ЖЭТФ, 2011, том 139, вып. 3, стр. 539–549

LONG-TIME RELAXATION PROCESSES IN THE NONLINEAR SCHRÖDINGER EQUATION

Yu. N. Ovchinnikov[a,b*], **I. M. Sigal**[c,d**]

[a] *Max-Planck Institute for Physics of Complex System 01187, Dresden, Germany*

[b] *Landau Institute for Theoretical Physics, Russian Academy of Sciences 117334, Moscow, Russia*

[c] *University of Toronto, Toronto Canada*

[d] *University of Notre Dame, IN USA*

Received June 30, 2010

The nonlinear Schrödinger equation, known in low-temperature physics as the Gross–Pitaevskii equation, has a large family of excitations of different kinds. They include sound excitations, vortices, and solitons. The dynamics of vortices strictly depends on the separation between them. For large separations, some kind of adiabatic approximation can be used. We consider the case where an adiabatic approximation can be used (large separation between vortices) and the opposite case of a decay of the initial state, which is close to the double vortex solution. In the last problem, no adiabatic parameter exists (the interaction is strong). Nevertheless, a small numerical parameter arises in the problem of the decay rate, connected with an existence of a large centrifugal potential, which leads to a small value of the increment. The properties of the nonlinear wave equation are briefly considered in the Appendix A.

1. INTRODUCTION

The nonlinear Schrödinger equation is probably the simplest model where both the wave-like and "particle"-like excitations–vortices and solitons — exist. The interaction of vortices with one another and with sound-like excitations leads to a nontrivial dynamics of vortices.

We consider the simplest equation of the type

$$i\frac{\partial\psi}{\partial t} = -\left\{\frac{\partial^2\psi}{\partial\mathbf{r}^2} + (1-|\psi|^2)\psi\right\} \tag{1}$$

in the two-dimensional space. The energy E corresponding to the state ψ can be defined as

$$E = \frac{1}{2}\int d^2\mathbf{r}\left\{\left(\frac{\partial\psi}{\partial\mathbf{r}}\right)^2 + \frac{1}{2}(1-|\psi|^2)^2\right\}. \tag{2}$$

The absolute minimum of E is reached on the state

$$\psi = \exp(i\chi), \tag{3}$$

where χ is a constant phase.

The ground state is degenerate with respect to multiplication by an arbitrary phase factor. This degeneracy leads to the existence of sound-like excitations.

To find these excitations, we set

$$\psi = \exp(i\chi_0)(1 + u_1 + iu_2), \tag{4}$$

where $u_{1,2}$ are real and

$$|u_{1,2}| \ll 1.$$

It follows from Eqs. (1) and (4) that

$$\frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = -2u_1\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{\partial^2}{\partial\mathbf{r}^2}\begin{pmatrix} -u_2 \\ u_1 \end{pmatrix}. \tag{5}$$

The general solution of Eq. (5) is a linear superposition of the type

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = A_{k,\omega}\begin{pmatrix} \frac{k^2}{\omega}\sin(\omega t - \mathbf{k}\cdot\mathbf{r}) \\ \cos(\omega t - \mathbf{k}\cdot\mathbf{r}) \end{pmatrix}, \tag{6}$$

where ω and $\mathbf{k}$ satisfy the equation

$$\omega^2 = k^2(2 + k^2). \tag{7}$$

*E-mail: ovc@itp.ac.ru

**E-mail: sigal@math.toronto.edu

For small values of the frequency ω, sound-like excitations therefore exist in the system with

$$\omega = \sqrt{2}\,k. \tag{8}$$

It follows from Eq. (1) that the total energy E (Eq. (2)) is conserved and the energy flow $\mathbf{S}_E$ is given by

$$\mathbf{S}_E = -\frac{1}{2}\left\{\frac{\partial\psi}{\partial t}\frac{\partial\psi^*}{\partial\mathbf{r}} + \frac{\partial\psi^*}{\partial t}\frac{\partial\psi}{\partial\mathbf{r}}\right\}. \tag{9}$$

In a plane wave with $|k| \ll 1$, the main contribution to the energy flow is due to the imaginary part of the perturbation u_2 (Eq. (4)).

Equation (1) also has time-independent solutions (vortices) of the form

$$\psi_n(\mathbf{r}) = \exp(in\varphi)\,f_n(r), \tag{10}$$

where $r = |\mathbf{r}|$.

If $\psi_n(\mathbf{r})$ is a solution of Eq. (1), then $\tilde{\psi}_n(\mathbf{r})$ and $\tilde{\psi}_n^*(\mathbf{r})$ are also solutions of Eq. (1), where

$$\tilde{\psi}_n(\mathbf{r}) = \psi_n(\mathbf{r}-\mathbf{a})\exp(i\chi_0), \tag{11}$$

where $\mathbf{a}$ and χ_0 are some constants. Solutions of type (10) are characterized by a discrete topological charge n.

2. NONLINEAR ADIABATIC THEORY

We now suppose that we have a system of vortices with a large separation,

$$|\mathbf{r}_i - \mathbf{r}_j| \gg 1,$$

from one another. Our task is to derive a system of equations for the positions $\mathbf{r}_i(t)$ of the centers of vortices. For this, we define the action A as

$$A = \int L\,dt,$$
$$L = E - \frac{i}{4}\int d^2\mathbf{r}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right). \tag{12}$$

The second term in Eq. (12) can be regarded as a dynamical one,

$$A_D = \frac{1}{2}\int dt\int d^2\mathbf{r}|\psi|^2\frac{\partial\chi}{\partial t}, \tag{13}$$

where

$$\psi = |\psi|\exp(i\chi). \tag{14}$$

For large distances between the vortices,

$$|\mathbf{r}_i - \mathbf{r}_j| \gg 1,$$

in the leading approximation, the phase χ is equals to the sum of phases of the individual vortices:

$$\chi = \sum_i \chi_i, \quad \chi_i = n_i\,\mathrm{arctg}\left(\frac{y-y_i}{x-x_i}\right). \tag{15}$$

In the same approximation, the modulus of ψ is given by

$$|\psi|^2 = 1 - \left(\frac{\partial\chi}{\partial\mathbf{r}}\right)^2. \tag{16}$$

Equations (13), (15), and (16) allow writing the dynamical part of the action A_D as

$$L_D = \frac{1}{2}\int\limits_{D_R} d^2\mathbf{r}\frac{\partial\chi}{\partial t} - \frac{1}{2}\int\limits_{D_R} d^2\mathbf{r}(1-|\psi|^2)\frac{\partial\chi}{\partial t}. \tag{17}$$

The integral in Eq. (17) is taken over a circle D_R of a large radius R. In the approximation given by Eq. (15), to the logarithmic accuracy, the last term is equal to the quantity

$$\pi\sum_{j\neq k} n_j^2 n_k\frac{\partial\mathbf{r}_{jk}}{\partial t}\frac{J\mathbf{r}_{jk}}{|\mathbf{r}_{jk}|^2}\ln|\mathbf{r}_{jk}|, \tag{18}$$

where

$$\mathbf{r}_{jk} = \mathbf{r}_j - \mathbf{r}_k$$

and

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{19}$$

is the simplectic matrix. Compared to the first term, this term has the parametric smallness r_{ij}^{-2} and is neglected in what follows. For the first term in Eq. (17), we obtain

$$L_D = -\sum_j \frac{n_j}{2|\mathbf{r}_j|}\frac{\partial\mathbf{r}_j}{\partial t}J\mathbf{r}_j\int\limits_0^{2\pi}d\varphi\,\times$$
$$\times\int\limits_0^R dr\,r\frac{r\cos\varphi - r_j}{r^2 + r_j^2 - 2rr_j\cos\varphi}. \tag{20}$$

The integral in Eq. (20) is given by

$$\int_0^{2\pi} d\varphi \frac{r\cos\varphi - r_j}{r^2 + r_j^2 - 2rr_j\cos\varphi} =$$

$$= \frac{i}{2r_j}\left\{2\pi i + \frac{r^2 - r_j^2}{2r_j}\oint_{|z|=1}\frac{dz}{(z - r/r_j)(z - r_j/r)}\right\} =$$

$$= \begin{cases} 0, & r > r_j, \\ -\dfrac{2\pi}{r_j}, & r < r_j. \end{cases} \quad (21)$$

As a result, we obtain

$$A_D = \frac{\pi}{2}\sum_j n_j \int dt \frac{\partial \mathbf{r}_j}{\partial t} J\mathbf{r}_j =$$

$$= \frac{\pi}{2}\sum_j \int dt \left(\mathbf{n}_j \left[\mathbf{r}_j \cdot \frac{\partial \mathbf{r}_j}{\partial t}\right]\right) dt, \quad (22)$$

where $\mathbf{n}_j = n_j(0,0,1)$.

In the next approximation, an outgoing wave arises due to the vortex motion. This phenomenon can be taken into account with the help of an additional term δS in the action,

$$\delta S = \frac{1}{2}\int dt \int d^2\mathbf{r} \left\{ \left(\frac{\partial \varphi_s}{\partial \mathbf{r}}\right)^2 - \frac{1}{2}\left(\frac{\partial \varphi_s}{\partial t}\right)^2 - \right.$$

$$\left. - \frac{\partial \varphi}{\partial t}\frac{\partial \chi}{\partial t} - \frac{1}{2}\left(\frac{\partial \chi}{\partial t}\right)^2 \right\}, \quad (23)$$

where φ_s is a single-valued scalar function, connected with small vibrations of the phase.

The action in (12) and (22) should take an extremal value on a trajectory. This leads to the equation [1]

$$n_j \frac{\partial \mathbf{r}_j}{\partial t} = -\frac{1}{\pi} J \frac{\partial E}{\partial \mathbf{r}_j}. \quad (24)$$

In the leading approximation of large distances between vortices, the energy E can be written as

$$E = \sum_i E_i + \sum_{i \neq j} E_{ij}, \quad (25)$$

where E_i is the self energy of a separate vortex and E_{ij} is the pair interaction energy. From Eqs. (2) and (15),

$$E_{ij} = \frac{1}{2}\int_{|r|<R} d^2\mathbf{r} \left(\frac{\partial \chi_i}{\partial \mathbf{r}}\frac{\partial \chi_j}{\partial \mathbf{r}}\right) =$$

$$= \pi n_i n_j \ln\left(\frac{R}{|\mathbf{r}_i - \mathbf{r}_j|}\right). \quad (26)$$

Equation of motion (24) is independent of the cut off distance R. There are some special configurations of vortices, where all quantities $\partial E/\partial \mathbf{r}_i$ vanish in the approximation (26). We call such configurations forceless. For forceless configurations, it is necessary to evaluate the energy E in the next approximation in the distances $|\mathbf{r}_i - \mathbf{r}_j|$. The interaction energy in such a case can not be presented as a pair interaction energy. Equation (24) continues to hold for that case.

3. EMISSION

Equation of motion (24) does not take the emission of excitations of type (6) into account. To do so we write Eq. (1) in the form

$$\begin{gathered} |\psi|^2 = 1 - \left(\frac{\partial \chi}{\partial \mathbf{r}}\right)^2 - \frac{\partial \chi}{\partial t} + \frac{1}{|\psi|}\frac{\partial^2|\psi|}{\partial \mathbf{r}^2}, \\ -\frac{\partial |\psi|^2}{\partial t} = 2|\psi|^2\frac{\partial^2 \chi}{\partial \mathbf{r}^2} + 2\frac{\partial \chi}{\partial \mathbf{r}}\frac{\partial |\psi|^2}{\partial \mathbf{r}}. \end{gathered} \quad (27)$$

From the system of equations (27), we obtain the following equation for the phase χ:

$$\frac{\partial^2 \chi}{\partial t^2} - 2\frac{\partial^2 \chi}{\partial \mathbf{r}^2} =$$

$$= -2\frac{\partial \chi}{\partial \mathbf{r}}\frac{\partial}{\partial \mathbf{r}}\left[\left(\frac{\partial \chi}{\partial \mathbf{r}}\right)^2 + \frac{\partial \chi}{\partial t} - \frac{1}{|\psi|}\frac{\partial^2|\psi|}{\partial \mathbf{r}^2}\right] -$$

$$- 2(1 - |\psi|^2)\frac{\partial^2 \chi}{\partial \mathbf{r}^2} - \frac{\partial}{\partial t}\left[\left(\frac{\partial \chi}{\partial \mathbf{r}}\right)^2 - \frac{1}{|\psi|}\frac{\partial^2|\psi|}{\partial \mathbf{r}^2}\right]. \quad (28)$$

We solve Eq. (28) treating the right-hand side as a perturbation. In the first approximation, we therefore set the right-hand side of Eq. (28) equal to zero and search χ of the form

$$\chi = \chi_0 + \tilde{\chi}, \quad \chi_0 = \sum_i \chi_i(\mathbf{r} - \mathbf{r}_i(t)), \quad (29)$$

where the functions χ_i are given in Eq. (15) and $\mathbf{r}_i(t)$ are solutions of Eq. (24). In the first approximation, we have

$$\tilde{\chi} = -\int dt_1 \times$$

$$\times \int d^2\mathbf{r}_1 G(t - t_1, \mathbf{r} - \mathbf{r}_1)\frac{\partial^2 \chi_0(\mathbf{r}_1, t_1)}{\partial t_1^2}, \quad (30)$$

where the Green's function

$$G(t, \mathbf{r}) = \frac{\theta(t)}{2\pi\sqrt{2}}\begin{cases} 0, & \sqrt{2}t < |\mathbf{r}|, \\ \dfrac{1}{\sqrt{2t^2 - \mathbf{r}^2}}, & \sqrt{2}t > |\mathbf{r}|, \end{cases} \quad (31)$$

is a solution of the equation

$$\frac{\partial^2 G}{\partial t^2} - 2\frac{\partial^2 G}{\partial \mathbf{r}^2} = \delta(t-t_1)\delta(\mathbf{r}-\mathbf{r}_1). \quad (32)$$

We now consider a system of two vortices with equal topological "charge" $n = 1$. We suppose that the distance between vortices is R_0. Then in the approximation Eq. (24), vortices rotate relative the "middle" point with the frequency

$$\omega = \frac{4}{R_0^2}. \quad (33)$$

It follows from Eqs. (15), (29), and (33) that

$$\frac{\partial \chi_0}{\partial t} = \frac{R_0^2\omega}{2} \times \frac{R_0^2/4 - r^2\cos(2(\varphi-\omega t))}{r^4 + R_0^4/16 - r^2(R_0^2/2)\cos(2(\varphi-\omega t))}. \quad (34)$$

Expression (34) can be expanded into a series in $\cos(2k(\varphi-\omega t))$, $k = 0, 1, 2\ldots$ Simple calculations give

$$\frac{\partial \chi_0}{\partial t} = I_2\cos(2(\varphi-\omega t)) + \ldots + I_{2n}\cos(2n(\varphi-\omega t)), \quad (35)$$

where

$$I_2 = \frac{R_0^2\omega}{2}\begin{cases} -\dfrac{1}{r^2}, & r > R_0/2, \\ \dfrac{16r^2}{R_0^4}, & r < R_0/2. \end{cases} \quad (36)$$

The contribution from other harmonics to $\tilde{\chi}$ is small and is not included.

From Eqs. (30), (32), and (35), in the wave zone $r \gg \omega^{-1}$, we obtain

$$\tilde{\chi} = \frac{2^{1/4}(\pi\omega)^{1/2}}{2r^{1/2}}\cos\left[\sqrt{2}\,\omega\left(r-\sqrt{2}\,t\right)+2\varphi-\frac{\pi}{4}\right] \times \int\limits_0^\infty d\rho_1\rho_1 I_2(\rho_1,\omega)J_2\left(\sqrt{2}\,\omega\rho_1\right), \quad (37)$$

where J_2 is the Bessel function and

$$\mathbf{r} = r(\cos\varphi, \sin\varphi) \quad (38)$$

is the observation point.

In Eq. (37), only the domain where $\rho_1 \gg R_0/2$ is essential and simple calculations give

$$\tilde{\chi} = -\frac{2^{1/4}(\pi\omega)^{1/2}\omega R_0^2}{8r^{1/2}} \times \cos\left[\sqrt{2}\omega\left(r-\sqrt{2}t\right)+2\varphi-\frac{\pi}{4}\right]. \quad (39)$$

Similarly, it is possible to find all higher harmonics of $\tilde{\chi}$. With the help of Eqs. (30), (34), and (35), in the wave zone, we obtain

$$\tilde{\chi} = \sum_{n=1}^\infty \chi_{2n}\cos\left(\sqrt{2}n\omega\left(r-\sqrt{2}t\right)+2n\varphi-\frac{\pi}{4}\right), \quad (40)$$

$$\chi_{2n} = \frac{(-)^{n+1}2^{1/4}\sqrt{\pi n\omega}}{2\sqrt{r}}\int\limits_0^\infty d\rho_1\rho_1 I_{2n}\left(\sqrt{2}n\omega\rho_1\right).$$

The first term in Eq. (40) coincides with expression (37). Only large values of $\rho_1 \sim (n\omega)^{-1} \gg R_0$ are essential in the integral in Eq. (40). In this domain from Eqs. (34) and (35), we obtain

$$I_{2n} \approx -2\omega\left(\frac{R_0^2}{4r^2}\right)^n. \quad (41)$$

Finally, we obtain

$$\chi_{2n} = \frac{(-)^n 2^{1/4}\sqrt{\pi n\omega}}{\sqrt{r}}\,\frac{\omega(n\omega)^{2(n-1)}}{\Gamma(2n)}\left(\frac{R_0^2}{8}\right)^n. \quad (42)$$

It follows from Eq. (42) that the amplitude of higher harmonics very rapidly decreases as n increases.

The right-hand side of Eq. (24) is localized near the circle of the radius $\rho = R_0/r$. Essential distances in the kernel in Eqs. (30) and (37) are of the order of the wave length $\rho \sim \omega^{-1}$. As a result, the correction to the value of $\tilde{\chi}$ given by Eq. (40) due to right-hand side of Eq. (28) is small. The order of magnitude of this correction can be estimated as follows. In the leading approximation, we have

$$\left(\frac{\partial\chi}{\partial\mathbf{r}}\right)^2 = \frac{4r^2}{r^4+R_0^4/16-r^2(R_0^2/2)\cos(2(\varphi-\omega t))} = \frac{16}{R_0^2}\left\{\frac{r^4+R_0^4/16}{|r^4-R_0^4/16|}-1\right\}\cos(2(\varphi-\omega t)) + \ldots \quad (43)$$

Using Eqs. (37) and (43), we obtain the correction to the phase $\tilde{\chi}$ that comes from the term

$$\frac{\partial}{\partial t}\left(\frac{\partial\chi}{\partial\mathbf{r}}\right)^2$$

in Eq. (28):

$$\delta\tilde{\chi} \approx \frac{\omega^{1/2}}{r^{1/2}}(\omega R_0^2)\omega^2\ln R_0 \times \cos\left[\sqrt{2}\omega(r-\sqrt{2}t)+2\varphi-\frac{\pi}{4}\right]. \quad (44)$$

Because of the small expansion parameter $\omega^2\ln R_0$, this correction is smaller than the leading contribution given by Eq. (34).

We now consider the correction to the phase $\tilde{\chi}$ associated with the term

$$\frac{\partial \chi_2}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} \left(\frac{\partial \chi_1}{\partial \mathbf{r}}\right)^2 + \frac{\partial \chi_1}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} \left(\frac{\partial \chi_2}{\partial \mathbf{r}}\right)^2.$$

This term and the similar one originate from the term

$$\frac{\partial \chi}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} \left(\frac{\partial \chi}{\partial \mathbf{r}}\right)^2$$

in Eq. (28).

We find

$$\frac{\partial \chi_2}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} \left(\frac{\partial \chi_1}{\partial \mathbf{r}}\right)^2 + \frac{\partial \chi_1}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} \left(\frac{\partial \chi_2}{\partial \mathbf{r}}\right)^2 = \\ = \frac{-2r^2 R_0^2 \sin[2(\varphi-\omega t)]}{[r^4+R_0^4/16-r^2R_0^2/2\cos(2(\varphi-\omega t))]^2} = \frac{16}{r^2R_0^2} \times \\ \times \left[1 - \frac{r^4+R_0^4/16}{|r^4-R_0^4/16|}\right] \sin(2(\varphi-\omega t)) + \ldots \quad (45)$$

From Eqs. (28), (37), and (47) we obtain

$$\delta\tilde{\chi} \approx \frac{\omega^{1/2}}{r^{1/2}}(\omega^2 \ln R_0) \times \\ \times \sin\left[\sqrt{2}\omega(r-\sqrt{2}t) + 2\varphi - \varphi_0\right]. \quad (46)$$

This correction is of the same order as the one in Eq. (44). The correction in Eq. (40) to the phase $\tilde{\chi}$ decays at large distances as $r^{-1/2}$ and leads to emission of sound-like excitations. As a result, the distance between two vortices increases with time. To find the dependence $R_0(t)$ we first derive the energy flow $\langle \mathbf{S}_E \rangle$ averaged over the period of motion. It follows from Eq. (9) that

$$\langle \mathbf{S}_E \rangle = \frac{\pi\omega^5 R_0^4}{32} \frac{\mathbf{r}}{r^2} = \frac{32\pi}{R_0^6} \frac{\mathbf{r}}{r^2}. \quad (47)$$

The energy conservation law and Eqs. (26) and (47) give the following equation for the quantity $R_0(t)$ [2]:

$$\frac{\partial R_0}{\partial t} = \frac{32\pi}{R_0^5}. \quad (48)$$

The general solution of Eq. (48) is

$$R_0^6(t) = R_0^6(t_0) + 32 \cdot 6\pi(t-t_0), \quad (49)$$

which means that $R_0^6(t)$ is a linear function of time. Equation (49) is valid only in the range $R_0 \gg 1$.

4. DECAY OF A DOUBLE VORTEX

We now consider the opposite limit case $R_0 \ll 1$, the initial stage of the decay of a double vortex. In this range, we can search for a solution of Eq. (1) in the form

$$\psi = f_2(r)e^{2i\varphi} + f_0(r)e^{-i\lambda t} + f_4^*(r)e^{4i\varphi}e^{i\lambda^* t}, \quad (50)$$

where f_2 is a double vortex solution of Eq. (1) and $f_{0,4}$ are small, $|f_{0,4}| \ll 1$. In the linear approximation we obtain

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f_0}{\partial r}\right) - (1-2f_2^2)f_0 + f_2^2 f_4 = \lambda f_0, \\ -\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f_4}{\partial r}\right) + \frac{16}{r^2}f_4 - \\ -(1-2f_2^2)f_4 + f_2^2 f_0 = -\lambda f_4. \quad (51)$$

We should find solution of system (51) subject to the boundary condition that $f_{0,4}$ are finite for $r \to 0$, and an outgoing wave exists for $r \to \infty$.

The function $f_2(r)$ is a solution of the Eq. (10) with the topological charge $n = 2$. For $r \ll 1$, we have (see Appendix B)

$$f_2(r) = Br^2 - \frac{B}{12}r^4 + \frac{B}{384}r^6 \ldots \quad (52)$$

and

$$\begin{pmatrix} f_0 \\ f_4 \end{pmatrix} = C_1 \begin{pmatrix} 1 - \frac{1+\lambda}{4}r^2 + \frac{(1+\lambda)^2}{64}r^4 - \frac{r^6}{36}\left(\frac{(1+\lambda)^3}{64} - 2B^2\right) + O(r^8) \\ \frac{B^2}{20}r^6 + O(r^8) \end{pmatrix} + C_2 \begin{pmatrix} \frac{B^2}{100}r^{10} + O(r^{12}) \\ r^4 - \frac{1-\lambda}{20}r^6 + O(r^8) \end{pmatrix}, \quad (53)$$

where $C_{1,2}$ are arbitrary constants. We put below $C_1 = 1$. The coefficients B, C_2 are found in Appendix B.

In the range $r \gg 1$, we have (see Appendix B)

$$f_2(r) = 1 - \frac{2}{r^2} - \frac{6}{r^4} - \frac{68}{r^6} + \frac{C}{\sqrt{r}} \exp\left(-\sqrt{2}r\right) \quad (54)$$

and

$$\begin{pmatrix} f_0 \\ f_4 \end{pmatrix} = C_3 \frac{\exp(-S)}{\sqrt{r}} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + C_4 \frac{\exp(-i\tilde{S})}{\sqrt{r}} \begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix}, \quad (55)$$

where $C_{3,4}$ are constants. The functions $S, \tilde{S}$, and $\alpha, \tilde{\alpha}, \beta, \tilde{\beta}$ are found in Appendix B:

$$S = \left(1 + \sqrt{1+\lambda^2}\right)^{1/2} r + \frac{\frac{1}{8} + \frac{2(1-2\lambda)}{\sqrt{1+\lambda^2}}}{r\sqrt{1+\sqrt{1+\lambda^2}}} \times \left(1 - \frac{1}{2r\sqrt{1+\sqrt{1+\lambda^2}}}\right), \quad (56)$$

$$\tilde{S} = \left(\sqrt{1+\lambda^2} - 1\right)^{1/2} r + \frac{-\frac{1}{8} + \frac{2(1-2\lambda)}{\sqrt{1+\lambda^2}}}{r\sqrt{\sqrt{1+\lambda^2}-1}} \times \left(1 + \frac{i}{2r\sqrt{\sqrt{1+\lambda^2}-1}}\right),$$

and

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ \lambda + \sqrt{1+\lambda^2} \end{pmatrix} + \frac{4+2\lambda}{(1+\lambda^2)r^2} \left(1 - \frac{2\sqrt{1+\sqrt{1+\lambda^2}}}{r\sqrt{1+\lambda^2}}\right) \times \begin{pmatrix} -\left(\lambda + \sqrt{1+\lambda^2}\right) \\ 1 \end{pmatrix}, \quad (57)$$

$$\begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix} = \begin{pmatrix} -\left(\lambda + \sqrt{1+\lambda^2}\right) \\ 1 \end{pmatrix} - \frac{4+2\lambda}{(1+\lambda^2)r^2} \left(1 + \frac{2i\sqrt{\sqrt{1+\lambda^2}-1}}{r\sqrt{1+\lambda^2}}\right) \times \begin{pmatrix} 1 \\ \lambda + \sqrt{1+\lambda^2} \end{pmatrix}.$$

We note that the equation

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \tilde{f}_0}{\partial r}\right) - \tilde{f}_0(1 - 2f_2^2(r)) = \tilde{\lambda}\tilde{f}_0 \quad (58)$$

has a negative eigenvalue

$$\tilde{\lambda} = -0.399689. \quad (59)$$

On the other hand, in the range of values of r where the function f_2 is essentially different from 1 the centrifugal potential in Eq. (51) for the function f_4 is large. As a result, the functions f_0 and f_4 in Eq. (51) overlap weakly, and we can hope that Eq. (51) has an eigenvalue of λ with a small imaginary part and with the real part close to $\tilde{\lambda}$. Matching the numerical solution of Eq. (51) starting from values given by (53) for $r \ll 1$ with values given by Eq. (55) for $r \gg 1$, we obtain all coefficients (λ, C_2, C_3, C_4). These coefficients are given in Appendix B. Specifically,

$$\lambda = -0.443673 + i\,0.004937. \quad (60)$$

The imaginary part of λ is nearly hundred times smaller then its real part. Such a high quality of oscillation in the system is related to the high value of the centrifugal potential in the "localization" range of the function f_0.

5. EXITATIONS OF THE SOLITON TYPE

We next consider the third-type exitations, solitons. For a long-wavelength soliton of small amplitude, we obtain from system of Eqs. (27) that [2]

$$\frac{\partial^2\chi}{\partial t^2} - 2\frac{\partial^2\chi}{\partial \mathbf{r}^2} = -2\frac{\partial}{\partial t}\left(\frac{\partial\chi}{\partial \mathbf{r}}\right)^2 - 2\frac{\partial\chi}{\partial t}\frac{\partial^2\chi}{\partial \mathbf{r}^2} - \frac{1}{2}\frac{\partial^2}{\partial t^2}\frac{\partial^2\chi}{\partial \mathbf{r}^2}. \quad (61)$$

We search a solution of Eq. (60) in the form

$$\chi = \chi(x - vt). \quad (62)$$

For the quantity

$$z = \frac{\partial\chi}{\partial\tilde{x}}, \quad \tilde{x} = x - vt, \quad (63)$$

we then obtain the equation

$$(v^2 - 2)z = 3vz^2 - \frac{v^2}{2}\frac{\partial^2 z}{\partial \tilde{x}^2}. \quad (64)$$

This equation with the boundary condition $z \to 0$ as $\tilde{x} \to \pm\infty$ is equivalent to the equation

$$(v^2 - 2)z^2 - 2vz^3 + \frac{v^2}{2}\left(\frac{\partial z}{\partial \tilde{x}}\right)^2 = 0. \quad (65)$$

The solution of Eq. (65) is

$$z = -\frac{A}{\operatorname{ch}^2(\beta\tilde{x})}, \quad (66)$$

where

$$\beta^2 = \frac{2 - v^2}{2v^2}, \quad A = \frac{2 - v^2}{2v}. \quad (67)$$

Equations (61) and (65) are valid only in the range

$$0 < \sqrt{2} - v \ll 1. \quad (68)$$

6. CONCLUSIONS

The dynamics of vortex states essentially depends on the type of the equation. It is quite different for the nonlinear Schrödinger and wave equations. For the nonlinear Schrödinger equation, there exists an adiabatic parameter, the distance $|\mathbf{r}_{i,j}|$ between vortices. In the leading approximation in this parameter, emission of sound-like exitations is weak and the equation of motion of the vortices is given by simple Eqs. (24), of the first order in time. In such an approximation, the energy of a vortex state is conserved. In the next approximation, the motion of vortices can lead to emission of sound-like exitations. We note that even if the adiabatic parameter is missing (the initial state of the double vortex decays), the increment is numerically small due to a large value of the centrifugal potential. Two vortices of opposite vorticity (charge) at large distances move together with a velocity smaller than the sound velocity ($\sqrt{2}$ in our case) and hence do not radiate. If the distance between such vortices is smaller than some critical value ($r_{cr} \sim 1$), then such vortices collapse, annihilating each other and producing a shock wave [2]. Recently, the interest in vortex dynamics in the time-dependent Schrödinger equation was increasing very rapidly. Numerical simulations of different vortex configurations have been reported [3], in particular, the time dependence of the distance $R_0(t)$ between two equal-charge vortices (see Eq. (51)) was confirmed to a high accuracy. Also a collapse of two vortices of the opposite charge and shock wave formation at $r < r_{cr}$ were found [3]. Equation (1) also has solutions in the form of a two-dimensional soliton [4–6].

The research of Yu. N. O. was supported under the grant EOARD № 097006 and the Programm of DFS, RAS. The research of I. M. S. was supported under the NSERC (grant № NA 7901).

APPENDIX A

In this Appendix, we find an action for the equation

$$-\frac{\partial^2\psi}{\partial t^2} = -\left\{\frac{\partial^2\psi}{\partial \mathbf{r}^2} + (1-|\psi|^2)\psi\right\}. \qquad \text{(A.1)}$$

We suppose that in the state given by the function ψ, there is a certain number of vortices with zeros at points $\mathbf{a}_j$ and with vorticities n_j. If the distances between the vortices are large, $|\mathbf{a}_i - \mathbf{a}_j| \gg 1$ for each pair $i \neq j$, then the vortices can be considered particles. The dynamics of the vortex motion can be described with the help of the action

$$A = \int L\, dt \qquad \text{(A.2)}$$

with

$$L = \frac{1}{2}\int d^2\mathbf{r}\left\{\left|\frac{\partial\psi}{\partial\mathbf{r}}\right|^2 + \frac{1}{2}(1-|\psi|^2)^2 - \left|\frac{\partial\psi}{\partial t}\right|^2\right\}. \qquad \text{(A.3)}$$

We suppose that we have only one moving vortex. Because Eq. (A.1) is Lorentz invariant, a solution for the moving vortex can be found with the help of the Lorentz transformation

$$t' = \frac{t - vx}{\sqrt{1-v^2}}, \quad x' = \frac{x - vt}{\sqrt{1-v^2}}, \quad y = y', \quad z = z', \qquad \text{(A.4)}$$

where v is the vortex velocity. It means that vortices are "heavy" with the masses

$$m = E_{n_j}, \qquad \text{(A.5)}$$

where

$$E_{n_j} = \pi n_j^2 \ln\left(\frac{R}{|n_j|}\right) + C(n_j) \qquad \text{(A.6)}$$

is the energy of the vortex state (see [7]). In Eq. (A.6), R is radius of cut-off circle, and n_j is the vorticity of the vortex. In the nonrelativistic case ($v \ll 1$), we obtain from Eqs. (A.3) and (A.6) that

$$L = E(a) - \frac{1}{2}\sum_j E_{n_j}(\dot{\mathbf{a}}_j)^2 - \frac{1}{2}\int d^2\mathbf{r}\sum_{i\neq j}(\dot{\mathbf{a}}_i\nabla\varphi_i)(\dot{\mathbf{a}}_j\nabla\varphi_j), \qquad \text{(A.7)}$$

where φ_j is the phase of jth vortex, $\mathbf{a}_j$ is the position of a zero, and $a = \{\mathbf{a}_j, n_j\}$. The energy $E(a)$ is equal to [7]

$$E(a) = \sum_j E_{n_j} + \sum_{i\neq j} n_i n_j \ln\frac{R}{|\mathbf{a}_i - \mathbf{a}_j|} + \text{Rem}. \qquad \text{(A.8)}$$

The last term in Eq. (A.8) is of the order of $O\left(\ln a/a^2\right)$ ($a = \min|\mathbf{a}_i - \mathbf{a}_j|$, $i \neq j$), and it can not be presented as a pair interaction only.

For the last term in Eq. (A.7), we obtain

$$\int d^2\mathbf{r}(\dot{\mathbf{a}}_i\nabla\varphi_i)(\dot{\mathbf{a}}_j\nabla\varphi_j) = (\dot{\mathbf{a}}_i\cdot\dot{\mathbf{a}}_j)\int_0^R dr\, r\,\times$$

$$\times\int_0^{2\pi} d\varphi\frac{\sin^2\varphi}{r^2+a_{ij}^2-2|a_{ij}|^2\cos\varphi}+\frac{(J\dot{\mathbf{a}}_i a_{ij})(J\dot{\mathbf{a}}_j a_{ij})}{a_{ij}^2}\times$$

$$\times\int_0^{2\pi} d\varphi\int_0^{\infty} dr\frac{r\cos(2\varphi)-|a_{ij}|\cos\varphi}{r^2+a_{ij}^2-2|a_{ij}|r\cos\varphi}, \quad (\text{A.9})$$

where $a_{ij} = |\mathbf{a}_i - \mathbf{a}_j|$, $|a_{i,j}| = \sqrt{(a_{ij})^2}$.

The integrals in Eq. (A.9) can easy be calculated, with the result

$$L = E(a) - \frac{\pi}{2}\left(\sum_j n_j\dot{\mathbf{a}}_j\right)^2\ln R -$$

$$-\frac{\pi}{2}\sum_j\left(|n_j^2\ln\frac{1}{|n_j|}+\frac{C(n_j)}{\pi}\right)\dot{\mathbf{a}}_j^2 -$$

$$-\frac{\pi}{2}\sum_{i\neq j} n_i n_j(\dot{\mathbf{a}}_i\cdot\dot{\mathbf{a}}_j)\left(\frac{1}{2}+\ln\frac{1}{|a_{ij}|}\right)+$$

$$+\frac{\pi}{2}\sum_{i\neq j} n_i n_j\frac{(J\dot{\mathbf{a}}_i a_{ij})(J\dot{\mathbf{a}}_j a_{ij})}{a_{ij}^2}. \quad (\text{A.10})$$

Equation (A.1) also has excitations of two types: vibrations of the phase, and vibrations of the modulus of ψ. The spectrum of the former excitations is $\omega^2 = k^2$, and of the second ones, $\omega^2 = 2 + k^2$. In the low-frequency limit, we can describe these excitations with the additional term in the action

$$\delta A = \frac{1}{2}\int dt\,\times$$

$$\times\left\{\left(\frac{\partial\varphi_s}{\partial\mathbf{r}}\right)^2-\left(\frac{\partial\varphi_s}{\partial t}\right)^2-2\frac{\partial\varphi_s}{\partial t}\frac{\partial\varphi_0}{\partial t}\right\}, \quad (\text{A.11})$$

where φ_s is a single-valued scalar function that gives the change of the phase function ψ due to small vibrations of the phase, and φ_0 is the phase of vortices. The lagrangian L in Eq. (A.10) contains standard terms of the form of charge–charge and current–current interactions.

But it also contain two nonstandard terms. One is given by the term Rem in Eq. (A.8), and second by the last term in Eq. (A.10). Rem leads to a multiparticle interaction. The additional term given by Eq. (A.11) leads to the excitation of waves due to the vortex motion. For large distances between vortices, this term can be taken into account with the help of the perturbation theory.

The equation of motion for a "particle" is

$$\frac{\delta A}{\delta\mathbf{a}_j} = 0. \quad (\text{A.12})$$

This equation leads to the following equation of motion for "particles":

$$-2n_j\sum_{i\neq j}\frac{n_i a_{ji}}{a_{ji}^2}+n_j\ln R\sum_i n_i\ddot{\mathbf{a}}_i+$$

$$+\ddot{\mathbf{a}}_j\left(n_j^2\ln\frac{1}{|n_j|}+\frac{C(n_j)}{\pi}\right)+$$

$$+n_j\sum_{i\neq j} n_i\frac{\partial}{\partial t}\left[\dot{\mathbf{a}}_i\left(\frac{1}{2}+\ln\frac{1}{|a_{ij}|}\right)\right]+$$

$$+\sum_{i\neq j} n_i n_j(\dot{\mathbf{a}}_i\cdot\dot{\mathbf{a}}_j)\frac{a_{ji}}{a_{ji}^2}-$$

$$-2n_j\sum_{i\neq j} n_i\frac{a_{ji}(J\dot{\mathbf{a}}_i a_{ji})(J\dot{\mathbf{a}}_j a_{ji})}{a_{ji}^4}+$$

$$+n_j\sum_{i\neq j} n_i\frac{(J\dot{\mathbf{a}}_j)(J\dot{\mathbf{a}}_i a_{ji})+(J\dot{\mathbf{a}}_i)(J\dot{\mathbf{a}}_j a_{ji})}{a_{ji}^2}-$$

$$-n_j\sum_{i\neq j} n_i\frac{\partial}{\partial t}\left[\frac{(Ja_{ji})(J\dot{\mathbf{a}}_i a_{ji})}{a_{ji}^2}\right]=0. \quad (\text{A.13})$$

We now consider the simplest case of two vortices with

$$-n_2 = n_1 = 1.$$

It is easy to see that there exist solutions of the system of Eqs. (A.12) of the type

$$-\mathbf{a}_2 = \mathbf{a}_1 = a(\cos\omega t, \sin\omega t). \quad (\text{A.14})$$

Inserting expression (A.14) in Eq. (A.13), we obtain

$$\omega^2 = \left\{a^2\left[2\ln R + 1 + \frac{C(1)}{\pi}+\ln\frac{1}{2a}\right]\right\}^{-1}. \quad (\text{A.15})$$

The velocity of motion given by Eq. (A.15) is small only due to the large value of the rest mass.

According to Eq. (A.3), the energy inside the domain D is given by

$$E_D = \frac{1}{2}\int_D d^2\mathbf{r}\,\times$$

$$\times\left\{\left(\frac{\partial\psi}{\partial\mathbf{r}}\right)^2+\frac{1}{2}(1-|\psi|^2)^2+\left(\frac{\partial\psi}{\partial t}\right)^2\right\}. \quad (\text{A.16})$$

Hence, we have

$$\frac{\partial E_D}{\partial t} = \frac{1}{2}\int_S d\mathbf{s}\left\{\frac{\partial\psi}{\partial t}\frac{\partial\psi^*}{\partial\mathbf{r}}+\frac{\partial\psi^*}{\partial t}\frac{\partial\psi}{\partial\mathbf{r}}\right\}. \quad (\text{A.17})$$

It follows from Eq. (A.17) that the energy flow density $\mathbf{S}_E$ is equal to

$$\mathbf{S}_E = -\frac{1}{2}\left\{\frac{\partial\psi}{\partial t}\frac{\partial\psi^*}{\partial\mathbf{r}} + \frac{\partial\psi^*}{\partial t}\frac{\partial\psi}{\partial\mathbf{r}}\right\}. \quad \text{(A.18)}$$

At large distances from vortices, we find

$$\mathbf{S}_E = -\frac{\partial\varphi_s}{\partial t}\frac{\partial\varphi_s}{\partial\mathbf{r}}. \quad \text{(A.19)}$$

The equation of motion for the phase φ_s follows from Eq. (A.11),

$$\frac{\partial^2\varphi_s}{\partial t^2} - \frac{\partial^2\varphi_s}{\partial\mathbf{r}^2} = -\frac{\partial^2\varphi_0}{\partial t^2}. \quad \text{(A.20)}$$

This equation is solved by

$$\varphi_s(t) = -\int\limits_{-\infty}^{t} dt_1 \int d^2\mathbf{r}_1 G(t-t_1, \mathbf{r}-\mathbf{r}_1) \times \\ \times \frac{\partial^2\varphi_0(\mathbf{r}_1, t_1)}{\partial t_1^2}, \quad \text{(A.21)}$$

where Green's function G is the solution of the equation

$$\frac{\partial^2 G}{\partial t^2} - \frac{\partial^2 G}{\partial\mathbf{r}^2} = \delta(t-t_1)\delta(\mathbf{r}-\mathbf{r}_1). \quad \text{(A.22)}$$

The explicit form of G is

$$G(t,\mathbf{r}) = \frac{\theta(t)}{2\pi}\begin{cases} 0, & t < |\mathbf{r}| \\ \dfrac{1}{\sqrt{t^2-\mathbf{r}^2}}, & t > |\mathbf{r}|. \end{cases} \quad \text{(A.23)}$$

Rotation of two vortices with different charge (Eq. A.13)) leads to emission of excitations. As a result, two vortices of different charge collapse.

APPENDIX B

We find the solution of Eqs. (51) that is finite as $r \to 0$ and has the form of an outgoing wave for $r \to \infty$. The function $f_2(r)$ is a solution of Eq. (10) with topological charge $n = 2$.

For $r \ll 1$, the function $f_2(r)$ is given by expansion (52). The function $f_2(r)$ is an analytical function of r and series (52) is convergent inside the circle with the radius equal to the distance to the nearest pole of function $f_2(r)$ on the imaginary axis of r.

For large values of $r \gg 1$, we have (54). To obtain the coefficient B, we solve Eq. (10) numerically, starting from small values of r (Eq. (52)), and match it with an expression (54) for $r \gg 1$. In this way, we obtain the exact value of B and an approximate value of C. To obtain a more accurate value of coefficient C, we solve Eq. (10), starting from large values, $r \gg 1$, and match the solution such found with values of the function $f_2(r)$, given by expansion (52) for $r \sim 0.1$. In this way, we obtain the coefficients B and C with high accuracy:

$$B = 0.15289, \quad C = -16.69. \quad \text{(B.1)}$$

Using expansion (52), we find the general solution of the system of Eqs. (51), finite as $r \to 0$, in form (53). At large distances $r \gg 1$, we seek a solution of Eqs. (51) in form (55).

In the leading approximation, we then obtain

$$-\left(\frac{\partial S}{\partial r}\right)^2\begin{pmatrix}\alpha\\ \beta\end{pmatrix} + (\alpha+\beta)\begin{pmatrix}1\\ 1\end{pmatrix} = \lambda\begin{pmatrix}\alpha\\ -\beta\end{pmatrix}. \quad \text{(B.2)}$$

Multiplying both sides of Eq. (B.2) by $(-\beta, \alpha)$, we obtain the following equation for α and β:

$$\alpha^2 - \beta^2 + 2\lambda\alpha\beta = 0. \quad \text{(B.3)}$$

There are two linearly independent solutions, which satisfy boundary condition at the infinity. The first solution is

$$\alpha = 1, \quad \beta = \lambda + \sqrt{1+\lambda^2}, \\ \frac{\partial S}{\partial r} = \left(1 + \sqrt{1+\lambda^2}\right)^{1/2}. \quad \text{(B.4)}$$

The second solution is

$$\tilde\beta = 1, \quad \tilde\alpha = -\left(\lambda + \sqrt{1+\lambda^2}\right), \\ \frac{\partial\tilde S}{\partial r} = i\left(\sqrt{1+\lambda^2} - 1\right)^{1/2}. \quad \text{(B.5)}$$

Corrections to expressions (B.4) and (B.5) can be found in the usual way via the perturbation theory. For this, we write Eq. (B.2) taking the terms

$$-\left(\frac{\partial S}{\partial r}\right)^2\begin{pmatrix}\alpha\\ \beta\end{pmatrix} + 2\left(\frac{\partial S}{\partial r}\right)\frac{\partial}{\partial r}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} + \\ + \left(-\frac{1}{4r^2} + \frac{\partial^2 S}{\partial r^2} - \frac{\partial^2}{\partial r^2}\right)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} + (\alpha+\beta)\begin{pmatrix}1\\ 1\end{pmatrix} + \\ + \frac{16}{r^2}\begin{pmatrix}0\\ \beta\end{pmatrix} - \left(\frac{4}{r^2} + \frac{8}{r^4}\right)\begin{pmatrix}2\alpha+\beta\\ 2\beta+\alpha\end{pmatrix} = \lambda\begin{pmatrix}\alpha\\ -\beta\end{pmatrix} \quad \text{(B.6)}$$

in the asymptotic expansion into account.

For the first linearly independent solution (B.4), we find

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}1\\ \lambda+\sqrt{1+\lambda^2}\end{pmatrix} + \left(\frac{\gamma_1}{r^2}\right) \times \times \left(1+\frac{\gamma_2}{r}\right)\begin{pmatrix}-\left(\lambda+\sqrt{1+\lambda^2}\right)\\ 1\end{pmatrix}, \tag{B.7}$$

$$S = \left(1+\sqrt{1+\lambda^2}\right)^{1/2} r + \left(\frac{\gamma_3}{r}\right)\left(1+\frac{\gamma_4}{r}\right),$$

where $\gamma_{1,2,3,4}$ are constants. From the first two equations in (B.7), we easily obtain the useful relations

$$\alpha^2-\beta^2 = -2\left(\lambda+\sqrt{1+\lambda^2}\right) \times \times \left[\lambda+\frac{2\gamma_1}{r^2}\left(1+\frac{\gamma_2}{r}\right)\right],$$

$$\frac{\beta}{\alpha} = \left(\lambda+\sqrt{1+\lambda^2}\right) \times \times \left[1+\frac{2\gamma_1}{r^2}\left(1+\frac{\gamma_2}{r}\right)\sqrt{1+\lambda^2}\right], \tag{B.8}$$

$$\alpha\beta = \left(\lambda+\sqrt{1+\lambda^2}\right)\left[1-\frac{2\lambda\gamma_1}{r^2}\left(1+\frac{\gamma_2}{r}\right)\right].$$

Multiplying Eq. (B.7) by the $(-\beta,\alpha)$ and using Eq. (B.8), we obtain the coefficiets $\gamma_{1,2}$:

$$\gamma_1 = \frac{2(2+\lambda)}{1+\lambda^2}, \qquad \gamma_2 = -\sqrt{1+\sqrt{1+\lambda^2}}\,\frac{2}{\sqrt{1+\lambda^2}}. \tag{B.9}$$

Inserting the values of the functions α, β and S from Eq. (B.7) in the first Eq. (B.6), we obtain the coefficients $\gamma_{3,4}$:

$$\gamma_3 = \frac{1}{\sqrt{1+\sqrt{1+\lambda^2}}}\left[\frac{1}{8}+\frac{2(1-2\lambda)}{\sqrt{1+\lambda^2}}\right], \qquad \gamma_4 = -\frac{1}{2\sqrt{1+\sqrt{1+\lambda^2}}}. \tag{B.10}$$

We similarly find for the second linearly independent solution (B.5)

$$\begin{pmatrix}\tilde\alpha\\ \tilde\beta\end{pmatrix} = \begin{pmatrix}-\left(\lambda+\sqrt{1+\lambda^2}\right)\\ 1\end{pmatrix} + + \frac{\tilde\gamma_1}{r^2}\left(1+\frac{\tilde\gamma_2}{r}\right)\begin{pmatrix}1\\ \lambda+\sqrt{1+\lambda^2}\end{pmatrix}, \tag{B.11}$$

$$\tilde S = i\left\{r\sqrt{\sqrt{1+\lambda^2}-1}+\frac{\tilde\gamma_3}{r}\left(1+\frac{\tilde\gamma_4}{r}\right)\right\}.$$

From the first two equations (B.11) we obtain the useful relations

$$\tilde\alpha^2-\tilde\beta^2 = 2\left(\lambda+\sqrt{1+\lambda^2}\right) \times \times \left[\lambda-\frac{2\tilde\gamma_1}{r^2}\left(1+\frac{\tilde\gamma_2}{r}\right)\right],$$

$$\frac{\tilde\beta}{\tilde\alpha} = -\frac{1}{\lambda+\sqrt{1+\lambda^2}} \times \times \left[1+\frac{2\tilde\gamma_1}{r^2}\sqrt{1+\lambda^2}\left(1+\frac{\tilde\gamma_2}{r}\right)\right], \tag{B.12}$$

$$\tilde\alpha\tilde\beta = -\left(\lambda+\sqrt{1+\lambda^2}\right)\left(1+\frac{2\lambda\tilde\gamma_1}{r^2}\left(1+\frac{\tilde\gamma_2}{r}\right)\right).$$

As previously, multiplying Eqs. (B.6) by $(-\tilde\beta,\tilde\alpha)$ and using Eqs. (B.12), we easily obtain the coefficients $\tilde\gamma_{1,2}$:

$$\tilde\gamma_1 = -\frac{2(2+\lambda)}{1+\lambda^2}, \qquad \tilde\gamma_2 = \frac{2i}{\sqrt{1+\lambda^2}}\sqrt{\sqrt{1+\lambda^2}-1}. \tag{B.13}$$

Inserting the values of the functions $\tilde\alpha$, $\tilde\beta$ and $\tilde S$ from Eq. (B.11) in the first Eq. (B.6), we obtain the coefficients $\tilde\gamma_{3,4}$:

$$\tilde\gamma_3 = \frac{1}{\sqrt{\sqrt{1+\lambda^2}-1}}\left[-\frac{1}{8}+\frac{2(1-2\lambda)}{\sqrt{1+\lambda^2}}\right], \qquad \tilde\gamma_4 = \frac{i}{2\sqrt{\sqrt{1+\lambda^2}-1}}. \tag{B.14}$$

As a result, for $r \gg 1$, the general solution of Eqs. (51) that satisfies the boundary condition as $r\to\infty$ can be presented in the form

$$\begin{pmatrix} f_0 \\ f_4 \end{pmatrix} = \frac{C_3}{\sqrt{r}} \exp\left\{ -\left[r\sqrt{1+\sqrt{1+\lambda^2}} + \frac{1/8 + 2(1-2\lambda)/\sqrt{1+\lambda^2}}{r\sqrt{1+\sqrt{1+\lambda^2}}} \left(1 - \frac{1}{2r\sqrt{1+\sqrt{1+\lambda^2}}} \right) \right] \right\} \times$$
$$\times \left\{ \begin{pmatrix} 1 \\ \lambda + \sqrt{1+\lambda^2} \end{pmatrix} + \frac{2(2+\lambda)}{(1+\lambda^2)r^2} \left(1 - \frac{2\sqrt{1+\sqrt{1+\lambda^2}}}{r\sqrt{1+\lambda^2}} \right) \begin{pmatrix} -(\lambda + \sqrt{1+\lambda^2}) \\ 1 \end{pmatrix} \right\} +$$
$$+ \frac{C_4}{\sqrt{r}} \exp\left\{ -i \left[r\sqrt{\sqrt{1+\lambda^2} - 1} + \frac{(-1/8 + 2(1-2\lambda)/\sqrt{1+\lambda^2})}{r\sqrt{\sqrt{1+\lambda^2}-1}} \left(1 + \frac{i}{2r\sqrt{\sqrt{1+\lambda^2}-1}} \right) \right] \right\} \times$$
$$\times \left\{ \begin{pmatrix} -(\lambda + \sqrt{1+\lambda^2}) \\ 1 \end{pmatrix} - \frac{2(2+\lambda)}{(1+\lambda^2)r^2} \left(1 + \frac{2i\sqrt{\sqrt{1+\lambda^2}-1}}{r\sqrt{1+\lambda^2}} \right) \begin{pmatrix} 1 \\ \lambda + \sqrt{1+\lambda^2} \end{pmatrix} \right\}. \quad \text{(B.15)}$$

To obtain the value of complex coefficients λ, C_2, C_3, and C_4, we solve Eqs. (51) numerically, starting from expression (53) for $r \ll 1$ and matching the solution thus found with expression (B.15) for $r \gg 1$. As a result, we obtain the value of the coefficients λ, C_2, and C_4 with high accuracy and the approximate coefficient C_3 (in front of the exponentially decreasing term as $r \to \infty$ in expression (B.15)). To obtain a more accurate value of C_3, we solve Eqs. (51), starting from expression (B.15) for $r \gg 1$ and matching this solution with expression (53) for $r \sim 1$. As a result, we obtain high-precision values for all the coefficients λ, C_2, C_3, and C_4:

$$\begin{gathered} \lambda = -0.443673 + i\,0.004937, \\ C_2 = -0.00734 + i\,0.0001494, \\ C_3 = 99.88 - i\,33.12, \\ C_4 = 0.12618 + i\,0.0275. \end{gathered} \quad \text{(B.16)}$$

REFERENCES

1. Yu. N. Ovchinnikov and I. M. Sigal, Nonlinearity **11**, 1277 (1998).

2. Yu. N. Ovchinnikov and I. M. Sigal, Nonlinearity **11**, 1295 (1998).

3. Yanzhi Zhang, Weizhu Bao, and Qiang Du, *Numerical Simulation of Vortex Dynamics in Ginzburg–Landau–Schrödinger Equation*, preprint of the National University of Singapore, Singapore (2007).

4. P. H. Roberts and J. Grant, J. Phys. A **4**, 55 (1971); C. A. Jones, S. J. Putterman, and P. H. Roberts, J. Phys. A **19**, 2991 (1986).

5. S. V. Manakov, V. E. Zakharov, A. A. Bordag, A. R. Its, and V. B. Matveev, Phys. Lett. A **63**, 205 (1977).

6. E. A. Kuznetsov and J. J. Rasmussen, Phys. Rev. E **51**, 4479 (1995).

7. Yu. N. Ovchinnikov and I. M. Sigal, Zh. Eksp. Teor. Fiz. **126**, 11249 (2004).