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PERFECT FLUID AND SCALAR FIELD IN THE REISSNER–NORDSTRÖM METRIC

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We describe the spherically symmetric steady-state accretion of perfect fluid in the Reissner–Nordström metric. We present analytic solutions for accretion of a fluid with linear equations of state and of the Chaplygin gas. We also show that under reasonable physical conditions, there is no steady-state accretion of a perfect fluid onto a Reissner–Nordström naked singularity. Instead, a static atmosphere of fluid is formed. We discuss a possibility of violation of the third law of black hole thermodynamics for a phantom fluid accretion.

1. INTRODUCTION

The problem of matter accretion onto compact objects in Newtonian gravity was formulated within the self-similar treatment by Bondi [1]. In the framework of general relativity, the steady-state spherical symmetric flow of test gas onto a Schwarzschild black hole was investigated by Michel [2]. Detailed studies of spherically symmetric accretion of different types of fluids onto black holes were further undertaken in a number of works [3] (see also review [4]).

In this paper, we study perfect fluids and scalar fields in the Reissner–Nordström (RN) metric. We describe spherically symmetric steady-state accretion of a test perfect fluid with a general equation of state onto a nonrotating charged black hole. We find analytic solutions for accretion of a perfect fluid with a linear equation of state and of the Chaplygin gas onto an RN black hole. When a phantom fluid accretes onto a black hole, the latter loses its mass. This result is consistent with the findings in Ref. [5] on the phantom accretion onto a Schwarzschild black hole.

We find that under reasonable physical assumptions, a perfect fluid does not accrete onto the RN naked singularity, i.e., when $M^2 < Q^2$, where M is the mass and Q is the electric charge of the naked singularity. Namely, steady-state accretion onto a naked singularity is only possible in two unphysical cases. In the first case, the accreting fluid is superluminal and an additional boundary condition on the central singularity is specified. In the second case, the fluid may be stiff or subluminal, but we have to postulate that the inflow and outflow coexist in the space–time manifold, and the solution passes somehow through a singular point. We show that instead of a steady-state accretion, a static atmosphere around a naked singularity is formed[1)].

We also show that the extreme state of an electrically charged black hole is reached in a finite time due to phantom fluid accretion, when gravitational back reaction of the accreting fluid is neglected. We argue, however, that the test fluid approximation may be violated when the RN black hole or naked singularity is almost extreme. This implies that back reaction of the fluid on the background geometry may prevent transformation of a black hole into a naked singularity, in accordance with the third law of black hole thermodynamics [7].

The paper is organized as follows. In Sec. 2, we construct the general formalism for steady-state spherically symmetric accretion of a test perfect fluid in the

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1) A similar result for the Kerr naked singularity was found in [6] using numerical methods.

RN metric. In Sec. 3, we give an alternative description of the accretion in terms of a scalar field. In Sec. 4, we apply the results of the previous sections to particular examples of perfect fluid, namely, we study accretion of a fluid with a linear equation of state and accretion of the Chaplygin gas. A static atmosphere of fluids around a naked singularity is described in Sec. 5. A black hole approaching the extreme state by accretion of phantom fluid and the possibility of violation of the third law of thermodynamics are discussed in Sec. 6. We conclude in Sec. 7.

2. STEADY-STATE ACCRETION

In this section, we study spherically symmetric steady-state accretion of a test perfect fluid with a general equation of state in the RN metric. We closely follow the approach in [5] to gas accretion in the Schwarzschild metric.

The RN metric is given by

$$ds^2 = fdt^2 - f^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{1}$$

where

$$f = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}.$$

Here, M is the black hole (or naked singularity) mass, and Q is its total charge. It is convenient to introduce dimensionless coordinates,

$$\tau \equiv \frac{t}{M}, \quad x \equiv \frac{r}{M},$$

and the dimensionless electric charge of the black hole $e \equiv Q/M$. In the case $e^2 < 1$, the equation $f(x) = 0$ has two roots

$$x_\pm = 1 \pm \sqrt{1 - e^2}.$$

The larger root, x_+, corresponds to the event horizon of the RN black hole, and $x = x_-$ is the so-called Cauchy (or inner) horizon. In the opposite case, $e^2 > 1$, the RN metric (1) describes a naked singularity without an event horizon. The marginal case $e^2 = 1$ corresponds to an extreme black hole.

The energy–momentum of a perfect fluid is

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu}, \tag{2}$$

where ρ and p are the fluid energy density and pressure respectively, and $u^\mu = dx^\mu/ds$ is the fluid four-velocity normalized by $u^\mu u_\mu = 1$. We assume that the pressure is an arbitrary function of the density alone, $p = p(\rho)$. To find integrals of motion, we use the projection of the equation for the energy–momentum tensor conservation onto the 4-velocity, $u_\mu T^{\mu\nu}{}_{;\nu} = 0$. This gives the continuity equation

$$u^\mu \rho_{,\mu} + (\rho + p)u^\mu{}_{;\mu} = 0. \tag{3}$$

Integrating (3) once, we find the integral of motion (the energy conservation)

$$ux^2 n = -A, \tag{4}$$

where

$$n \equiv \exp\left[\int\limits_{\rho_\infty}^{\rho} \frac{d\rho'}{\rho' + p(\rho')}\right],$$

$u = dr/ds < 0$ in the case of inflow motion (accretion), and $A > 0$ is a constant of integration, which is related to the radial energy flux.

Integration of the time component of the conservation law $T^{\mu\nu}{}_{;\nu} = 0$ gives another integral of motion (the relativistic Bernoulli equation),

$$(\rho + p)(f + u^2)^{1/2} x^2 u = C_1, \tag{5}$$

where $u \equiv dr/ds$ and C_1 is a constant of integration. From (4) and (5), we can easily obtain that

$$\frac{\rho + p}{n}(f + u^2)^{1/2} = C_2, \tag{6}$$

where

$$C_2 \equiv \frac{-C_1}{A} = \frac{\rho_\infty + p(\rho_\infty)}{n(\rho_\infty)},$$

with ρ_∞ being the energy density at infinity.

Equations (4) and (6) along with the equation of state $p = p(\rho)$ form a closed system for accretion onto an RN black hole (or naked singularity). This system is to be supplied with appropriate boundary conditions. The obtained system of equations describes accretion of a perfect fluid with a general equation of state $p = p(\rho)$, and may be applied, in particular, to accretion of the Chaplygin gas [8] or dark energy described by the generalized linear equation of state [9].

The constant C_2 is fixed by the boundary condition at infinity. Fixing A in (4) and, respectively, the flux is more tricky. This is provided by the physical requirement of a smooth transition through the critical sound point (see the details, e. g., in [2]). The resulting solution should be continuous from infinity down to the black hole horizon. Following [2], we find relations at the critical point,

$$u_*^2 = \frac{x_* - e^2}{2x_*^2}, \quad c_s^2(\rho_*) = \frac{x_* - e^2}{2x_*^2 - 3x_* + e^2}, \tag{7}$$

where $c_s(\rho) \equiv (\partial p/\partial \rho)^{1/2}$ is the speed of sound, and the subscript "$*$" indicates that the values are taken at the critical point. It follows from (7) that

$$x_*^{\pm} = \frac{1+3c_*^2}{4c_*^2}\left\{1 \pm \left[1 - \frac{8c_*^2(1+c_*^2)}{(1+3c_*^2)^2}e^2\right]^{1/2}\right\}, \qquad (8)$$

where $c_* \equiv c_s(x_*)$. Critical points exist only if

$$e^2 \le \frac{\left(1+3c_*^2\right)^2}{8c_*^2\left(1+c_*^2\right)}.$$

It is worthwhile to note that in contrast to the case of a Schwarzschild black hole, there are formally two different critical points, corresponding to the plus and minus signs in (8). We also note that $x_*^- \to 0$ as $e \to 0$.

Depending on the values of e and c_s one can identify the following five cases.

1. $e < 1$, $c_s^2 < 1$ ($c_s^2 = 1$). In this case, the event and the Cauchy horizons exist, $x^+ > x^-$, as well as both critical points; the outer critical point is outside the event horizon, $x_*^+ > x^+$ ($x_*^+ = x^+$), the inner critical point is between the event and the Cauchy horizons, $x^- < x_*^- < x^+$ ($x_*^- = x_-$).

2. $e < 1$, $c_s^2 > 1$. Similarly to the previous case, the event and the Cauchy horizons, and both critical points exist; but the outer critical point is in between the event and the Cauchy horizons, $x^- < x_*^- < x^+$ ($x_*^+ = x_- = x_+$); the inner critical point is inside the Cauchy horizon, $x_*^- < x^+$.

3. $e = 1$. The event and the Cauchy horizons coincide, $x^+ = x^- = 1$, and both critical points exist: in the subluminal case, $x_*^+ > 1$ and $x_*^- = 1$; for a stiff fluid, $c_s^2 = 1$, we find $x_*^{\pm} = 1$; in the superluminal case, $x_*^+ = 1$ and $x_*^- < 1$.

4. $1 < e < 3/2\sqrt{2}$. The RN metric describes a naked singularity (the horizons are absent). Critical points exist for two different branches,

$$c_s^2 \le \frac{-4e^2+3-4e\sqrt{e^2-1}}{8e^2-9} \ \text{(subluminal)},$$

or

$$c_s^2 \ge \frac{-4e^2+3+4e\sqrt{e^2-1}}{8e^2-9} \ \text{(superluminal)}.$$

5. $e \ge 3/2\sqrt{2}$. The RN metric describes a naked singularity. In contrast to the previous case, the critical points exist only for subluminal branch.

In Fig. 1, the critical radii are shown as functions of the speed of sound for several values of e.

Substituting the value of x_*^+ from (8) in the first relation in (7) and then substituting x_* and u_* expressed

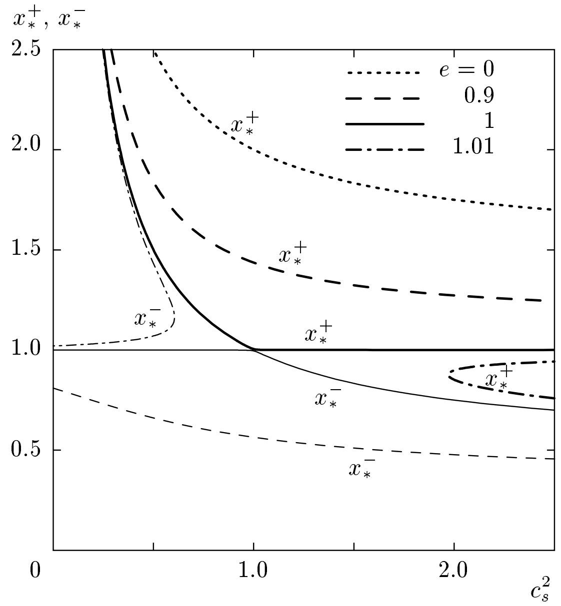

Fig. 1. The outer critical radius x_*^+ (thick lines) and inner critical radius x_*^- (thin lines) are shown as functions of the sound speed c_s for several values of the electric charge $e = Q/M$. We note that the outer critical radius coincides with the event horizon, $x_*^+ = 1$, for the extreme black hole ($e = 1$) in the case $c_s \ge 1$

in terms of c_* in (6), we find a closed equation for ρ at the critical point,

$$\frac{\rho_*+p_*}{\rho_\infty+p_\infty}\,\frac{n_\infty}{n_*} = \frac{1+3c_*^2+\mathcal{D}}{\sqrt{2\left[1+3c_*^2+4e^2c_*^2(c_*^2-1)+\mathcal{D}\right]}}, \qquad (9)$$

where

$$\mathcal{D} = \sqrt{\left(1+3c_*^2\right)^2 - 8e^2c_*^2(c^2+1)}.$$

For $e = 0$, Eq. (9) reduces to the equation for the critical point in the case of the Schwarzschild black hole [5].

The black hole mass changes at the rate $\dot{M} = -4\pi r^2 T_0{}^r$ due to fluid accretion. With the help of (4) and (6), this expression can be written as

$$\dot{M} = 4\pi A M^2[\rho_\infty + p_\infty]. \qquad (10)$$

It is clear from this equation that the accretion of phantom energy, defined by the condition $\rho_\infty + p(\rho_\infty) < 0$, is always accompanied with a decrease in the black hole mass. This is in accordance with previous findings [5]. We stress that the result is valid for any equation of state $p = p(\rho)$ with $\rho + p(\rho) < 0$.

3. PERFECT FLUID AS A SCALAR FIELD

It is well known that the dynamics of a relativistic perfect fluid in the absence of vorticity can be described in terms of a scalar field. In particular, a stiff fluid corresponds to a canonical massless scalar field. To describe more complicated equations of state, we introduce a generalized noncanonical scalar field Lagrangian of the form

$$\mathcal{L} = \mathcal{L}(X), \quad X \equiv \frac{1}{2}\partial_\mu\phi\partial^\mu\phi. \tag{11}$$

The energy–momentum tensor corresponding to Lagrangian (11) is

$$T_{\mu\nu} = \mathcal{L}_X \nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}\mathcal{L},$$

where the subscript X denotes the derivative with respect to X. The correspondence between the scalar field and the perfect fluid with energy–momentum tensor (2) is achieved by the identification (see, e. g., [10])

$$u_\mu \equiv \frac{\nabla_\mu\phi}{\sqrt{2X}},$$

where the pressure p coincides with the Lagrangian density of the scalar field, $p = \mathcal{L}(X)$, and the energy density is

$$\rho\,(X) = 2X\mathcal{L}_{,X} - \mathcal{L}.$$

The sound speed can be expressed as

$$c_s^2 = \frac{\mathcal{L}_{,X}}{\rho_{,X}} = \left(1 + 2X\frac{\mathcal{L}_{XX}}{\mathcal{L}_X}\right)^{-1}.$$

Apart from the energy density ε and pressure p, we can formally define the "particle number density"

$$n \equiv \exp\left(\int \frac{d\rho}{\rho + p}\right) = \sqrt{X}\mathcal{L}_{,X}$$

and the enthalpy

$$h \equiv \frac{\rho + p}{n} = 2\sqrt{X}.$$

Equations of motion following from (11) are

$$\partial_\mu\left(\sqrt{-g}\,\mathcal{L}_X\, g^{\mu\nu}\partial_\nu\phi\right) = 0. \tag{12}$$

A steady-state flow is described by the ansatz

$$\phi(t, x) = a_\infty t + \psi(x), \tag{13}$$

where the constant a_∞ defines the "cosmological" value of $\dot\phi$ at spatial infinity. For ansatz (13), it is easy to see that

$$X = \frac{1}{2}\left(\frac{a_\infty^2}{f} - f\psi'^2\right),$$

and equation of motion (12) can be integrated once to give

$$x^2 f\mathcal{L}_X\psi'(x) = \sqrt{2}A. \tag{14}$$

Equation (14) is in fact another form of (3), written in terms of the scalar field. Moreover, Eq. (14) is an algebraic equation for the function ψ'. Therefore, the general solution must contain A, which should be determined via an analog of critical point (7). From (12), we can find ψ'' in terms of ψ' (this expression also contains $\mathcal{L}_X$ and $\mathcal{L}_{XX}$). The critical point is then found by equating both the nominator and the denominator of the obtained expression to zero. As a result, we obtain

$$\psi_*'^{\,2} = a_\infty^2\frac{x_* f_*'}{f^2\,(x_* f_*' + 4f_*)}, \quad f_*\psi_*'^{\,2}\mathcal{L}_{XX} = \mathcal{L}_X, \tag{15}$$

which is another form of (7). We now have three equations (14) and (15) which can be used to find ψ_*', x_*, and A. This procedure is fully equivalent to fixing the critical point for the fluid accretion. This description is very useful for some particular tasks.

In particular, we analyze (14) in the limit $x \to 0$. We have

$$2X \sim \frac{x^2}{e^2}B^2 - \frac{e^2}{x^2}\psi'^2.$$

Because $X > 0$ for the fluid, this leads to

$$X \to 0, \quad \psi'^2 \to 0 \quad \text{as} \quad x \to 0. \tag{16}$$

On the other hand, it follows from (14) that

$$\mathcal{L}_X\psi' \to \text{const}, \quad x \to 0. \tag{17}$$

Combining (16) and (17), we conclude that the fluid reaches $x = 0$ during a steady-state accretion only if $\mathcal{L}_X \to \infty$ as $X \to 0$. This means, in particular, that a fluid described by the linear equation of state with $\alpha \le 1$ does not reach the central singularity at $x = 0$ if $e \ne 0$.

4. ACCRETION ONTO A BLACK HOLE

In this section, we present and discuss several analytic solutions for steady-state accretion of a perfect fluid onto a charged black hole.

4.1. Linear equation of state

As the first example, we consider the linear equation of state

$$p = \alpha(\rho - \rho_0), \tag{18}$$

where α and ρ_0 are constants. This equation was introduced in [5] (see also [9]) to avoid hydrodynamical instability for a perfect fluid with negative pressure. The constant α in (18) determines the squared speed of sound of small perturbations, $\alpha = c_s^2$, and it must be positive. We note that (18) can be regarded as a linear approximation to the general nonlinear equation of state $p = p(\rho)$ around some point $\rho = \rho_1$. Therefore, the results in this section can be applied to a generic equation of state if $|\rho - \rho_1|$ is small enough.

Using (7) and (8), we can use (4) to calculate the dimensionless constant A for the linear equation of state as

$$A = \alpha^{1/2} x_*^2 \left(\frac{2\alpha x_*^2}{x_* - e^2} \right)^{(1-\alpha)/2\alpha}. \qquad (19)$$

The velocity and the energy density as functions of the radius are determined by solving (4) and (6),

$$\begin{aligned} f + u^2 &= \left(-\frac{ux^2}{A} \right)^{2\alpha}, \\ \frac{\rho + p}{\rho_\infty + p_\infty} &= \left(-\frac{A}{ux^2} \right)^{1+\alpha}. \end{aligned} \qquad (20)$$

It is possible to express the solutions of the above equations through known analytic functions for specific values of α, namely, $\alpha = 1/4$, $1/3$, $1/2$, $2/3$, 1, $3/2$, and 2. Below, we present solutions corresponding to some particular values of α.

We first consider the case of the stiff fluid: $\alpha = 1$. For the radial velocity and the energy density we then find

$$u^2 = \frac{(x - x_-) x_+^4}{(x + x_+)(x^2 + x_+^2) x^2},$$

$$\rho = \frac{\rho_0}{2} + \left(\rho_\infty - \frac{\rho_0}{2} \right) \frac{(x + x_+)(x^2 + x_+^2)}{(x - x_-) x^2}.$$

The density at the horizon is

$$\rho_+ = \frac{\rho_0}{2} + \left(\rho_\infty - \frac{\rho_0}{2} \right) \frac{2x_+}{\sqrt{1 - e^2}}. \qquad (21)$$

We note that the energy density diverges at the event horizon x_+ of an extreme black hole, $e = 1$.

Solutions for a thermal photon gas, $\alpha = 1/3$, can be found accordingly. Indeed, the radial distribution of the energy density in this case is

$$\rho = \frac{\rho_0}{4} + \left(\rho_\infty - \frac{\rho_0}{4} \right) \left(\frac{1 + 2z}{3f} \right)^2,$$

where

$$z = \begin{cases} \cos \dfrac{2\pi - \beta}{3}, & x_+ \le x \le x_*, \\[2ex] \cos \dfrac{\beta}{3}, & x > x_* \end{cases}$$

and

$$\beta = \arccos \left(1 - \frac{27}{2} A^2 \frac{f^2}{x^4} \right).$$

Phantom energy in this particular case corresponds to the choice $\rho_0 > 4\rho_\infty$. At the event horizon $x = x_+$, we have

$$\rho_+ = \rho(x_+) = \frac{\rho_0}{4} + \left(\rho_\infty - \frac{\rho_0}{4} \right) \frac{A^2}{x_+^4}.$$

The case of a superluminal fluid is also worth studying. As an example, we take $\alpha = 2$. The inflow then consists of two hydrodynamical branches:

$$\begin{aligned} u_{1,2} &= \frac{1}{\sqrt{2}} \frac{A^2}{x^4} \sqrt{1 \pm \sqrt{1 + 4f \frac{x^8}{A^4}}}, \\ \rho_{1,2} &= \left(\frac{A}{u_{1,2} x^2} \right)^3. \end{aligned} \qquad (22)$$

At the outer and inner horizons, we find

$$u_1(x_\pm) = \frac{A^2}{x_\pm^4}, \quad u_2(x_\pm) = 0.$$

The energy density diverges at r_-, and the solution does not exist for $r < r_-$. The behavior of superluminal fluids ($c_s > 1$) is quite unusual. Apart from the transonic solution in (22), there is an infinite family of solutions that are regular at $r > 0$ and are parameterized by A with $A > A_*$. These solutions consist of a single hydrodynamical branch, and the sonic horizon is absent. A solution with $A > A_*$ allows probing the singularity of a black hole with small perturbations. In fact, it is not clear how to choose the "correct" physical solution for a superluminal fluid[2)].

Contrary to accretion of a superluminal fluid, a solution for a subluminal fluid exists only above some minimal radius r_{min}, $0 < r_{min} < r_-$, and hence the inflowing fluid does not reach the central singularity (see Sec. 3). The energy density of the fluid has the maximum at r_{min}. For example, $r_{min} = 2(\sqrt{2} - 1)M$ and $\rho(r_{min}) = (8/3)^2 (12\sqrt{2} + 17)\rho_\infty$ in the case of accretion of a fluid with $\alpha = 1/3$ (thermal photon gas) onto the extremely charged black hole.

[2)] One can argue, however, that all these problems are due to the unphysical choice of the equation of state (18). We note that $\rho \to 0$ as $x \to 0$. The equation of state (18) is unphysical for $\alpha \ne 1$ at $\rho \to 0$, due to the pathological behavior of the equations of motion for ψ in the limit $\rho \to 0$, as it was shown in [10]. To cure the model in (18) with $\alpha \ne 1$ for small densities, one can modify the equation of state, such that $p \to \rho$ as $\rho \to 0$. For example, in terms of the scalar field, the Lagrangian

$$\mathcal{L} = (\sigma + X)^{3/4} - \sigma, \qquad (23)$$

with small σ satisfies this requirement, also giving a "superluminal" fluid with $p = 2\rho$ for large densities.

We note that similar behavior was found for geodesic motion of test particles with a nonzero mass [11, 12] in the RN metric. In particular, the radial component of the 4-velocity for parabolic radial geodesics (i. e., for particles with zero velocity at infinity) is

$$u_p(x) = \pm\frac{\sqrt{2x-e^2}}{x}. \tag{24}$$

The particle bounces at $r_{min} = Q^2/2M$ and $u_p(r_{min}) = 0$, but $|u_p'(r_{min})| = \infty$ according to (24).

The corresponding solutions for an accreting subluminal fluid are singular at $r = r_{min}$, namely, $u'(r_{min}) = \infty$ and $\rho'(r_{min}) = -\infty$ (although both the 4-velocity and the energy density are finite at $r = r_{min}$). As a result, continuity equation (3) is ill-defined at $r = r_{min}$. In what follows, we assume that (i) the fluid can have double-valued solutions, and hence inflow and outflow solutions can coexist at the same point of the manifold, and (ii) the fluid passes through the singularity in the solution at $r = r_{min}$. Formally, these assumptions imply that we can match solutions for the inflow and outflow at r_{min}, such that $\rho_{inflow}(x) = \rho_{outflow}(x)$ and $u_{inflow}(x) = -u_{outflow}(x)$. A physical interpretation is then as follows: the fluid accretes onto a black hole, then it bounces at r_{min} and flows outwards to the asymptotically flat internal spacetime. Because the inflow and the outflow are symmetric by construction, we present the results for the inflow only.

The resulting distribution for the energy density $\rho(x)$ for the thermal photon gas is shown in Fig. 2. In Fig. 3, the corresponding distributions for the radial component of the 4-velocitiy are shown. In Fig. 4, we plot the radial 3-velocity $v(x)$ with respect to local static observers. We note that $v(x)$ is equal to the speed of sound, $v(r_{min}) = c_s$, at the minimal radius r_{min} for a generic equation of state.

In Fig. 5, we depict a part of the Carter–Penrose diagram for the the RN metric [13, 14], containing an accreting fluid. This diagram is symmetric and time-reversible due to the stationarity of the process. We note that for "astrophysical" black holes formed by gravitational collapse of massive objects, the internal space–times are absent and the inflowing fluid can be expected to modify the metric inside the event horizon (see, e. g., [15–25] and the references herein).

In the Carter–Penrose diagram, the streamlines of the outflowing fluid intersect the inflowing ones in the region $r_{min} < r < r_-$ (note the intersecting dashed lines in Fig. 5). As we discussed before, we assume the inflow and outflow do not interact and they freely pass through each other (similarly to the motion of test particles). If the fluid is viscous, the picture should be

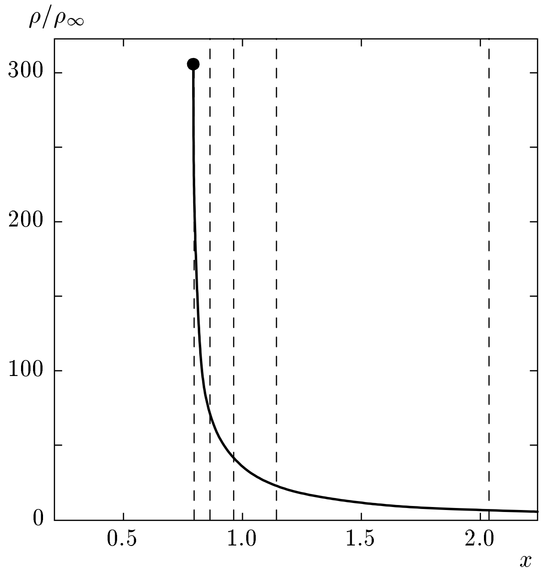

Fig. 2. Energy density $\rho(x)$ for the inflowing fluid with $\alpha = 1/3$ (thermal photon gas) in the RN metric with the charge $e = 0.99$. After reaching the bounce point (marked by the dot) at the minimal radius r_{min}, the fluid expands to the internal asymptotically flat universe. $x_*^+ = 2.04$, $x_+ = 1.14$, $x_*^- = 0.96$, $x_- = 0.86$, $x_{min} = 0.79$

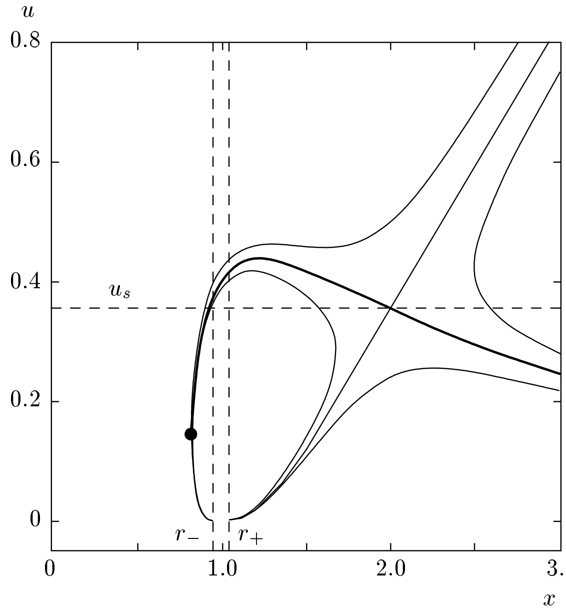

Fig. 3. Radial 4-velocity $u(r)$ (thick curve) for the inflowing fluid with $\alpha = 1/3$ (thermal photon gas) in the RN metric with the charge $e = 0.999$. Thin curves correspond to the unphysical hydrodynamical branches and u_s is the 4-velocity at the critical (sound) point

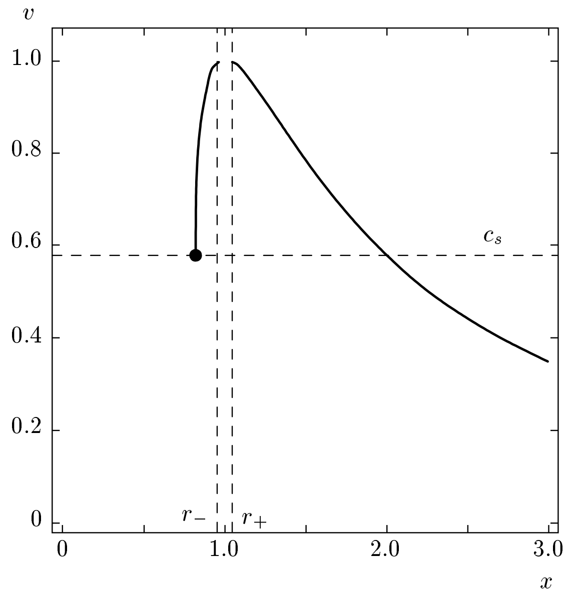

Fig. 4. Radial 3-velocity $v(x)$ for the inflowing fluid ($\alpha = 1/2$, $e = 0.999$) with respect to the local static observers in the R-regions $r_+ < r < \infty$ and $0 < r < r_-$. In the T-region $r_- < r < r_+$, the local static observers do not exist, and hence the 3-velocity is undefined

modified (at least for $r < r_-$, but not for $r > r_*$), because intersecting streamlines interact. The resulting flow may become time dependent, turbulent or/and accompanied by formation of shocks.

4.2. Chaplygin gas

Another analytically solvable example we consider here is the Chaplygin gas,

$$p = -\frac{\alpha}{\rho},$$

where a constant $\alpha > 0$ corresponds to a hydrodynamically stable fluid. The Chaplygin gas with $\rho^2 < \alpha$ represents phantom energy with a superluminal speed of sound. The opposite case, $\rho^2 > \alpha$, corresponds to dark energy with $\rho + p > 0$ and $0 < c_s^2 < 1$.

We find the following relations at the critical point:

$$f_* = \frac{\xi - 1}{\xi}, \qquad x_*^{\pm} = \xi \left[1 \pm \sqrt{1 - \frac{e^2}{\xi}} \right], \quad A = \frac{x_*^2}{\sqrt{\xi}}, \tag{25}$$

where $\xi = \rho_\infty^2/\alpha$. The sonic point exists and the accretion is transonic for $\xi \geq e^2$, i.e., when the square root in (25) is real. We note that for the nonphantom Chaplygin gas, this is always the case. On the other hand, in the phantom case, the critical point is absent for some range of parameters, implying that a physical solution does not exist. This, however, is merely a consequence of pathological behavior of the Chaplygin gas in the phantom regime. For the radial dependence of the energy density and the radial 4-velocity u, we find

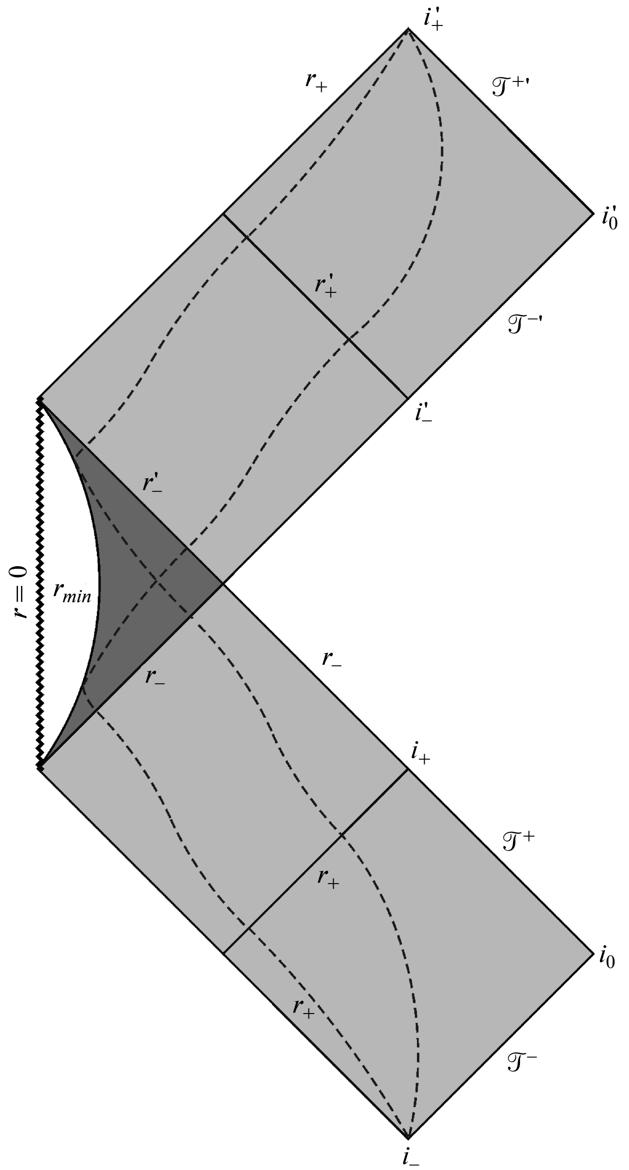

Fig. 5. Carter–Penrose diagram of the Reissner–Nordström metric containing a steady-state accreting fluid. The fluid streamlines are shown by dashed lines. The minimal radius r_{min} is a bounce point for inflowing fluid

$$u = -\frac{A}{x^2}\sqrt{\frac{\xi - 1}{\xi(\rho/\rho_\infty)^2 - 1}}, \tag{26}$$

$$\frac{\rho}{\rho_\infty} = \sqrt{\frac{f - A^2(\xi - 1)x^{-4}}{\xi(f-1)+1}}. \tag{27}$$

The value of the energy density at the event horizon is $\rho(r_+)/\rho_\infty = A/x_+^2$. In the special case $\xi = 1$, solution (26) corresponds to the vacuum state with $p = -\rho = -\rho_\infty$ and $u = 0$. The energy density of the nonphantom Chaplygin gas diverges at the inner critical point

$$x_{min} = x_*^- = \xi\left(1 - \sqrt{1 - e^2/\xi}\right).$$

5. SOLUTIONS FOR A NAKED SINGULARITY

As was discussed in Sec. 3, only "superluminal" fluids reach a naked singularity in steady-state accretion. More precisely, when formulated in terms of a scalar field, a solution well-behaved at $r > 0$ exists only if the Lagrangian satisfies the relation $d\mathcal{L}/dX \to \infty$ as $X \to 0$[3)]. In this case one can specify the second boundary condition for accretion at the singularity, $r = 0$.

In the case of a "subluminal" fluid, the critical solution for steady-state accretion exists not for all r but only for $r > r_{min}$. This is in fact similar to the case of an RN black hole, when a fluid bounces from the singularity, as was discussed in Sec. 4. The radial 4-velocity as a function of r is similar to that for the RN black hole, plotted in Fig. 3. But the 3-velocity does not have a gap with undefined values, in contrast to the case of the black hole. Thinking in terms of a superfluid, the solution for the critical flow can be interpreted as two physical solutions: the inflow and the outflow, matched at the point r_{min}. We note, however, that in the case of a black hole, the matching point r_{min} (where the solution becomes singular) is hidden by the horizon, while in the case of an RN naked singularity, the singular matching point is reachable by a static observer. It should be expected that an arbitrarily small viscosity of the fluid drastically changes the solution, because the inflowing and outflowing components of the fluid interact in the whole space–time. We can therefore conclude that for any realistic fluid, the steady-state accretion does not occur for the RN singularity.

[3)] As was discussed in Sec. 3, the condition $d\mathcal{L}/dX \to$ const as $X \to 0$ must hold for the fluid to be nonpathological. Therefore, strictly speaking, a "nonpathological" superluminal fluid does not reach a naked singularity either.

5.1. Static fluid atmosphere

It is interesting that in contrast to the black hole case, a static solution for a naked singularity can be constructed. Such a solution describes a static light atmosphere with zero influx. Indeed, from (6), assuming $u = 0$, we find a static distribution of a test perfect fluid around the RN naked singularity

$$\frac{\rho + p}{\rho_\infty + p(\rho_\infty)}\exp\left[-\int\limits_{\rho_\infty}^{\rho}\frac{d\rho'}{\rho' + p(\rho')}\right] = f^{-1/2}.$$

In the particular case of linear equation of state (18), we obtain

$$\rho(r) = \frac{\alpha\rho_0}{1+\alpha} + \left(\rho_\infty - \frac{\alpha\rho_0}{1+\alpha}\right)f^{-(1+\alpha)/2\alpha} \tag{28}$$

for a static atmosphere. The energy density of ordinary matter (with $\rho_0 = 0$ and $\alpha > 0$) approaches zero at the singularity, $\rho \propto x^{1+1/\alpha}$ as $x \to 0$. The phantom energy density is finite at $x = 0$, and hence the phantom fluid "overcomes" the naked singularity repulsiveness.

In the case $e^2 > 1$, setting $u = A = 0$ in Eq. (26), we find a static distribution of the Chaplygin gas around a naked singularity.

5.2. Static scalar field atmosphere

We note that the solutions for a static atmosphere of the fluid considered above corresponds to the following solution in terms of the scalar field,

$$\frac{\partial\phi}{\partial t} = \text{const}, \quad \frac{\partial\phi}{\partial r} = 0.$$

However, zero energy flux,

$$T_0^1 = -f\mathcal{L}_X\partial_0\phi\,\partial_1\phi = 0,$$

is also achieved by setting $\partial_0\phi = 0$. Then the equation of motion becomes

$$\frac{\partial}{\partial r}\left(r^2\mathcal{L}_X f\frac{\partial\phi}{\partial r}\right) = 0. \tag{29}$$

We restrict ourselves to the canonical scalar field, $\mathcal{L}(X) = X$. The respective solutions of (29) for an RN black hole and a naked singularity are

$$\phi(x) = \frac{\xi_1}{M(x_+ - x_-)}\ln\left|\frac{x - x_+}{x - x_-}\right| + \xi_2,$$

$$\phi(x) = \frac{\xi_1}{M\sqrt{e^2-1}}\text{arctg}\left[\frac{x-1}{\sqrt{e^2-1}}\right] + \xi_2, \tag{30}$$

where ξ_1 and ξ_2 are constants. We note that $\phi(1) = 0$ in (30) for any $e \neq 0$, but $\phi(0)$ is not necessarily zero. The energy density of the scalar field is $T_0^0 = \xi_1^2/2r^4 f$. In the case of an RN black hole, it diverges at the horizon, while for a naked singularity, the energy density is singular at $r = 0$. However this singularity is integrable and the mass of the scalar field atmosphere is finite inside any finite r.

6. APPROACH TO THE EXTREME STATE

A black hole can approach the extreme state by capturing particles with an electric charge and/or angular momentum, but an infinite time is required to reach the extreme state [7, 26, 27]. This is a manifestation of the third law of the black hole thermodynamics [7]. We note that during accretion of neutral phantom energy, the electric charge of the RN black hole is unchanged, $Q = \text{const}$, while the black hole mass decreases. As a result, the black hole approaches a near-extreme state because the ratio $e = Q/M(t)$ increases. In the test fluid approximation, the black hole reaches the extreme state in a finite time $t = t_{NS}$ defined by the relation $Q = M(t_{NS})$. Indeed, using (10), the time t_{NS} for a black hole with the initial mass $M = M(0)$ and the electric charge $Q = \text{const}$ can be calculated from the equation

$$\int_0^{t_{NS}} \dot{m}\, dt = Q - M(0). \tag{31}$$

If we neglect the cosmological evolution of ρ_∞, then in the particular case of phantom with the stiff equation of state ($c_s = 1$), it follows from (10), (19) and (31) that

$$t_{NS} = \frac{e_0^3 - 3e_0^2 + 2 - 2(1 - e_0^2)^{3/2}}{3e_0^4}\,\tau, \tag{32}$$

where $e_0 = Q/M(0)$ and $\tau = -\{4\pi[\rho_\infty + p(\rho_\infty)]M(0)\}^{-1}$ is the characteristic accretion time.

The finiteness of the time t_{NS} in (32) implies violation of the third law of the black hole thermodynamics in the considered test fluid approximation[4)].

We note that in deriving the above result, we assumed that the fluid does not back-react. But this assumption may not be valid for the near-extreme black holes/naked singularities. Indeed, in the case $\alpha \geq 1$, the energy density of the accreting fluid diverges at the horizon, as the black hole approaches the extreme state. This can be seen from (19), (21), and (20). Similarly, violation of the test fluid approximation occurs at the radius $r = M$ for the static atmosphere around a near-extreme naked singularity due to the divergence of the energy density, as can be verified from Eqs. (28). It is worth noting that in the case of a near-extreme Kerr–Newman naked singularity, the energy density diverges at $r = M$ for an atmosphere of a fluid [29].

Meanwhile, when $0 < \alpha < 1$, the energy density of the accreting fluid remains finite even for the extreme black hole. It can nevertheless be argued that the test fluid approximation is violated for the following reason. The test fluid approximation is valid if the back reaction of an accreting fluid is small. But for an almost extreme black hole, with $|m - e| \ll m$, the back reaction can be calculated from the perturbed Einstein equations,

$$\delta G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{33}$$

where $\delta G_{\mu\nu}$ is the deviation of the Einstein tensor due to the presence of the accreting fluid with the energy–momentum tensor $T_{\mu\nu}$. Even if the perturbation of the metric calculated from (33) is small, the presence of the fluid may have a drastic effect on the metric in the limit as $M \to Q$. Hence, the back reaction effects in the case of near-extreme black holes must be considered carefully, even if the accreting fluid has a small energy–momentum tensor. The back reaction of the accretion flow may prevent conversion of a black hole into a naked singularity[5)]. This question, however, is beyond the scope of this paper, and we leave it for future investigation.

7. CONCLUSION

We have studied the steady-state distribution of a test perfect fluid with a general equation of state, $p = p(\rho)$, and a scalar field in the Reissner–Nordström metric. Similarly to the case of steady-state accretion of a perfect fluid onto a Schwarzschild black hole, the corresponding solution for the accretion exists also in the case of the RN black hole. On the other hand, no steady-state accretion of a perfect fluid exists onto the RN naked singularity, unless the double-valued velocity, energy density, and the pressure of a fluid are introduced in order to describe the inflow and the outflow

4) Possibility for a black hole to be transformed into a naked singularity by phantom accretion was first discussed in [28].

5) The importance of back reaction was discussed in [30] in the context of absorption of scalar particles with a large angular momentum by a near-extreme black hole.

occurring in the same points of space–time. Instead of a steady-state accretion, a static atmosphere of the fluid is formed around a naked singularity. For both the black hole and the naked singularity, we found analytic solutions of the problem of the steady state configurations of perfect fluids with an arbitrary equation of state $p = p(\rho)$. As particular cases, we studied a fluid with the linear equation of state, $p = \alpha(\rho - \rho_0)$ and the Chaplygin gas, $p = \alpha/\rho$. We also found a static distribution of a scalar field around the RN naked singularity.

When the accreting fluid is phantom, $\rho + p < 0$, the mass of the RN black hole decreases. This result is in agreement with the previous findings [5, 31]. This poses a question of whether it is possible to convert an RN black hole into a naked singularity by accretion of phantom. Under the assumptions we made, such a conversion is possible, because the accreting phantom decreases the black hole mass, while the electric charge of the black hole remains the same. The conversion of an RN black hole into a naked singularity in the case of accretion of exotic matter with a negative energy density $\rho < 0$ was already studied in [25, 32]. It is interesting to verify the possibility of similar conversion in the case of a phantom fluid with a positive energy density $\rho > 0$ by taking back reaction into account, which, as we expect, plays an important role in the case of near-extreme states. We leave this question for future study.

Although the test fluid approximation seems to break down for the near-extreme state of the black hole/naked singularity, we stress that for the far-from-extreme state of a black hole (in particular, for the Schwarzschild solution), the parameters of the perfect fluid and the boundary condition at the infinity can be tuned such that the test fluid approximation describes the accretion process well.

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