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RESONANCE REFLECTION BY THE ONE-DIMENSIONAL ROSEN–MORSE POTENTIAL WELL IN THE GROSS–PITAEVSKII PROBLEM

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We consider the quantum above-barrier reflection of a particle by the one-dimensional Rosen–Morse potential well, for the nonlinear Schrödinger equation (the Gross–Pitaevskii equation) with a small nonlinearity. The most interesting case is realized in resonances when the reflection coefficient is exactly equal to zero for the linear Schrödinger equation. Then the reflection is determined by only a small nonlinear term in the Gross–Pitaevskii equation. The simple analytic expression is obtained for the reflection coefficient produced only by the nonlinearity. The analytic condition is found for the common action of the potential well and the nonlinearity to produce the zero reflection coefficient. The reflection coefficient is also derived analytically in the vicinity of a resonance shifted by the nonlinearity.

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1. INTRODUCTION

Quantum tunneling in physical systems is a hot topic. The most direct way to study these physical properties is to find the exact solutions of the Schrödinger equation that dominates the system dynamics. However, only in a few cases with the simplest potentials, like a square well, the Schrödinger equation can be solved exactly. In most circumstances, exact solutions are difficult to obtain due to not only the effect of the external field exerted on particles but also the interaction of particles. The most direct generalization of the single-particle case is the tunnelling of the mean field through a barrier in the Gross–Pitaevskii, or the nonlinear Schrödinger equation [1]. We emphasize that this is a nonlinear tunneling problem in the mean-field approximation. There have been various theoretical studies. From the theoretical standpoint, the main complication in the description of a quasistationary scattering process of particles obviously comes from the presence of the atom–atom interaction. In leading order, the effect of this interaction is included in the nonlinear term in the Schrödinger-like Gross–Pitaevskii equation for the wave function. The dynamics of solutions of this equation is very complex and rich. The phenomena of coherence, macroscopic tunneling, vortex formation, instabilities, focusing, and blowup are all concepts related to the nonlinear nature of the systems.

A convenient theoretical approach is based on the one-dimensional Gross–Pitaevskii equation

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V(x)+g|\psi(x,t)|^2\right)\psi(x,t),$$

which describes the dynamics in the mean-field approximation at low temperatures. Another important application is the propagation of electromagnetic waves in nonlinear media. The ansatz

$$\psi(x,t) = \exp(-i\mu t/\hbar)\psi(x)$$

reduces the Gross–Pitaevskii equation to the corresponding time-independent (stationary) nonlinear Schrödinger equation

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)+g|\psi(x)|^2\right)\psi(x) = \mu\psi(x) \quad (1)$$

with the chemical potential μ.

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To consider solutions in a finite trap, the Rosen–Morse potential

$$V(x) = \frac{U_0}{\operatorname{ch}^2(\alpha x)} \qquad (2)$$

is studied in this paper. This potential yields analytic solutions, unlike harmonic traps. It also provides a model for a finite-depth trap. The treatment of transport within this mean-field theory reveals new interesting phenomena arising from the nonlinearity of the equation. As was already shown in Refs. [2, 3], a complex solution of Eq. (1) with the one-dimensional square-well potential is given in terms of the Jacobi elliptic functions $\operatorname{dn}(x)$. The Gross–Pitaevskii equation for a one-dimensional finite square-well potential was studied in Ref. [4] in terms of incoming and outgoing waves. The transmission coefficient T becomes equal to one periodically as a function of the chemical potential μ. Thus, there is the total transparency of the potential barrier at resonances.

The resonance line shape was investigated in recent paper [5]. The stationary nonlinear Schrödinger equation or the Gross–Pitaevskii equation for one-dimensional potential scattering was studied in that paper. The nonlinear transmission function exhibits a distorted profile, which differs from the Lorentzian one found in the linear case. This nonlinear profile function is analyzed and related to Siegert-type complex resonances. It is shown that the characteristic nonlinear profile function can be conveniently described in terms of skeleton functions depending on a few parameters. These skeleton functions also determine the decay behavior of the underlying resonance state.

Macroscopic quantum tunneling of Bose–Einstein condensates in a finite potential well has been considered in Ref. [6]. The nonlinearity, which is proportional to both the number of atoms and the interaction strength, can transform bound states into quasibound ones. The latter have a finite lifetime due to tunneling through the barriers at the borders of the well. They predict the lifetime and stability properties for repulsive and attractive condensates in one, two, and three dimensions, for both the ground state and the excited soliton and vortex states.

Resonance solutions of the nonlinear Schrödinger equation, the tunneling lifetime, and fragmentation of trapped condensates were investigated in Ref. [7]. It is shown there how the lifetimes and energies of resonance states can be calculated by applying the complex scaling transformation to the nonlinear Schrödinger equation. It is essential to first apply the complex scaling transformation to the full Hamiltonian and to subsequently derive the correct complex scaled nonlinear Schrödinger equation from the result. The nonlinear Schrödinger equation is physically relevant and amenable to numerical calculations. To analyze the results obtained by solving this equation, it is necessary to realize the close association of resonance phenomena with fragmentation of the system.

In Ref. [8], the hydrodynamic representation of the Gross–Pitaevskii and the nonlinear Schrödinger equations was used to analyze the dynamics of macroscopic tunneling processes. A tendency toward wave breaking and shock formation during the early stages of the tunneling process was observed. A blip in the density distribution appears on the outskirts of the barrier and may transform into a bright soliton under proper conditions.

A particle moving through a classically allowed region can be reflected by a potential without reaching a classical turning point. Above-barrier reflection also occurs when $U_0 < 0$ and the chemical potential $\mu > 0$. In the linear problem ($g = 0$ in Eq. (1)) with potential (2), the reflection coefficient R is determined by the expression (see [9, 10])

$$R = \frac{\cos^2\left(\dfrac{\pi}{2}\sqrt{1 - \dfrac{8mU_0}{\hbar^2\alpha^2}}\right)}{\operatorname{sh}^2\left(\dfrac{\pi k}{\alpha}\right) + \cos^2\left(\dfrac{\pi}{2}\sqrt{1 - \dfrac{8mU_0}{\hbar^2\alpha^2}}\right)}, \qquad (3)$$

where $k = \sqrt{2m\mu}/\hbar$. The inequality $8mU_0 < \hbar^2\alpha^2$ is suggested for the above-barrier transmission and reflection. We everywhere use the system of units $\hbar = m = \alpha = 1$. It follows that in linear problem, $R = 0$ when $1 - 8U_0 = (2n+1)^2$ with $n = 1, 2, 3, \ldots$ Hence, the reflection coefficient in this resonant case is determined only by the nonlinearity.

To avoid secular terms, we use the multiple-scale analysis for the derivation of the resonant reflection coefficient. This approach was used in studying Bose–Einstein solitons in highly asymmetric traps [11]. Quantum reflection of an incident soliton by an attractive sech-squared-shape potential

$$V(x) = -\frac{V}{\operatorname{ch}^2(x/x_0)}$$

(the Rosen–Morse potential) was analyzed numerically in [12]. The prediction was that quantum reflection can occur to a kind of macroscopic quantum objects, atomic matter-wave bright solitons. The pronounced switching between reflection and transmission is a characteristic behavior that should be observable for sufficiently well localized and deep potential wells, such as those created by a strongly focused red-detuned laser beam or a second, incoherent soliton.

It was shown in [13] that the well-known absolute transmission of the nonlinear system can occur in the Rosen–Morse potential. The authors investigated the atomic trap and transport of a Bose–Einstein condensate in one-dimensional waveguide with an obstacle potential of the sech-squared form. By applying a nonbalance condition, they obtained exact solutions of the system, which contain the bound states and transmission states.

The nonlinearity is assumed to be small, i.e., $g \ll \mu$. It is then possible to find a simple analytic expression by the multi-scale approach for the reflection coefficient for the Rosen–Morse potential in the vicinity of resonances. Just this is the goal of our work.

2. THE MODEL AND SOLUTIONS

We assume that a particle moves in the positive direction of the x axis. At $x \to -\infty$, there are both incident and reflecting waves, and at $x \to +\infty$, there is only the transmitted wave. For simplicity, we consider only the first resonance, i.e., $n = 1$. Then $U_0 = -1$.

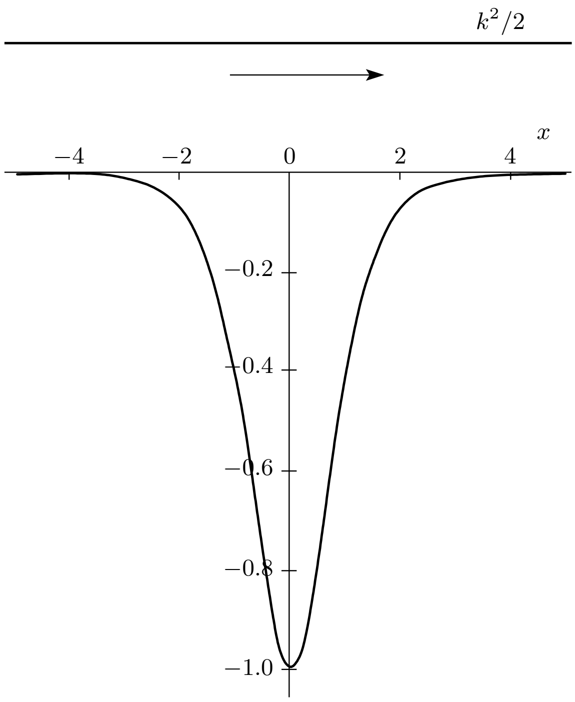

Fig. 1. The potential $-1/\operatorname{ch}^2 x$ for the first resonance ($n = 1$) provides total transmission

The potential corresponding to this resonance case is shown in Fig. 1. When $g = 0$, Eq. (1) takes the form

$$-\frac{1}{2}\frac{d^2\psi}{dx^2} - \frac{1}{\operatorname{ch}^2 x}\psi = \frac{k^2}{2}\psi. \tag{4}$$

The linearly independent solutions of Eq. (4) are

$$\begin{aligned} \psi_1(x) &= \frac{ik - \operatorname{th} x}{\sqrt{2ik\,(k^2+1)}}\exp(ikx), \\ \psi_2(x) &= \frac{ik + \operatorname{th} x}{\sqrt{2ik\,(k^2+1)}}\exp(-ikx). \end{aligned} \tag{5}$$

The Wronskian of these solutions is

$$\psi_2'(x)\psi_1(x) - \psi_1'(x)\psi_2(x) = 1. \tag{6}$$

We choose the unperturbed solution of Eq. (4) in the form $\psi_1(x)$. Then the transmission coefficient is

$$T = \left|\frac{\psi_1(+\infty)}{\psi_1(-\infty)}\right|^2 = 1, \tag{7}$$

i.e., there is no reflection at any value of k.

We now consider the case $g \neq 0$. The differential Gross–Pitaevskii equation is of the form

$$\frac{d^2\psi}{dx^2} + k^2\psi + \frac{2}{\operatorname{ch}^2 x}\psi = 2g|\psi|^2\psi. \tag{8}$$

In the iteration scheme, we introduce

$$\psi(x) = \psi_1(x) + \delta\psi(x),$$

where

$$\delta\psi(x) \ll \psi_1(x).$$

Then Eq. (8) implies the inhomogeneous linear differential equation

$$\frac{d^2\delta\psi}{dx^2} + k^2\delta\psi + \frac{2}{\operatorname{ch}^2 x}\delta\psi = 2g|\psi_1|^2\psi_1 = f(x), \tag{9}$$

where we set

$$f(x) = g\frac{k^2 + \operatorname{th}^2 x}{k\,(k^2+1)}\psi_1(x).$$

The particular solution of Eq. (9) is chosen as

$$\delta\psi(x) = \psi_2(x)\int_{\infty}^{x} f(x')\psi_1(x')\,dx' - \psi_1(x)\int_{0}^{x} f(x')\psi_2(x')\,dx'. \tag{10}$$

We now derive the first term in the right-hand side

$$J(x) = \psi_2(x)\int_{\infty}^{-\infty} f(x')\psi_1(x')\,dx' = = g\frac{(\operatorname{th}x + ik)\exp(-ikx)}{k(k^2+1)\left[2ik(k^2+1)\right]^{3/2}} I(k), \quad (11)$$

where we set

$$I(k) = = \int_{-\infty}^{\infty}\left(\operatorname{th}^4 x - 2ik\operatorname{th}^3 x - 2ik^3\operatorname{th}x - k^4\right)\exp(2ikx)\,dx.$$

We note that when $k \neq 0$,

$$\int_{-\infty}^{\infty}\exp(2ikx)\,dx = 0.$$

We now evaluate the integrals contained in I:

$$\int_{-\infty}^{\infty}\operatorname{th}x\exp(2ikx)\,dx = \frac{\pi i}{\operatorname{sh}(\pi k)}, \quad (12)$$

$$\int_{-\infty}^{\infty}\operatorname{th}^3 x\exp(2ikx)\,dx = \frac{\pi i\left(1-2k^2\right)}{\operatorname{sh}(\pi k)}, \quad (13)$$

$$\int_{-\infty}^{\infty}\operatorname{th}^4 x\exp(2ikx)\,dx = \frac{4\pi k\left(k^2-2\right)}{3\operatorname{sh}(\pi k)}. \quad (14)$$

Hence, the integral I is

$$I(k) = -\frac{2\pi k\left(k^2+1\right)}{3\operatorname{sh}(\pi k)} \quad (15)$$

and according to Eq. (11),

$$J(x) = g\frac{2\pi i(1-ik)\exp(-ikx)}{3\left[2ik(k^2+1)\right]^{3/2}\operatorname{sh}(\pi k)}. \quad (16)$$

The reflection coefficient is obtained from Eq. (5) and Eq. (16) by letting $x \to -\infty$:

$$R(k) = \left|\frac{J(x)}{\psi_1(x)}\right|^2 = \left[\frac{\pi g}{3k(k^2+1)\operatorname{sh}(\pi k)}\right]^2 \ll 1. \quad (17)$$

The reflection coefficient rapidly decreases as k increases. The condition of the applicability of this approach is $g \ll k^2$. The values $k \ll 1$ are also acceptable when $g \ll k^2$.

We now consider the second term in $\delta\psi$, Eq. (10):

$$K(x) = -\psi_1(x)\int_0^x f(x')\psi_2(x')\,dx'. \quad (18)$$

As $x \to \pm\infty$, the quantity $K(x)$ is determined by large values of $x' \gg 1$. Hence, $\operatorname{th}x' \approx 1$ in the integrand of Eq. (18). We obtain the secular term

$$K(x) = g\frac{ik \mp 1}{2ik^2\sqrt{2ik(k^2+1)}}x\exp(ikx). \quad (19)$$

On the other hand, it follows from Gross–Pitaevskii equation (8) as $x \to \pm\infty$ that

$$\frac{d^2\psi}{dx^2} + k^2\psi = 2g|\psi|^2\psi. \quad (20)$$

The solution of this equation in the form of a transmitted wave is (see Eq. (5))

$$\tilde{\psi}_1(x) = \frac{ik - \operatorname{th}x}{\sqrt{2ik(k^2+1)}}\exp(ik'x),$$

where

$$k' = \sqrt{k^2 - g/k} \approx k - g/2k^2,$$

$$\exp(ik'x) \approx \exp(ikx)\left[1 - i\frac{gx}{2k^2}\right].$$

The secular term in the transmitted wave at $|x| \gg 1$ is

$$\delta\tilde{\psi}_1(x) = \tilde{\psi}_1(x) - \psi_1(x) = = g\frac{ik \mp 1}{2ik^2\sqrt{2ik(k^2+1)}}x\exp(ikx). \quad (21)$$

This is just the same secular term as in Eq. (19) [14]. The secular term does not influence the reflection coefficient $R(k)$ in (17).

Hence, the total wave function of the nonlinear problem at the first resonance ($U_0 = -1$) has the form

$$\psi(x) = \tilde{\psi}_1(x) + g\frac{2\pi(ik-1)\exp(-ikx)}{3\left[2ik(k^2+1)\right]^{3/2}\operatorname{sh}(\pi k)}. \quad (22)$$

The reason for the occurence of secular terms is that as $x \to \pm\infty$, the inhomogeneous term $\exp(ikx)$ is simultaneously a solution of the homogeneous differential equation. In the reflecting wave $\exp(-ikx)$, the value of k also changes because of secular terms, but this does not affect the reflection coefficient in the terms of the first order in the small parameter $g/k^2 \ll 1$. Analogously, the condition $T + R = 1$ is satisfied with the accuracy of terms linear in this parameter.

3. TRANSMISSION RESONANCES IN THE GROSS–PITAEVSKII EQUATION

We now consider the vicinity of the first resonance, when $U_0 = -1+\gamma$ with $\gamma \ll 1$. In this case, the Gross–Pitaevskii equation takes the form (see Eq. (8))

$$\frac{d^2\psi}{dx^2} + k^2\psi + \frac{2-2\gamma}{\operatorname{ch}^2 x}\psi = 2g|\psi|^2\psi. \tag{23}$$

In the iteration scheme for solution of this equation, we write

$$\psi(x) = \psi_1(x) + \Delta\psi(x),$$

where

$$\Delta\psi(x) \ll \psi_1(x).$$

The inhomogeneous linear differential equation for $\Delta\psi(x)$ is

$$\frac{d^2\Delta\psi}{dx^2} + k^2\Delta\psi + \frac{2}{\operatorname{ch}^2 x}\Delta\psi = F(x) \tag{24}$$

where we set

$$F(x) = 2g|\psi_1|^2\psi_1 + \frac{2\gamma}{\operatorname{ch}^2 x}\psi_1. \tag{25}$$

Substituting $\psi_1(x)$ from Eq. (5) in Eq. (25), we obtain

$$F(x) = \left[g\frac{k^2+\operatorname{th}^2 x}{k\left(k^2+1\right)} + \frac{2\gamma}{\operatorname{ch}^2 x}\right] \times \frac{ik - \operatorname{th} x}{\sqrt{2ik\left(k^2+1\right)}}\exp\left(ikx\right). \tag{26}$$

The solution of Eq. (24) is given by (see Eq. (10))

$$\Delta\psi(x) = \psi_2(x)\int\limits_{\infty}^{x} F(x')\psi_1(x')\,dx' - \psi_1(x)\int\limits_{0}^{x} F(x')\psi_2(x')\,dx'. \tag{27}$$

As $x \to -\infty$, the first term in this equation can be represented in the form (see Eq. (18))

$$M(x) = \psi_2(x)\int\limits_{\infty}^{-\infty} F(x')\psi_1(x')\,dx' = J(x) + \frac{2\gamma(1-ik)\exp(-ikx)}{\left[2ik\left(k^2+1\right)\right]^{3/2}}L(k) \tag{28}$$

where

$$L(k) = \int\limits_{-\infty}^{\infty} \frac{\left(\operatorname{th} x - ik\right)^2}{\operatorname{ch}^2 x}\exp(2ikx)\,dx. \tag{29}$$

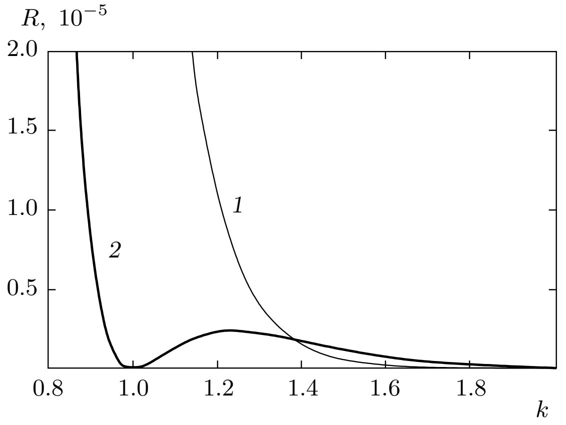

Fig. 2. The dependence of the reflection coefficient $R(k)$ derived in accordance with Eq. (32) on k; $g = 0.2$, $\gamma = 0$ (*1*) and 0.05 (*2*)

This integral can be evaluated similarly to $I(k)$:

$$L(k) = \frac{2\pi k\left(k^2+1\right)}{3\operatorname{sh}(\pi k)}. \tag{30}$$

Hence,

$$M(x) = \frac{2\pi i(1-ik)\exp(-ikx)}{3\operatorname{sh}(\pi k)\sqrt{2ik\left(k^2+1\right)}} \times \left[\frac{g}{2k\left(k^2+1\right)} - \gamma\right]. \tag{31}$$

The reflection coefficient is (see Eq. (18))

$$R(k) = \left|\frac{M(x)}{\psi_1(x)}\right|^2 = \left[\frac{\pi\left(g - 2\gamma k\left(k^2+1\right)\right)}{3k\left(k^2+1\right)\operatorname{sh}\left(\pi k\right)}\right]^2 \ll 1. \tag{32}$$

The reflection coefficient $R(k)$ is zero under the condition

$$g = 2\gamma k\left(k^2+1\right). \tag{33}$$

In Fig. 2, the reflection coefficient $R(k)$ derived in accordance with Eq. (32) is represented as a function of k in the example where $g = 0.2$, $\gamma = 0$ and 0.05.

4. CONCLUSION

The scattering of Bose–Einstein condensate by the Rosen–Morse potential has been discussed in terms of stationary states of the Gross–Pitaevskii equation. Neglecting the mean-field interaction outside the potential, the incoming and outgoing waves and the reflection and transmission probabilities can be defined within the approximation of a weak nonlinear parameter. The vicinity of resonances has been investigated where the role of a weak nonlinearity is significant. A simple analytic expression for the reflection coefficient in the case where reflection is absent in the linear problem and also the reflection coefficient in the vicinity of resonances of the linear problem have been obtained. New positions of resonances were found where the reflection coefficient is zero in the presence of both nonlinearity and some small detuning from resonance in the linear problem.

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