} is diagonal, and its nonzero components are

$$g_{IK} = \begin{cases} e^{2\gamma}, & I = K = 0, \\ -e^{2\gamma}, & 0 < I = K < d_0, \\ -1, & I = K = d_0, \\ -e^{2\beta}, & I = K = \varphi. \end{cases}$$

The curvature of the metric on the brane caused by matter is supposed to be much smaller than the curvature of the bulk caused by the brane formation.

3.2. Regularity conditions

If the effect of matter on the brane is neglected, then there is no physical reason for singularities, and the self-consistent structure of a topological defect should be regular. A necessary condition of regularity is the finiteness of all invariants of the Riemann curvature tensor. The nonzero components of the Riemann tensor are

$$R^{AB}{}_{CD} = \\ = \begin{cases} -\gamma'^2\left(\delta^A_C\delta^B_D - \delta^A_D\delta^B_C\right), & A, B, C, D < d_0, \\ -\beta'\gamma', \quad A = C = \varphi, & B, D < d_0, \\ -\left(\gamma''+\gamma'^2\right)\delta^B_D, \; A = C = d_0, & B, D < d_0, \\ -\left(\beta''+\beta'^2\right), \; A = C = d_0, & B = D = \varphi, \end{cases} \tag{10}$$

where the prime denotes d/dl. One of the invariants of the Riemann tensor is the Kretchmann scalar $K = R^{AB}{}_{CD}R^{CD}{}_{AB}$, which is the sum of all nonzero $R^{AB}{}_{CD}$ squared, and hence all the nonzero components of the Riemann tensor, and specifically

$$\gamma', \quad \gamma'' + \gamma'^2, \quad \beta'\gamma', \quad \beta'' + \beta'^2 \tag{11}$$

must be finite. $r = 0$ is a singular point of the cylindrical coordinate system. The absence of a curvature singularity in the center follows from the last condition in (11). Let

$$\beta'' + \beta'^2 = c < \infty \quad \text{at} \quad l = 0. \tag{12}$$

Integrating (12) in the vicinity of the center, we have

$$\beta' = \frac{1}{l} + \frac{1}{3}cl + O\left(l^3\right). \tag{13}$$

Relation (13) ensures the correct ($= 2\pi$) circumference-to-radius ratio, or, equivalently, $dr^2 = dl^2$ as $l \to 0$. The quantity $\beta'\gamma'$ is finite at $l = 0$ if

$$\gamma' = O\left(l\right) \quad \text{or smaller as} \quad l \to 0. \tag{14}$$

3.3. Vector order parameter

Our aim is to consider the order parameter as a vector in extra dimensions directed normally from the Minkowski hypersurface. In the cylindrical coordinate system of extra dimensions, the only nonzero component of the vector order parameter is

$$\phi_{d_0} \equiv \phi. \tag{15}$$

In the space–time with metric (9), the covariant derivative

$$\phi_{I;K} = \delta^{d_0}_I\delta^{d_0}_K\phi' - \frac{1}{2}\delta_{IK}g'_{II}\phi \tag{16}$$

is a symmetric tensor:

$$\phi_{I;K} = \phi_{K;I}.$$

Hence,

$$\phi^I_{;K}\phi^{;K}_I = \phi^I_{;K}\phi^K_{;I},$$

and Lagrangian (5) takes the form

$$L_d = A\left(\phi' + \frac{1}{2}\phi\sum_K g^{KK}g'_{KK}\right)^2 + \\ + \widetilde{B}\left(\phi'^2 + \frac{1}{4}\phi^2\sum_L\left(g^{LL}g'_{LL}\right)^2\right) - V\left(-\phi^2\right), \tag{17}$$

which contains only two arbitrary constants A and $\widetilde{B} = B + C$. In (17), we set $g^{d_0d_0} = -1$ in accordance with (9). But we should keep in mind that (17) cannot be used in (8). To derive energy–momentum tensor (8), we should use Lagrangian (5) and set $g^{d_0d_0} = -1$, $\left(g^{d_0d_0}\right)' = 0$ after the differentiation. Nevertheless, the field equation can be derived using (17) in the general formula

$$\frac{1}{\sqrt{-g}}\left(\frac{\partial\sqrt{-g}\,L_d}{\partial\phi'}\right)' - \frac{\partial L_d}{\partial\phi} = 0. \tag{18}$$

In the space–time with metric (9), the sums in (17) are

$$S_n = \frac{1}{2^n}\sum_K\left(g^{KK}g'_{KK}\right)^n = d_0\gamma'^n + \beta'^n, \\ n = 1, 2, \ldots, \tag{19}$$

and the determinant of the metric tensor is

$$g = (-1)^{D-1}e^{2(d_0\gamma+\beta)}. \tag{20}$$

3.4. Field equation

We consider the case $A \neq 0$, $\widetilde{B} = 0$ in what follows. The case $A = 0$, $\widetilde{B} \neq 0$ will be considered elsewhere. Substituting (17) with $A = 1/2$ and $\widetilde{B} = 0$ in (18), we obtain the following field equation in the case of vector order parameter:

$$[\phi' + (d_0\gamma' + \beta')\,\phi]' + \frac{\partial V}{\partial \phi} = 0. \qquad (21)$$

In the case of the multiplet of scalar fields, we had [1]

$$\phi'' + \phi'\,(d_0\gamma' + \beta') - \phi e^{-2\beta} + \frac{\partial V}{\partial \phi} = 0. \qquad (22)$$

Unlike (22), field equation (21) does not depend directly on β (and hence on the circular radius $r = \ln\beta$), but instead includes second derivatives of the metric tensor. In the flat space–time,

$$\gamma' = 0, \quad \beta' = \frac{1}{l}, \quad \beta'' = -\frac{1}{l^2}, \quad e^{-2\beta} = \frac{1}{l^2},$$

and both field equations reduce to

$$\phi'' + \frac{1}{l}\phi' - \frac{1}{l^2}\phi + \frac{\partial V}{\partial \phi} = 0. \qquad (23)$$

3.5. Energy-momentum tensor

The energy–momentum tensor in (8) inevitably contains second derivatives, but they can be eliminated with the aid of field equation (21). The final result of a rather tiresome derivation is

$$T_I^K = \frac{1}{2}\delta_I^K\,[\phi' + (d_0\gamma' + \beta')\,\phi]^2 + \delta_I^K V + \left(\delta_I^{d_0}\delta_{d_0}^K - \delta_I^K\right)\frac{\partial V}{\partial \phi}\phi. \qquad (24)$$

Unlike in the scalar multiplet case, energy–momentum tensor (24) contains not only the potential V but also its derivative $\partial V/\partial\phi$.

Correctness of (24) is checked by the derivation of the covariant divergence $T^K_{I;K}$ (actually, $T^K_{d_0;K}$). Again, with the aid of field equation (21), we confirm that $T^K_{d_0;K} = 0$.

3.6. Einstein equations

The same way as in [1], we use the Einstein equations in the form

$$R_I^K = \kappa^2 \widetilde{T}_I^K,$$

where R_I^K is the Ricci tensor,

$$R_I^K = \begin{cases} \delta_I^K\,[\gamma'' + \gamma'\,(d_0\gamma' + \beta')]\,, & I < d_0, \\ \delta_{d_0}^K\left[d_0\left(\gamma'' + \gamma'^2\right) + \beta'' + \beta'^2\right], & I = d_0, \\ \delta_{\varphi}^K\,[\beta'' + \beta'\,(d_0\gamma' + \beta')]\,, & I = \varphi, \end{cases}$$

and

$$\widetilde{T}_I^K = T_I^K - \frac{1}{d_0}\delta_I^K T = -\frac{1}{d_0}\delta_I^K\,[\phi' + (d_0\gamma' + \beta')\,\phi]^2 - \delta_I^K\frac{2}{d_0}V + \delta_I^K\left(\delta_I^{d_0} + \frac{1}{d_0}\right)\frac{\partial V}{\partial\phi}\phi.$$

In the case of the vector order parameter, the set of Einstein equations

$$\gamma'' + \gamma'\,(d_0\gamma' + \beta') = \kappa^2\left[-\frac{1}{d_0}\,[\phi' + (d_0\gamma' + \beta')\,\phi]^2 - \frac{2V}{d_0} + \frac{1}{d_0}\frac{\partial V}{\partial\phi}\phi\right], \qquad (25)$$

$$d_0\gamma'' + \beta'' + d_0\gamma'^2 + \beta'^2 = \kappa^2\left[-\frac{1}{d_0}\,[\phi' + (d_0\gamma' + \beta')\,\phi]^2 - \frac{2V}{d_0} + \left(1 + \frac{1}{d_0}\right)\frac{\partial V}{\partial\phi}\phi\right], \qquad (26)$$

$$\beta'' + \beta'\,(d_0\gamma' + \beta') = \kappa^2\left[-\frac{1}{d_0}\,[\phi' + (d_0\gamma' + \beta')\,\phi]^2 - \frac{2V}{d_0} + \frac{1}{d_0}\frac{\partial V}{\partial\phi}\phi\right] \qquad (27)$$

consists of three first-order equations for γ', β', and ϕ. Both γ and β enter Eqs. (25)–(27) not directly but via the derivatives. In the case of a scalar multiplet order parameter (see Eqs. (14)–(16) in [1]), β enters the Einstein equations directly, and the system of equations is of the fourth order.

Field equation (21) is not independent. It is a consequence of Einstein equations (25)–(27) due to the Bianchi identity.

3.6.1. First integral

Eliminating the second derivatives γ'' and β'' in (25)–(27), we obtain the relation

$$(d_0\gamma' + \beta')^2 - \left(d_0\gamma'^2 + \beta'^2\right) = -\kappa^2\left\{[\phi' + (d_0\gamma' + \beta')\,\phi]^2 + 2V\right\}, \qquad (28)$$

which can be considered a first integral of system (25)–(27).

3.6.2. Further simplification

Equations (25) and (27) have the same right-hand sides. Subtracting one from the other yields the equation

$$\left(\gamma' - \beta'\right)' + \left(\gamma' - \beta'\right)\left(d_0\gamma' + \beta'\right) = 0, \qquad (29)$$

which can be used instead of one of Eqs. (25) and (27). With the aid of relations (28) and (29) the complete set of equations can be reduced to a simpler form. Introducing new functions

$$U = \gamma' - \beta', \quad W = d_0\gamma' + \beta', \quad Z = \phi' + W\phi, \quad (30)$$

we obtain the set of four first-order equations

$$U' = -UW, \qquad (31)$$

$$W' = \kappa^2 \frac{d_0+1}{d_0}\left(\frac{\partial V}{\partial \phi}\phi - 2V - Z^2\right) - W^2, \qquad (32)$$

$$\phi' = Z - W\phi, \qquad (33)$$

$$Z' = -\frac{\partial V}{\partial \phi}. \qquad (34)$$

The functions β' and γ', and their combination

$$S_2 = d_0\gamma'^2 + \beta'^2$$

in (19) are expressed via U and W as follows:

$$\gamma' = \frac{U+W}{d_0+1}, \quad \beta' = \frac{W - d_0 U}{d_0+1}, \quad S_2 = \frac{d_0U^2 + W^2}{d_0+1}.$$

The first integral (28), rewritten in terms of U, W, and Z as

$$W^2 - U^2 = -\kappa^2 \frac{d_0+1}{d_0}\left\{Z^2 + 2V\right\},$$

allows simplifying (32) even further:

$$W' = \kappa^2 \frac{d_0+1}{d_0}\frac{\partial V}{\partial \phi}\phi - U^2.$$

The set of equations

$$\begin{aligned} U' &= -UW, \\ W' &= \kappa^2 \frac{d_0+1}{d_0}\frac{\partial V}{\partial \phi}\phi - U^2, \\ \phi' &= Z - W\phi, \\ Z' &= -\frac{\partial V}{\partial \phi} \end{aligned} \qquad (35)$$

is most convenient for both analytic and numerical analysis.

3.7. General analysis of the equations

Equations (25)–(27) are invariant under the addition of arbitrary constants to γ and β. Without loss of generality, we can set

$$\gamma(0) = 0. \qquad (36)$$

The requirement of regularity in the center dictates condition (13), and, if we do not consider configurations with an angle deficit (or surplus), we have

$$r = e^{\beta} = l \quad \text{as} \quad l \to 0. \qquad (37)$$

Integrating (29) with boundary conditions (36) and (37), we obtain

$$\gamma' - \beta' = -e^{-(d_0\gamma + \beta)}. \qquad (38)$$

It follows from (38) that $\beta' > \gamma'$ everywhere.

We recall that topological defects formed as multiplets of scalar fields [1] are of three types. Integral curves can terminate with:

a) an infinite circular radius $r(l)$ as $l \to \infty$;

b) a finite circular radius $r_\infty = r(\infty) = \text{const} < \infty$;

c) a second center $r = 0$ at some finite $l = l_c$.

In the vector order parameter case, the situation is different. Equation (38) allows proving that a regular configuration can terminate neither with a finite circular radius r_∞ as $l \to \infty$ nor in the second center.

We suppose for a moment that $r_\infty = \text{const} < \infty$. Then $\beta'(\infty) = 0$, and (38) reduces to

$$\gamma' = -\frac{1}{r_\infty}e^{-d_0\gamma} \quad \text{as} \quad l \to \infty.$$

After integration, we have

$$e^{d_0\gamma} = \frac{d_0}{r_\infty}\left(l_0 - l\right),$$

where l_0 is a constant of integration. The left-hand side is obviously positive, while the right-hand side becomes negative and infinitely large as $l \to \infty$. Hence, $r_\infty = \text{const} < \infty$ is impossible.

The second center is also impossible. In the vicinity of the second center, the left-hand side of (38) becomes large positive due to $-\beta'$, and the right-hand side remains negative.

We conclude that regular configurations of topological defects with the vector order parameter start at the center $l = 0$ and terminate at $l \to \infty$ with an infinitely increasing circular radius $r(l) \to \infty$.

It follows from the requirement of regularity in (14) that $\gamma' = \gamma_0'' l$ as $l \to 0$. From the first integral (28), we find the relation between γ_0'', ϕ_0', and V_0:

$$\gamma_0'' = -\frac{\kappa^2}{d_0}\left(2\phi_0'^2 + V_0\right), \qquad (39)$$

where V_0 is the value of the potential at the center $l = 0$. In both cases (scalar multiplet and vector order parameter), the value $\phi_0' = \phi'(0)$ is not restricted by the equations. The difference is that in the scalar multiplet case, ϕ_0' is uniquely fixed by the regularity requirement, and in the case of vector order parameter, ϕ_0' remains a free parameter.

3.8. Asymptotic behavior

The regularity condition requires that γ' be finite everywhere. Within the domain of regularity, γ' tends to a fixed finite value γ'_∞ as $l \to \infty$. As soon as $r(l) \to \infty$ as $l \to \infty$, we see from (38) that $\gamma' - \beta' \to 0$. Hence, $\beta'(\infty) = \gamma'_\infty$. The field $\phi(l)$ also tends to its finite value $\phi_\infty = \phi(\infty)$. It then follows from field equation (21) that $\partial V/\partial\phi \to 0$ as $l \to \infty$, i.e., the regular configuration terminates at an extremum of the potential $V(\phi)$. Let

$$V_\infty = V(\phi_\infty), \quad V'(\phi_\infty) = 0.$$

From the first integral (28), we find the limit value γ'_∞:

$$\gamma'_\infty = \sqrt{-\frac{2\kappa^2 V_\infty}{(d_0+1)\left[d_0 + (d_0+1)\kappa^2\phi_\infty^2\right]}}. \tag{40}$$

A necessary condition for the existence of regular configurations of topological defects with the vector order parameter is $V_\infty < 0$.

To find the asymptotic behavior of $\phi(l)$ and $W(l)$, we linearize Eqs. (35) as $l \to \infty$:

$$\phi = \phi_\infty + \delta\phi,$$

$$W = (d_0+1)\gamma'_\infty + \delta W,$$

$$\begin{gathered} \delta W' = \kappa^2 \frac{d_0+1}{d_0} V''_\infty \phi_\infty \delta\phi, \\ \delta\phi' = \delta Z - (d_0+1)\gamma'_\infty \delta\phi - \phi_\infty \delta W, \\ \delta Z' = -V''_\infty \delta\phi, \end{gathered} \tag{41}$$

where primes denote the derivatives d/dl ($\delta W' = d\delta W/dl, \ldots$) except

$$V''_\infty = \left.\frac{\partial^2 V}{\partial\phi^2}\right|_{\phi=\phi_\infty}.$$

Eliminating δZ and δW, we obtain a second-order linear homogeneous equation for $\delta\phi$,

$$\delta\phi'' + (d_0+1)\gamma'_\infty\delta\phi' + \frac{2\kappa^2|V_\infty|V''_\infty}{d_0(d_0+1)\gamma'^2_\infty}\delta\phi = 0.$$

If the extremum of the potential is a minimum ($V''_\infty > 0$), the nontrivial solution vanishes as $l \to \infty$:

$$\delta\phi = Ae^{\lambda_+ l} + Be^{\lambda_- l}, \tag{42}$$

where A and B are constants of integration, and both eigenvalues

$$\lambda_\pm = -\frac{(d_0+1)\gamma'_\infty}{2} \times \left(1 \mp \sqrt{1 - \frac{8\kappa^2|V_\infty|V''_\infty}{d_0(d_0+1)^3\gamma'^4_\infty}}\right) \tag{43}$$

are either negative or have negative real parts. The absence of increasing solutions is the reason why ϕ_0' remains a free parameter in the vector order parameter case.

The asymptotic behavior of the field $\phi(l)$ far from the center is determined by two constant parameters of the symmetry breaking potential near its extremum, V_∞ and V''_∞. If the extremum is a minimum, $V''_\infty > 0$, then the expression under the root can be both positive and negative. Therefore, $\phi(l)$ can tend to ϕ_∞ either smoothly or with oscillations. In the space of physical parameters, the boundary between smooth and oscillating solutions is determined by the relation

$$\frac{8\kappa^2|V_\infty|V''_\infty}{d_0(d_0+1)^3\gamma'^4_\infty} = 1. \tag{44}$$

Oscillating behavior of the field $\phi(l)$ induces oscillations of β' and γ'. If γ' changes sign, then $\gamma(l)$ can have minima. We recall that γ acts as a gravitational potential, and hence the matter can be trapped near the minima of $\gamma(l)$.

Usually, $\phi = 0$ is a maximum of the potential $V(\phi)$. This is also an extremum, $\partial V/\partial\phi = 0$ at $\phi = 0$. Regular configurations that start at the center $l = 0$ with $\phi(0) = 0$ can also terminate with $\phi_\infty = 0$ as $l \to \infty$. In this case, $V''_\infty = V''(0) < 0$, and linear set (41) reduces to the following asymptotic equation for $\phi(l)$:

$$\phi'' + (d_0+1)\gamma'_\infty\phi' - |V''_\infty|\phi = 0.$$

Its general solution is a linear combination of functions vanishing and growing as $l \to \infty$,

$$\phi = Ae^{-\lambda_+ l} + Be^{-\lambda_- l},$$

$$\lambda_\pm = \frac{(d_0+1)\gamma'_\infty}{2} \pm \sqrt{\frac{(d_0+1)^2\gamma'^2_\infty}{4} + |V''_\infty|}, \quad l \to \infty.$$

The regularity requirement demands that the increasing solutions be excluded from consideration. This can be done at the expense of ϕ_0'. Regular solutions terminating at a maximum of the potential can exist only at some fixed values of ϕ_0'.

3.9. Boundary conditions

The complete set of equations determining the structure of a topological defect in the case of a vector order parameter, Eqs. (25), (27), and (28), is of the third order with respect to the three unknowns γ', β', and ϕ. The simple solution is uniquely determined by the values of these three functions at any regular point. The center $l = 0$ is a singular point of the cylindrical coordinate system. The condition $\phi(0) = 0$ is satisfied for both symmetries (high and broken). β' is infinite at $l = 0$. We have to set the boundary conditions very close to the center, but not exactly at $l = 0$.

For numerical analysis, it is convenient to deal with a system of four first-order equations solved for derivatives (35). The symmetry breaking potential $V(\phi)$ enters Eqs. (35) only via its derivative $\partial V/\partial\phi$. If we leave only the leading terms in the boungary conditions,

$$U = -\frac{1}{l}, \quad W = \frac{1}{l} \quad \text{as} \quad l \to 0,$$

then we lose any information about the absolute value of the potential. The value $V_0 = V(0)$ appears in the next approximation. Using expansion (13) of β' in the vicinity of the center and Eq. (38), we express c in terms of γ_0'':

$$c = -(d_0 - 2)\,\gamma_0''.$$

To preserve the complete information about the symmetry breaking potential, we have to write the boundary conditions as $l \to 0$ as follows

$$U = \frac{1}{3}(d_0+1)\,\gamma_0'' l - \frac{1}{l}, \quad W = \frac{2}{3}(d_0+1)\,\gamma_0'' l + \frac{1}{l}, \quad \phi = \phi_0' l, \quad Z = 2\phi_0'. \tag{45}$$

The values γ_0'', ϕ_0', and V_0 are not independent. They are connected with each other by Eq. (39).

3.10. Solutions in case $\partial V/\partial\phi = 0$

If the potential $V = V_0$ is independent of ϕ, then it actually plays the role of the cosmological constant $\Lambda = \kappa^2 V_0$. The peculiarity of the vector order parameter is that Eqs. (35) lose the information about the potential if $\partial V/\partial\phi \equiv 0$. V_0 is present only in boundary conditions (45). Equations (35) with $\partial V/\partial\phi \equiv 0$ and boundary conditions (45) have the analytic solution

$$U = -\frac{\sqrt{C}}{\operatorname{sh}\left(\sqrt{C}\,l\right)}, \quad W = \sqrt{C}\operatorname{cth}\left(\sqrt{C}\,l\right),$$

$$\phi(l) = \frac{2\phi_0'}{\sqrt{C}}\operatorname{th}\frac{\sqrt{C}\,l}{2},$$

where

$$C = 2(d_0+1)\,\gamma_0'' = -\frac{2(d_0+1)}{d_0}\left(2\kappa^2\phi_0'^2 + \Lambda\right). \tag{46}$$

The solution is regular if $C \geq 0$, i.e., $\Lambda \leq -2\kappa^2\phi_0'^2$. For $g_{00} = e^{2\gamma}$ and $r = e^{\beta}$, we find

$$g_{00}(l) = e^{2\gamma} = \left(\operatorname{ch}\frac{\sqrt{C}l}{2}\right)^{4/(d_0+1)},$$

$$r(l) = \frac{2\operatorname{sh}\left(\dfrac{\sqrt{C}\,l}{2}\right)}{\sqrt{C}}\left(\operatorname{ch}\frac{\sqrt{C}l}{2}\right)^{-(d_0-1)/(d_0+1)}.$$

The slope ϕ_0' remains arbitrary. If $\phi_0' = 0$, this solution reduces to the one found earlier (see [1] and [3]) in the special case $\phi \equiv 0$. The point is that the Einstein equations with a negative cosmological constant have a nontrivial solution (with a nonzero order parameter) even without a symmetry breaking potential.

The necessary condition for regular solutions with broken symmetry is the existence of extremum points of $V(\phi)$, where $\partial V/\partial\phi = 0$. In case $V = \text{const}$, the condition $\partial V/\partial\phi = 0$ is satisfied identically, and the order parameter ϕ can formally tend to any ϕ_∞ as $l \to \infty$. The above analytic solution shows that the existence of a negative cosmological constant is sufficient for the symmetry breaking of a uniform flat bulk.

The special case $C = +0$ in (46) with $\gamma_0'' = 0$ and

$$\phi_0' = \pm\sqrt{-\frac{\Lambda}{2\kappa^2}} \tag{47}$$

corresponds to the flat bulk $g_{00}(l) = 1$, and $r(l) = l$.

3.11. Weak curvature of space–time

The limit $\kappa^2 \to 0$ is the transition to a flat space–time. The functions β' and γ' reduce to $\beta' = l^{-1}$ and $\gamma' = 0$. Field equation (21) reduces to (23), which is the usual equation for the order parameter in cylindrical coordinates in a flat space–time. The symmetry breaking potential V is a function of ϕ^2, and hence $\partial V/\partial\phi \sim \phi$ and (23) has a trivial solution $\phi = 0$ corresponding to the symmetric (unbroken) state. The nontrivial solutions that start with $\phi(0) = 0$, $\phi'(0) \neq 0$ and terminate with $\phi = \phi_m$ at an extremum of the potential ($\partial V(\phi_m)/\partial\phi = 0$) describe the states of broken symmetry. Equation (23) is nonlinear. However, depending on the form of the potential $V(l)$, it can also have a sequence of nontrivial solutions $\phi_n(l)$, $n = 0, 1, 2\ldots$, with zero boundary conditions $\phi(0) = \phi(\infty) = 0$

on both ends. The discrete sequence of derivatives $\lambda_n = \phi'_n(0)$ forms the eigenvalues for the eigenfunctions $\phi_n(l)$. The functions $\phi_n(l)$ change sign n times. The nontrivial solutions of the field equation with $\phi'(0)$ within the interval $(\lambda_n, \lambda_{n+1})$ change sign $n+1$ times.

The principal difference between Eqs. (21) and (23) is that the coefficient $(d_0\gamma' + \beta')$ at ϕ' in a curved space–time does not vanish as $l \to \infty$. If $\phi = \phi_m$ is a minimum of $V(\phi)$, then $V''(\phi_m) > 0$, and the linearized field equation (23) in the case of a flat space–time as $l \to \infty$ reduces to

$$\phi'' + V''(\phi_m)(\phi - \phi_m) = 0$$

and describes nonvanishing oscillations. In a curved space–time, the oscillations vanish as $l \to \infty$ in accordance with (42).

Further detailed analysis is done with the aid of numerical integration.

4. NUMERICAL ANALYSIS

4.1. Regular solutions in the space of parameters

The numerical integration of Eqs. (35) is performed for the "Mexican hat" potential taken in the same form as in [1]:

$$V = \frac{\lambda\eta^4}{4}\left[\varepsilon + \left(1 - \frac{\phi^2}{\eta^2}\right)^2\right]. \tag{48}$$

Potential (48) has three extremum points: a maximum at $\phi = 0$ and two minima at $\phi = \pm\eta$. At the limit values of the order parameter, we have

$$1)\quad V'_\infty = 0, \quad V''_\infty = 2\eta^2, \quad \phi_\infty = \pm\eta,$$

$$2)\quad V'_\infty = 0, \quad V''_\infty = -\eta^2, \quad \phi_\infty = 0.$$

The dimensionless parameter ε moves the "Mexican hat" up and down. It is equivalent to adding a cosmological constant. The energy of spontaneous symmetry breaking is characterized by $\eta^{2/(D-2)}$, and

$$a = \frac{1}{\sqrt{\lambda}\eta} \tag{49}$$

determines the length scale, as usual. In most cases, a is associated with the core radius of a topological defect. Without loss of generality, we set $a = 1$ in computations. The strength of the gravitational field is characterized by the dimensionless parameter

$$\Gamma = \kappa^2\eta^2. \tag{50}$$

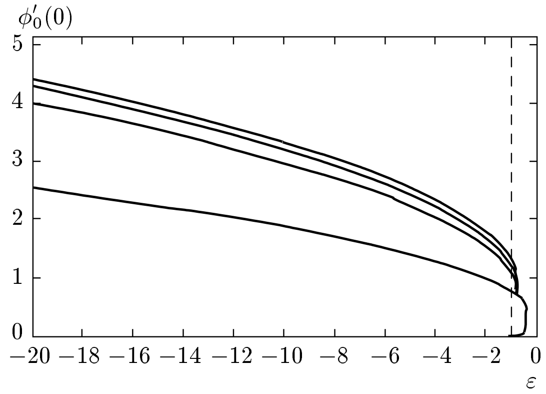

Fig. 1. The domain of regular configurations in the plane (ε, ϕ'_0) for $d_0 = 4$ and $\Gamma = 1$. The upper curve is the boundary of the existence of regular solutions. Other curves separate the regions with different signs of ϕ_∞. Below the lower curve, $\phi(l)$ does not change sign. Between the first and the second curves from bottom, the order parameter changes its sign once. Between the second and the third curves, it changes sign twice, and so on. The curves quickly condense to the upper curve

In the case of a vector order parameter, the state of broken symmetry is controlled by four parameters d_0, ε, Γ, and ϕ'_0. The main difference is that in the scalar multiplet case, regular configurations with given d_0, ε, and Γ existed only for a fixed value of ϕ'_0, but with a vector order parameter, regular configurations with given d_0, ε, and Γ exist within some interval $0 < \phi'_0 \le \phi'_{0\,\max}$, whose upper boundary $\phi'_{0\,\max}$ depends on d_0, ε, and Γ. This additional parametric freedom allows forgeting about the so-called "fine tuning" of the physical parameters.

For visual demonstration, it is worth fixing $d_0 = 4$ and one of the other three parameters. Then the domain of existence of regular solutions can be presented as a map in the plane of two remaining parameters.

Figure 1 shows the domain of regular configurations in the plane (ε, ϕ'_0) for $d_0 = 4$ and $\Gamma = 1$. Depending on the values of ε and ϕ'_0, the order parameter $\phi(l)$ tends to $+\eta$, 0, or $-\eta$ as $l \to \infty$. The sequence of curves $f_n(\varepsilon)$ in Fig. 1 are those where $\phi(l) \to 0$ as $l \to \infty$. They separate the domains with different signs of ϕ_∞. Below the bottom curve $f_1(\varepsilon)$, where $0 < \phi'_0 < f_1(\varepsilon)$, the order parameter $\phi(l)$ does not change sign. Between $f_1(\varepsilon) < \phi'_0 < f_2(\varepsilon)$, it changes the sign once. In the domain $f_2(\varepsilon) < \phi'_0 < f_3(\varepsilon)$, it changes the sign twice, and so on. The curves $f_n(\varepsilon)$ rapidly condense to the upper curve $f_\infty(\varepsilon)$ as $n \to \infty$. $f_\infty(\varepsilon)$ is the upper boundary of the existence of regular solutions (in the

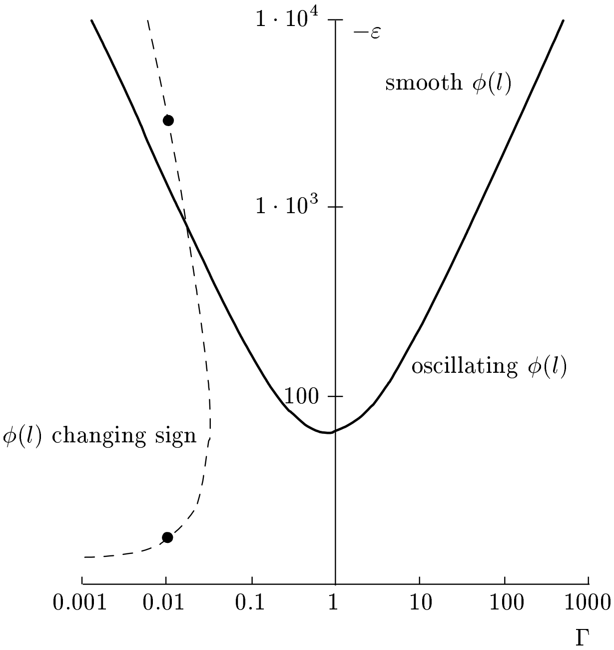

Fig. 2. Map of regular solutions in the plane $(\Gamma, -\varepsilon)$ for $\phi_0' = \pm\sqrt{-(\varepsilon+1)/8}$, $d_0 = 4$. The solid curve separates the regions of smooth (above) and oscillating (below) behavior of the order parameter as $l \to \infty$. To the left of the dashed curve, the order parameter changes sign

particular case $d_0 = 4$ and $\Gamma = 1$).

The curves in Fig. 1 are those where

$$\phi_\infty\left(\phi_0', \varepsilon, d_0 = 4, \Gamma = 1\right) = 0. \tag{51}$$

Similar curves can be shown for fixed ϕ_0' in the plane (ε, Γ). For instance, the dashed line in Fig. 2 is the first of the curves $\phi_\infty(\phi_0' = \pm\sqrt{-(\varepsilon+1)/8}, \varepsilon, d_0 = 4, \Gamma) = 0$, where the order parameter tends to zero as $l \to \infty$. The value $\phi_0' = \pm\sqrt{-(\varepsilon+1)/8}$ in (47) corresponds to $\gamma_0'' = 0$ in (39). This is the case $C = 0$ in (46), and hence the symmetry breaking of the flat bulk is entirely caused by the potential $V(\phi)$, not by the cosmological constant. To the right of the dashed line, $\phi(l)$ does not change sign.

For potential (48), boundary line (44) between the oscillating and smooth $\phi(l)$ is

$$-\varepsilon_b = 16\frac{(1+G)^2}{G}, \quad G = \frac{d_0+1}{d_0\Gamma}. \tag{52}$$

It is shown in Fig. 2 (solid line). Below the solid line, the order parameter $\phi(l)$ tends to its limit value ϕ_∞ with damped oscillations (see Fig. 3), and above this curve, without oscillations (see Fig. 4). The curves in Fig. 3 correspond to the close vicinity of the lower black point on the dashed curve in Fig. 2, and the curves in Fig. 4 correspond to the vicinity of the upper black point.

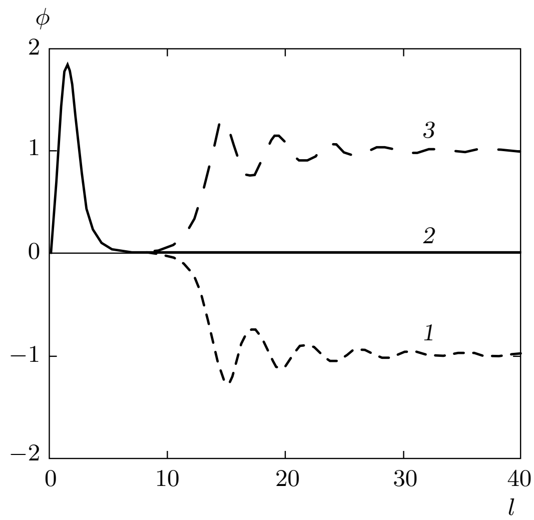

Fig. 3. Oscillating solutions in the close vicinity of the lower point on the dashed curve in Fig. 2: $\varepsilon = -17.413$ (*1*), -17.403 (*2*), -17.390 (*3*)

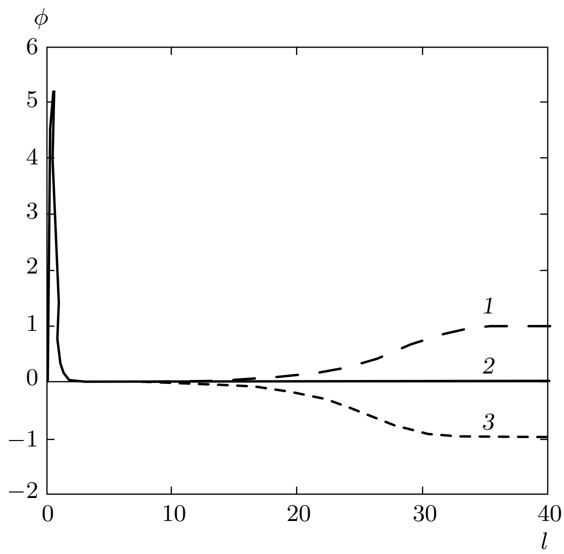

Fig. 4. Smooth solutions in the close vicinity of the upper point on the dashed curve in Fig. 2: $\varepsilon = -2900$ (*1*), -2893 (*2*), -2880 (*3*)

4.2. Neutral quantum particle in the space–time with metric (9)

A neutral spinless quantum particle is described by a scalar wave function χ with the Lagrangian

$$L_\chi = \frac{1}{2} g^{AB} \chi^*_{,B} \chi_{,A} - \frac{1}{2} m_0^2 \chi^* \chi. \qquad (53)$$

In a uniform bulk (while the symmetry is not broken), χ describes a free particle with mass m_0 and spin zero in the D-dimensional space–time. In the broken-symmetry space–time with metric (9), χ satisfies the Klein–Gordon equation

$$\frac{1}{\sqrt{-g}} \left(\sqrt{-g} g^{AB} \chi_{,A} \right)_{,B} + m_0^2 \chi = 0. \qquad (54)$$

All coordinates except $x^{d_0} = l$ are cyclic variables, and the conjugate momenta are quantum numbers. The wave function in a quantum state is

$$\chi \left(x^A \right) = X(l) \exp \left(-i p_\mu x^\mu + i n \varphi \right), \qquad (55)$$

where $p_\mu = (E, \mathbf{p})$ is the d_0-momentum within the brane and n is the integer angular momentum conjugate to the circular extra-dimensional coordinate φ. $X(l)$ satisfies the equation [1]

$$X'' + W X' + \left(p^2 e^{-2\gamma} - n^2 e^{-2\beta} - m_0^2 \right) X = 0. \qquad (56)$$

The eigenvalues of $p^2 = E^2 - \mathbf{p}^2$ compose the spectrum of squared masses, as observed on the brane. The quantum number n is the integer proper angular momentum of the particle. From the standpoint of the observer on the brane, it is the internal momentum, identical to the spin of the particle.

After the substitution

$$dl = e^\gamma dx, \quad X(l) = \frac{y(x)}{\sqrt{f(x)}},$$

$$f(x) = \exp \left\{ -\frac{1}{2} \left[(d_0 - 1) \gamma + \beta \right] \right\},$$

Eq. (56) takes the form of the Schrödinger equation

$$y_{xx} + \left[p^2 - V_g(x) \right] y = 0. \qquad (57)$$

The gravitational potential

$$V_g(x) = e^{2\gamma} \left(e^{-2\beta} n^2 + m_0^2 \right) + \\ + \frac{1}{2} \frac{1}{\sqrt{f}} \frac{d}{dx} \left(\frac{1}{\sqrt{f}} \frac{df}{dx} \right) \qquad (58)$$

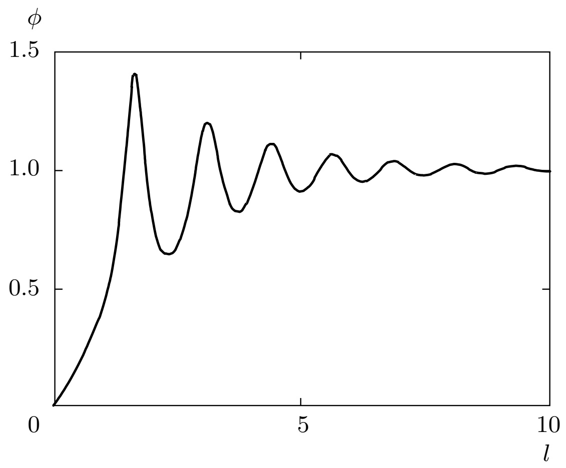

Fig. 5. A solution with the oscillating order parameter $\phi(l)$. Here, $d_0 = 4$, $\varepsilon = -2$, $\Gamma = 10$, and $\phi'_0 = \sqrt{-(\varepsilon+1)/8}$

determines the trapping properties of particles to the brane. In terms of U, W, and ϕ in (30), the dependence of gravitational potential (58) on the distance l is

$$V(l) = e^{2\gamma} \left(e^{-2\beta} n^2 + m_0^2 \right) + \\ + \frac{e^{2\gamma}}{4} \frac{(d_0 W - U)(U + (d_0 + 2) W)}{(d_0 + 1)^2} + \\ + \frac{e^{2\gamma}}{2} \left[\kappa^2 \frac{\partial V}{\partial \phi} \phi + \frac{U (W - d_0 U)}{d_0 + 1} \right]. \qquad (59)$$

4.3. Oscillations

In terms of (52), eigenvalues (43) are expressed as

$$\lambda_\pm = -\sqrt{-\frac{\varepsilon}{8(G+1)}} \times \\ \times \left[1 \pm \sqrt{1 + \frac{16}{\varepsilon G} (G+1)^2} \right]. \qquad (60)$$

The oscillations display themselves the stronger, the smaller is $|\varepsilon|$. In the limit cases of small and large Γ, the oscillation frequencies

$$|\mathrm{Im}\,\lambda| = \begin{cases} \sqrt{2}, & \Gamma \to 0, \\ \sqrt{2\left(1 + \dfrac{1}{d_0}\right)\Gamma}, & \Gamma \to \infty, \end{cases}$$

are independent of ε as $l \to \infty$.

Oscillations of the order parameter $\phi(l)$ (see Fig. 5) induce oscillations of gravitational potential (58). At

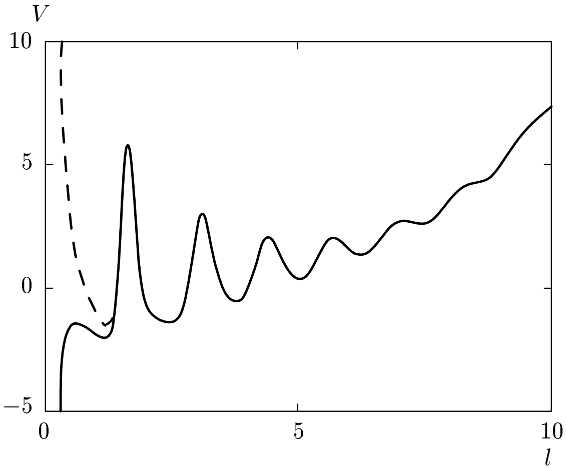

Fig. 6. The gravitational potential $V(l)$ in (59) for the same set of the parameters as in Fig. 5, $d_0 = 4$, $\varepsilon = -2$, $\Gamma = 10$, and $\phi_0' = \sqrt{-(\varepsilon+1)/8}$. The initial mass of a test particle is $m_0 = 0$. The solid curve corresponds to the angular momentum $n = 0$ and the dashed one to $n = \pm 1$

$|\varepsilon| \sim 1$ and $\Gamma \gg 1$, the gravitational potential has many points of minimum (see Fig. 6).

The length scale a in (49) remains an arbitrary parameter of the theory. The physical interpretation is different in the limit cases of large and small a. If a is extremely large, each minimum of the potential $\gamma(l)$ forms its own brane. If the potential barrier is high, the branes are separated from one another. In the opposite limit, when the scale length a is extremely small, all points of minimum are located within one common brane, and in the Kaluza–Klein spirit, the points of minimum are beyond the resolution of modern devices.

Low-energy particles can be trapped by the points of minimum of potential (58). Neutral spinless particles identical in the bulk, acquire different masses and angular momenta when trapped at different minimum points. If the scale length a is extremely small, then for an observer within the brane, they appear as different particles with integer spins.

Most elementary particles have half-integer spins. The simple case of spontaneous symmetry breaking considered above cannot relate the origin of half-integer spins to extra-dimensional angular momenta.

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