ЖЭТФ, 2011, том 140, вып. 2 (8), стр. 387–400

MAGNETIC FIELD CORRELATIONS IN A RANDOM FLOW WITH STRONG STEADY SHEAR

I. V. Kolokolov, V. V. Lebedev*[*], G. A. Sizov

Landau Institute for Theoretical Physics, Russian Academy of Sciences
119334, Moscow, Russia

Moscow Institute of Physics and Technology
141700, Dolgoprudnyi, Moscow Region, Russia

Received October 28, 2011

We analyze the magnetic kinematic dynamo in a conducting fluid where a stationary shear flow is accompanied by relatively weak random velocity fluctuations. The diffusionless and diffusion regimes are described. The growth rates of the magnetic field moments are related to the statistical characteristics of the flow describing divergence of the Lagrangian trajectories. The magnetic field correlation functions are examined, and their growth rates and scaling behavior are established. General assertions are illustrated by the explicit solution of a model where the velocity field is short-correlated in time.

1. INTRODUCTION

The subject of the paper is the magnetic dynamo, that is, the magnetic field generation by hydrodynamic motions in a conducting medium. We theoretically investigate the effect in a conducting fluid (plasma, electrolyte) where a random hydrodynamic flow is excited. The principal example of such a flow is hydrodynamic turbulence (see, e.g., Refs. [1, 2]) responsible for the magnetic field generation in different geophysical and astrophysical phenomena [3–10]. We consider the case where the magnetic field grows from small initial fluctuations and examine the evolution stage of a sufficiently weak magnetic field, which allows neglecting the feedback from the magnetic field to the flow. The stage where the flow is independent of the magnetic field is called kinematic. The kinematic approach becomes invalid when the increasing magnetic field begins to affect the fluid motion essentially. In this case, the velocity field is strongly influenced by the Lorentz force, and hence the induction dynamics is no longer linear. In most cases, this leads to saturation of the magnetic field fluctuations maintained by the hydrodynamic flow. Although the magnetic field cannot be described by a linear equation in this regime, the kinematic stage produces magnetic structures similar to those occurring at the saturation state (see, e.g., Ref. [11]). A possible explanation of this fact is related to strong intermittency of the magnetic field, which implies that the feedback is concentrated in restricted space regions where the magnetic field is anomalously strong.

We assume that the random flow exciting the dynamo is statistically homogeneous in space and time. Usually, it is assumed in addition that the flow is statistically isotropic. If the velocity field is short-correlated in time, then it is possible to derive closed equations for the magnetic induction correlation functions [12]. The corresponding pair correlation function has been analyzed in Refs. [13, 14]. The complete statistical description of the magnetic field for a short-correlated smooth statistically isotropic flow was given in Ref. [15], where the growth rates and the structure of spatial correlation functions were found. However, it is interesting to consider random flows with an average shear flow, which are widespread in astrophysical applications. Such flows are statistically anisotropic and need a special analysis. Here, we examine the case where a steady shear flow is complemented by a relatively weak random component. We focus on the analysis of growth rates of moments of the magnetic field (magnetic induction), the degree of its anisotropy, and the structure of the magnetic field correlation functions. Our goal is to relate the magnetic statistical characteristics to those of the flow, thus revealing the most universal features of the dynamo effect. The general

[*]E-mail: lwlebede@gmail.com

assertions are illustrated by a model where the velocity field is short-correlated in time and can be solved analytically.

An additional motivation for our research comes from dynamics of polymer solutions that is in many respects similar to magnetohydrodynamics [16–18]. In particular, we have in mind the so-called coil–stretch transition [19] (see also Refs. [20, 21]) that is an analog of the dynamo effect in polymer solutions. A decade ago, the elastic turbulence was discovered [22–24], which is a chaotic hydrodynamic motion of polymer solutions that can be realized even at small Reynolds numbers, in contrast to the traditional hydrodynamic turbulence. Elastic turbulence is a natural framework for applying an extension of the dynamo theory to polymer solutions.

The behavior of the magnetic field moments at the kinematic stage in the presence of a strong shear flow was established in Ref. [25]. But to examine the spatial structure of the magnetic field, we must know its correlation functions, and these are studied in this paper in the framework of the general scheme used in Ref. [25]. To verify our general predictions, we examine the dynamo effect in the framework of an analytically solvable model where random flow a short-correlated in time is excited on the background of a strong stationary shear flow.

The structure of this paper is as follows. In Sec. 2, we introduce basic relations needed to analyze the magnetic field correlations and dynamics. We present the general dynamic equation, give its formally exact solution, and discuss statistical properties of the quantities entering this solution. In Sec. 3, moments and correlation functions of the magnetic field are investigated. We relate its growth rates to the growth rates of the separation between two close fluid particles and establish the principal spatial structure of the correlation functions. Section 4 is devoted to the model where the fluctuating component of the flow is short-correlated in time. We establish the growth rates for the model and analyze the pair correlation function in detail. The obtained results are in agreement with our general assertions. In Sec. 5, we outline our main results and discuss their possible applications and extensions.

2. BASIC RELATIONS

We consider the magnetic field evolution in a conducting fluid (plasma or electrolyte) where hydrodynamic motions are excited. Then the magnetic field dynamics is governed by the equation [26]

$$\partial_t \mathbf{B} = (\mathbf{B}\cdot\boldsymbol{\nabla})\boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{\nabla})\mathbf{B} + \kappa\nabla^2\mathbf{B}, \tag{2.1}$$

where $\mathbf{B}$ is the magnetic induction, $\boldsymbol{v}$ is the flow velocity, and κ is the magnetodiffusion coefficient, inversely proportional to the electrical conductivity of the medium. The flow is assumed to be incompressible, $\nabla\cdot\boldsymbol{v} = 0$. We also assume that the magnetodiffusion term in Eq. (2.1) is small in comparison with those related to the flow. We consider the case where the magnetic field is relatively weak and therefore its feedback to the flow is negligible. Then relation (2.1) is a linear equation determining the magnetic field evolution in a prescribed velocity field; this regime is called kinematic.

The hydrodynamic motion excited in the fluid is assumed to be random (turbulent) and the velocity statistics is assumed to be homogeneous in space and time. We examine the magnetic field growth from initial weak fluctuations distributed statistically homogeneously in space at the initial instant $t = 0$. The correlation length of the initial fluctuations l is assumed to be smaller than the velocity correlation length η. If we consider hydrodynamic turbulence, then the role of the velocity correlation length is played by the Kolmogorov scale. At scales less than η, the velocity field $\boldsymbol{v}$ can be considered smooth. The magnetic growth (dynamo) can be characterized by moments of the magnetic induction that exponentially increase with time t:

$$\left\langle |\mathbf{B}(t)|^{2n} \right\rangle \propto \exp(\gamma_n t). \tag{2.2}$$

Here, angular brackets denote averaging over space. Exponential laws (2.2) are characteristic of the kinematic dynamo because Eq. (2.1) is linear in the magnetic induction $\mathbf{B}$ in this case.

One of our goals is to express the growth rates γ_n in Eq. (2.2) via statistical characteristics of the flow. The natural measure for the growth rates γ_n is the so-called Lyapunov exponent of the flow, λ, equal to the average logarithmic divergence rate of close fluid particles. A special question concerns the n dependence of γ_n. If the magnetic induction statistics is Gaussian, then $\gamma_n \propto n$. Deviations from the linear law signal the intermittency of the magnetic field. The intermittency implies that high moments of the magnetic field are determined by rare strong fluctuations.

There are two different regimes of the kinematic magnetic field growth. The first regime is realized if all characteristic scales of the magnetic field are much larger than the magnetic diffusion length $r_d = \sqrt{\kappa/\lambda}$. The assumed smallness of the diffusion coefficient implies the inequality $\eta \gg r_d$. We also assume that $l \gg r_d$; then the diffusion term in Eq. (2.1) is negligible

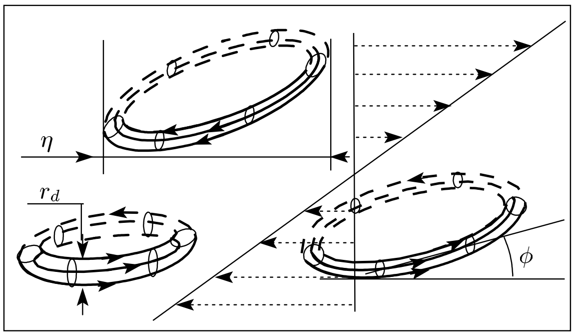

Fig. 1. Sketch of typical magnetic blobs during the diffusive kinematic stage

at the first stage of the magnetic evolution, which we call diffusionless. The magnetic field lines are deformed by the flow without reconnections in this regime. But distortions of the magnetic field by the flow inevitably lead to producing order-r_d scales in the field. After that, the magnetic diffusion is switched on and reconnections can occur. This second (diffusion) stage is characterized by the growth rates different from those describing the diffusionless regime.

We present a qualitative picture explaining the magnetic field evolution at the kinematic stage. The initial magnetic field distribution in space can be regarded as an ensemble of blobs of sizes of the order of l. Then the blobs are distorted by the flow, being stretched in one direction and compressed in another direction. In the isotropic case, the stretching and compression directions vary chaotically in space and time, but in our case, they are attached to the shear flow: the blobs are stretched mainly along the shear velocity and are compressed in the direction of the shear velocity gradient. At the first (diffusionless) stage, the blobs are deformed without intersections and the magnetic field induction grows as the separation between close fluid particles because Eq. (2.1) at $\kappa = 0$ coincides with the equation for the separations.

The diffusionless stage terminates when the characteristic blob width decreased to the diffusion length r_d. Then the diffusion is switched on, which leads to two effects. First, the diffusion prevents further shrinking the blob widths, which therefore remain of the order of r_d, whereas the blobs continue to be stretched in the direction of the shear velocity. Second, due to reconnections of the magnetic field lines allowed by diffusion, the blobs start to overlap. As a result, new blobs of a characteristic longitudinal size η are formed (Fig. 1). The magnetic induction in such blobs can be found by averaging the induction of a large number N of initial blobs, with the number N increasing exponentially with time. Averaging over a large number of random variables leads to the appearance of an exponentially small factor about $1/\sqrt{N}$ in the amplitude of the magnetic induction. In addition, the amplitudes of the initial blobs continue to increase with time as the separation between fluid particles. We conclude that at this second (diffusive) stage, the magnetic field is still increasing exponentially with time, but slower than at the first stage.

We consider the case where the steady shear constituent of the flow is much stronger than the random one. Quantitatively, the condition is written as the inequality $s \gg \lambda$, where s is the shear rate. Indeed, the Lyapunov exponent in a pure shear flow is zero, and its nonzero value is associated with the presence of a relatively weak random constituent of the flow. The distorted magnetic blobs are elongated mainly along the shear velocity. However, they are tilted with respect to the velocity direction due to presence of the random velocity component (see Fig. 1). The tilt exhibits the same dynamics as the direction of the polymer stretching in the same flow [21]. Therefore, the tilt angle ϕ (see Fig. 1) can be estimated as $\phi \sim \lambda/s$. The tilt angle determines the typical ratio of the magnetic field components $B_y/B_x \sim \lambda/s \ll 1$, where the x axis is directed along the shear velocity, which varies along the y axis. Thus, the ratio s/λ characterizes the anisotropy degree of the magnetic field.

2.1. Lagrangian dynamics

To analyze moments and correlation functions of the magnetic induction, we need a solution of magnetodynamic equation (2.1) for the induction field $\mathbf{B}(t)$ in terms of its initial value $\mathfrak{B}$, $\mathfrak{B} = \mathbf{B}(0)$. We here use a generalization of the scheme proposed in Ref. [27] and elaborated in Ref. [25], which uses the Lagrangian approach to fluid dynamics.

First, instead of solving Eq. (2.1) with the second-order Laplace operator, it is convenient to pass to the first-order equation

$$\partial_t \mathbf{B} = (\mathbf{B} \cdot \boldsymbol{\nabla})\boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{\nabla})\mathbf{B} + (\boldsymbol{\xi} \cdot \boldsymbol{\nabla})\mathbf{B}, \qquad (2.3)$$

where $\boldsymbol{\xi}(t)$ are white noises (Langevin forces) mimicking the magnetic diffusion. The means of the $\boldsymbol{\xi}$ are zero and their pair correlation function is

$$\lfloor \xi_i(t_1)\xi_j(t_2) \rfloor = 2\kappa\delta_{ij}\delta(t_1 - t_2), \qquad (2.4)$$

where "$\lfloor \dots \rfloor$" mean averaging over the $\boldsymbol{\xi}$ statistics and κ is the same diffusion coefficient as in Eq. (2.1). The solution of evolution equation (2.1) is given by the solution of Eq. (2.3) averaged over the $\boldsymbol{\xi}$ statistics for any velocity field $\boldsymbol{v}(t,\mathbf{r})$.

To prove the assertion, we find the increment of the magnetic field induction during a small time interval ϵ. A formal solution of Eq. (2.3) is

$$\mathbf{B}(t) = \mathrm{Texp}\left\{\int_{t-\epsilon}^{t} dt' \left[\hat{\Sigma} - \boldsymbol{v}\nabla + \boldsymbol{\xi}(t')\nabla\right]\right\}\mathbf{B}(t-\epsilon),$$

where Texp is the chronologically ordered exponential and $\hat{\Sigma}$ is the matrix of the velocity gradients, $\Sigma_{ji} = \partial_i v_j$. Expanding the exponent and averaging the result over the $\boldsymbol{\xi}$ statistics in accordance with Eq. (2.4), we find the increment

$$\mathbf{B}(t) - \mathbf{B}(t-\epsilon) = \epsilon(\mathbf{B}\cdot\boldsymbol{\nabla})\boldsymbol{v} - \epsilon(\boldsymbol{v}\cdot\boldsymbol{\nabla})\mathbf{B} + \epsilon\kappa\nabla^2\mathbf{B}$$

in the first order in ϵ. We note that cross terms are absent in this approximation because the averages of $\boldsymbol{\xi}$ are zero. The above increment is equivalent to the one obtained directly from Eq. (2.1).

Second, we solve Eq. (2.3) by the method of characteristics. The equation for the characteristic $\mathbf{R}$ is

$$\partial_t \mathbf{R} = \boldsymbol{v}(t,\mathbf{R}) + \boldsymbol{\xi}. \tag{2.5}$$

This equation describes a Lagrangian trajectory disturbed by Langevin forces. If the magnetic induction is written as $\mathbf{B}(t,\mathbf{r}) = \mathbf{b}(t,\mathbf{R})$, then the quantity $\mathbf{b}$ satisfies the equation $\partial_t b_j = \Sigma_{ji} b_i$, where $\hat{\Sigma}(t)$ is the velocity gradients matrix, $\Sigma_{ji} = \partial_i v_j$, taken at the time t and at the spatial point $\mathbf{R}(t)$. A solution of the equation can be written as $\mathbf{b}(t) = \hat{W}(t)\mathbf{b}(0)$, where the matrix $\hat{W}(t)$ is the chronologically ordered exponential

$$\hat{W}(t) = \mathrm{T}\exp\left\{\int_0^t dt'\,\hat{\Sigma}(t')\right\}. \tag{2.6}$$

The matrix $\hat{W}$, which we call the evolution matrix, can be treated as a solution of the equation $\partial_t \hat{W} = \hat{\Sigma}\hat{W}$ with the initial condition $\hat{W}(0) = 1$.

Finally, we find the formally exact solution of Eq. (2.1),

$$\mathbf{B}(t,\mathbf{r}) = \left\lfloor \hat{W}(t)\mathfrak{B}[\mathbf{R}(0)]\right\rfloor, \tag{2.7}$$

where, again, "$\lfloor\ldots\rfloor$" mean averaging over the $\boldsymbol{\xi}$ statistics determined by Eq. (2.4). To find $\mathbf{R}(0)$, we must solve Eq. (2.5) on the time interval $(0,t)$ with the boundary condition $\mathbf{R}(t) = \mathbf{r}$ posed at the final time. In other words, we should track the magnetic field back in time along the disturbed Lagrangian trajectories and include the factor $\hat{W}$ accumulated along the trajectory.

The evolution matrix $\hat{W}$ has some general properties that follow from definition (2.6). The determinant of $\hat{W}$ is equal to unity because the velocity gradient matrix $\hat{\Sigma}$ is traceless, $\mathrm{tr}\,\hat{\Sigma} = 0$, which is in turn a consequence of the incompressibility condition $\nabla\cdot\boldsymbol{v} = 0$. We introduce the symmetric matrix $\hat{W}\hat{W}^T$ (where the superscript "T" denotes transposition) and denote its eigenvalues as W_1^2, W_2^2, W_3^2, where all the quantities W_1, W_2, W_3 are positive. Because the determinant of $\hat{W}$ is equal to unity, we have $W_1W_2W_3 = 1$. We order the eigenvalues as $W_1 > W_2 > W_3$, then $W_1 > 1$ and $W_3 < 1$. At times $t \gg \lambda^{-1}$ we are interested in, typical values of $\ln W_1$ and $\ln W_3$ can be estimated as $\pm\lambda t$, and therefore W_1 is exponentially large and W_3 is exponentially small. The estimation for W_2 depends on the details of the flow statistics. In any case, $W_1 \gg W_2 \gg W_3$ at times $t \gg \lambda^{-1}$.

In the framework of the proposed formalism, correlation functions of the magnetic field $\mathbf{B}$ are to be calculated by averaging products of factors (2.7) taken at the respective points over the statistics of the noise $\boldsymbol{\xi}$, in addition to averaging over space. Thus, say, the one-time correlation function

$$F_{2n,i\ldots j}(\mathbf{r}_1,\ldots,\mathbf{r}_{2n}) = \langle B_i(\mathbf{r}_1)\ldots B_j(\mathbf{r}_{2n})\rangle, \tag{2.8}$$

has to be calculated in two steps. First, we substitute expression (2.7) in the right-hand side of (2.8) and then average the resulting product over the $\boldsymbol{\xi}$ statistics determined by Eq. (2.4); this averaging catches the magnetic diffusion. We emphasize that the fields $\boldsymbol{\xi}$ have to be treated as independent for all the $2n$ factors in the product. Second, we average the result over space. Averaging over scales less than or of the order of η (traced back to the initial time) gives statistics of the initial magnetic field fluctuations, and averaging over scales more than or of the order of η counts different realizations of $\hat{\Sigma}$. Therefore, the latter is equivalent to averaging over the velocity statistics. This logic was realized for the isotropic random flow in Ref. [15].

In the diffusionless regime, realized at $t \ll \lambda^{-1}\ln(l/r_d)$, we can neglect the diffusion effects. Then in calculating the moment $\langle|\mathbf{B}|^{2n}\rangle$, we can take the product of identical factors (2.7), where $\mathbf{R}$ is simply a Lagrangian trajectory terminated at the point $\mathbf{r}$ at time t. Then

$$|\mathbf{B}(\mathbf{r})|^{2n} \approx W_1^{2n}|\mathfrak{B}|^{2n},$$

where $\mathfrak{B}$ is taken at the origin of the Lagrangian trajectory. Here, just the factor W_1^{2n} is responsible for the exponential growth of the moments, and therefore we can restrict ourselves to the estimation

$$\lfloor |\mathbf{B}(\mathbf{r})|^{2n} \rfloor \sim W_1^{2n} \mathfrak{B}_0^{2n},$$

where $\mathfrak{B}_0$ is the characteristic value of the initial magnetic field fluctuations. In the diffusion regime, realized at $t \gg \lambda^{-1} \ln(l/r_d)$, the situation is somewhat more complicated.

We first consider the second moment. Then we deal with two trajectories, $\mathbf{R}$ and $\mathbf{R}'$, terminating at the same point $\mathbf{r}$ at time t, but characterized by independent noises $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$. The second moment can be written as the average,

$$\langle (\mathbf{B})^2 \rangle = \left\langle \left\lfloor \mathfrak{B}^T[\mathbf{R}(0)] \hat{W}^T \hat{W}' \mathfrak{B}[\mathbf{R}'(0)] \right\rfloor \right\rangle. \quad (2.9)$$

An appreciable contribution to the second moment is associated with the trajectories with $|\mathbf{R}(0) - \mathbf{R}'(0)| \lesssim l$. Because $|\mathbf{R}(0) - \mathbf{R}'(0)| \ll \eta$ and $|\mathbf{R}(t) - \mathbf{R}'(t)| = 0$, the difference $\Delta\mathbf{R} = \mathbf{R} - \mathbf{R}'$ remains much less than η at any time from the interval $(0, t)$ for such an event. From Eq. (2.5), expanding the velocity up to terms linear in $\Delta\mathbf{R}$, we then obtain

$$\partial_t \Delta\mathbf{R} = \hat{\Sigma} \Delta\mathbf{R} + \boldsymbol{\xi} - \boldsymbol{\xi}', \quad (2.10)$$

where $\hat{\Sigma}$ can be taken at any of the points $\mathbf{R}$ or $\mathbf{R}'$. The solution of Eq. (2.10) that is equal to zero at $t' = t$ is written as

$$\Delta\mathbf{R}(t') = \\ = -\hat{W}(t') \int_{t'}^{t} dt_1 \hat{W}^{-1}(t_1) [\boldsymbol{\xi}(t_1) - \boldsymbol{\xi}'(t_1)]. \quad (2.11)$$

To calculate the second moment, we should know the $\Delta\mathbf{R}(0)$ statistics. Because the separation $\Delta\mathbf{R}(0)$ is a linear combination of $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$, it should be treated as a Gaussian variable in averaging over the $\boldsymbol{\xi}$ statistics, and then its probability distribution function is completely determined by the matrix $\hat{M}$:

$$M_{ij} = \lfloor \Delta R_i(0) \Delta R_j(0) \rfloor = 4\kappa \int_0^t dt_1 \, W_{ik}^{-1} W_{jk}^{-1}. \quad (2.12)$$

The expression for $\hat{M}$ is derived from Eqs. (2.4) and (2.11). The matrix $\hat{M}$ is symmetric, and its eigenvalues are positive. We let the eigenvalues be denoted by m_1^2, m_2^2, m_3^2 and order the m as $m_1 > m_2 > m_3$; the inequalities become strong, $m_1 \gg m_2 \gg m_3$, if $\lambda t \gg 1$. We emphasize that the directions of the eigenvectors of $\hat{M}$ are "frozen" at $\lambda t \gg 1$ [28–31]. Then the integral determining m_1 is dominated by $t - t_1 \sim \lambda^{-1}$, and we arrive at the estimation $m_1 \sim r_d W_3^{-1}$. The integral determining m_3 is dominated by $t_1 \sim \lambda^{-1}$, and therefore $m_3 \sim r_d$. An estimation for m_2 depends on the time dependence of W_2. If W_2 increases, then m_2 remains of the order of r_d, but it grows like $m_2 \sim r_d W_2^{-1}$ if W_2 decreases.

We now find the probability that $\Delta\mathbf{R}(0)$ is less than l in the diffusion regime, when $t \gg \lambda^{-1} \ln(l/r_d)$. We can think in terms of the components of $\Delta\mathbf{R}(0)$ in the basis attached to the eigenvectors of $\hat{M}$. Because $m_1 \gg l$, the probability that the first component of $\Delta\mathbf{R}(0)$ is less than l is estimated as $l/m_1 \sim (l/r_d) W_3$. If W_2 increases with time, then both m_2 and m_3 are of the order of r_d and therefore the probability that the second and the third components of $\Delta\mathbf{R}(0)$ are less than l is close to unity. From Eq. (2.9), we then find

$$\lfloor |\mathbf{B}|^2 \rfloor \sim \mathfrak{B}_0^2 (l/r_d) W_1 W_2^{-1}, \quad (2.13)$$

where we used the relation $W_1 W_2 W_3 = 1$.

The situation with a decreasing W_2 is slightly different. In this case, $m_2 \gg l$ at the diffusive stage, and there appears an additional small probability that the second component $\Delta\mathbf{R}(0)$ is less than l. This probability can be estimated as $l/m_2 \sim (l/r_d) W_2$. We then obtain $\langle |\mathbf{B}|^2 \rangle \sim \mathfrak{B}_0^2 W_1 (l/r_d)^2$ instead of Eq. (2.13). But the integration over space (at the next step of averaging) kills the leading term due to the solenoidal nature of the magnetic field $\mathbf{B}$. Therefore, we have to take the next term in the probability distribution of $\Delta R_2(0)$ into account, which contributes an extra small factor $(l/m_2)^2$ to the probability. Thus, we arrive at

$$\lfloor |\mathbf{B}|^2 \rfloor \sim \mathfrak{B}_0^2 (l/r_d)^4 W_1 W_2^2. \quad (2.14)$$

We note that expressions (2.13) and (2.14) are equivalent to those obtained in the Fourier representation for the statistically isotropic case in Ref. [15]. But expressions (2.13) and (2.14), written for real space, are also correct for the anisotropic problem (which we are investigating), and are in fact more suitable for the problem.

We turn to higher moments. It can be seen that the principal contribution to the average $\lfloor |\mathbf{B}|^{2n} \rfloor$ is produced by configurations where the $2n$ points $\mathbf{R}_\alpha(0)$ are divided into n pairs with separations less than or of the order of l in each pair. Because of the independence of the white noises $\boldsymbol{\xi}_\alpha$, the probability of this event can be estimated as the product of probabilities for the second moment, that is,

$$\lfloor |\mathbf{B}|^{2n} \rfloor \sim \lfloor |\mathbf{B}|^2 \rfloor^n, \quad (2.15)$$

where the second moment is given by Eq. (2.13) or Eq. (2.14). We have ignored a combinatorial factor in Eq. (2.15) because we are interested in the time dependence of the moments.

2.2. Evolution matrix

The next step in finding the magnetic field moments is averaging over the velocity statistics. Before doing this, we should establish statistical properties of evolution matrix (2.6). Some universal properties of such matrices, which can be treated as products of a large number of random matrices, are well established [28–31]; the properties are revealed at $t \gg \lambda^{-1}$. But we examine a strongly anisotropic case, with the steady shear flow dominating. This requires modifying the consideration in Ref. [15], where the isotropic case (statistically isotropic flow) was investigated.

For the anisotropic problem, it is convenient to use the Gaussian decomposition of the evolution matrix $\hat{W} = \hat{T}_L \hat{\Delta} \hat{T}_R$, where $\hat{T}_L$ and $\hat{T}_R$ are the triangle matrices

$$\hat{T}_L = \begin{pmatrix} 1 & \chi & \chi_1 \\ 0 & 1 & \chi_2 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \hat{T}_R = \begin{pmatrix} 1 & 0 & 0 \\ \zeta_1 & 1 & 0 \\ \zeta_2 & \zeta_3 & 1 \end{pmatrix} \tag{2.16}$$

and $\hat{\Delta}$ is a diagonal matrix. Because both triangle matrices, $\hat{T}_L$ and $\hat{T}_R$, have unit determinants, the determinant of $\hat{\Delta}$ is also equal to unity.

The matrices are written in the reference frame attached to the shear flow: the axis x is directed along the shear velocity and the axis y is directed along the shear velocity gradient. Therefore, the shear velocity is written as $v_x = sy$, where s is the shear rate. For our flow, which is composed of a steady shear flow and a random component, the matrix of the velocity gradients $\Sigma_{ji} = \partial_i v_j$ is a sum of two terms related to the shear and the random components of the flow:

$$\Sigma_{ji}(t) = s\delta_{jx}\delta_{iy} + \sigma_{ji}(t). \tag{2.17}$$

The random matrix σ_{ji} is zero on average and should be characterized in terms of its correlation functions. The trace of the matrix is zero, $\operatorname{tr}\hat{\sigma} = 0$ (due to the flow incompressibility). We recall that the Lyapunov exponent λ of a purely shear flow is equal to zero. Therefore λ is sensitive to $\hat{\sigma}$, although the random flow is weaker than the steady one.

Substituting the decomposition $\hat{W} = \hat{T}_L \hat{\Delta} \hat{T}_R$ in the evolution equation $\partial_t \hat{W} = \hat{\Sigma}\hat{W}$, we find

$$\hat{T}_L^{-1}\hat{\Sigma}\hat{T}_L = \hat{T}_L^{-1}\partial_t \hat{T}_L + \partial_t\hat{\Delta}\hat{\Delta}^{-1} + \\ + \hat{\Delta}\partial_t\hat{T}_R\hat{T}_R^{-1}\hat{\Delta}^{-1}. \tag{2.18}$$

The respective terms in the right-hand side of Eq. (2.18) are a left off-diagonal matrix, a diagonal matrix, and a right off-diagonal matrix. Therefore, we obtain a closed (nonlinear) equation for the matrix $\hat{T}_L$ that leads to a homogeneous-in-time statistics of the matrix. Next, for components of the diagonal matrix $\partial_t\hat{\Delta}\hat{\Delta}^{-1}$, we obtain expressions that are random variables with statistics homogeneous in time. Therefore, the central limit theorem applies to $\ln\Delta_1$, $\ln\Delta_2$, and $\ln\Delta_3$ (where Δ_i are eigenvalues of $\hat{\Delta}$). Typically, the variables are linear in time t with the coefficients of the order of λ. The situation with the matrix $\hat{T}_R$ is slightly more complicated because of the exponential factors in the last term in Eq. (2.18). Therefore, some components of $\hat{T}_R$ behave exponentially with time like the factors.

From Eq. (2.18) for $\hat{T}_L$, based on the leading role of the shear term in expression (2.17), we obtain a hierarchy $\chi \gg \chi_1 \gg \chi_2$. Therefore, in the leading approximation in λ/s, the only component σ_{yx} is relevant and the equation for the matrix $\hat{T}_L$ is reduced to a single equation for the component χ,

$$\partial_t \chi = s - \chi^2 \sigma, \tag{2.19}$$

where $\sigma \equiv \sigma_{yx}$. We conclude that the variable χ has a statistics homogeneous in time, in accordance with our general expectations. We note that $\chi \sim s/\gamma \gg 1$, as follows from Eq. (2.19). Keeping the leading-in-χ contributions to the diagonal terms in Eq. (2.18), we obtain $\operatorname{diag}(\partial_t\hat{\Delta}\hat{\Delta}^{-1}) = (-\chi\sigma, \chi\sigma, 0)$. Therefore, in this approximation,

$$\operatorname{diag}\Delta = (e^{-\rho}, e^{\rho}, 1), \quad \partial_t\rho = \chi\sigma. \tag{2.20}$$

If $t \gg \lambda^{-1}$, then typically $\rho \sim \lambda t \gg 1$.

We conclude from the equations for ζ_1, ζ_2, and ζ_3 following from Eq. (2.18) that at $\lambda t \gg 1$, the variable ζ_1 is "frozen" at an order-of-unity level, whereas the variables ζ_2 and ζ_3 increase exponentially and can be estimated as e^{ρ}. However, the combination $\zeta_1\zeta_3 - \zeta_2$, entering $\hat{T}_R^{-1}$, is "frozen" at an order-of-unity level as well as ζ_1.

Based on the results obtained for the matrices $\hat{T}_L$, $\hat{\Delta}$, and $\hat{T}_R$ we find eigenvalues of the matrix $\hat{W}$. In the leading approximation in λ/s, we obtain $W_1 \sim e^{\rho}$,

$W_2 \sim 1$, and $W_3 \sim e^{-\rho}$. These expressions, together with Eq. (2.20), lead to the relation

$$\lambda \equiv \langle \partial_t \rho \rangle = \langle \chi \sigma \rangle, \tag{2.21}$$

where averaging is performed over the velocity statistics.

Because ρ is given by an integral over time of a random quantity whose statistics is homogeneous in time (see Eqs. (2.19) and (2.20)), the ρ statistics has some universal features at $\lambda t \gg 1$. Namely, the probability distribution function (PDF) of ρ can be written in a self-similar form [32]

$$P(\rho) \propto \exp[-tS(\rho/t)], \tag{2.22}$$

where S is the so-called Kramer function (or entropy function). Expression (2.22) is a manifestation of PDFs for the so-called intensive variables (see, e.g., [33]). Expression (2.22) implies that relative fluctuations of ρ decrease as t increases.

We consider moments of the divergence of close Lagrangian trajectories in our random flow. The equation governing the separation $\Delta\mathbf{R}$ between the trajectories is $\partial_t \Delta R_j = \Sigma_{ji} \Delta R_i$; it can be obtained from Eq. (2.10) by letting $\boldsymbol{\xi} \to 0$. A solution of the equation is $\Delta\mathbf{R}(t) = \hat{W} \Delta\mathbf{R}(0)$. Therefore, at $t \gg \lambda^{-1}$, we arrive at the estimation $\Delta R(t) \sim \Delta R(0) e^{\rho}$. Then the moments of $\Delta\mathbf{R}$ can be calculated in the saddle-point approximation (justified by the inequality $\lambda t \gg 1$):

$$\langle |\Delta\mathbf{R}|^n \rangle = \int d\rho\, P(\rho) |\Delta\mathbf{R}|^n \propto \exp(\lambda_n t), \tag{2.23}$$

$$\lambda_n = -S(\psi_n) + n\psi_n, \quad \text{where} \quad S'(\psi_n) = n. \tag{2.24}$$

Thus, the exponents λ_n are determined by statistical properties of the Lagrangian trajectories. We note that the Lyapunov exponent λ can be formally expressed via λ_n as $\lambda = (d\lambda_n/dn)_{n=0}$.

General statistical properties of the separation $\Delta\mathbf{R}$ for the random flow with strong average shear were established in Ref. [21], in the context of the single-polymer dynamics in such a flow. A strong intermittency of $\Delta R(t)$ is expected, which is revealed in a large-n growth of λ_n that is faster than linear, because the linear law $\lambda_n \propto n$ is characteristic of the Gaussian statistics of $\Delta R(t)$.

3. CORRELATION FUNCTIONS

To find the time dependence of the magnetic field moments, we have to additionally average expression (2.15) over space, which is equivalent to averaging over the ρ statistics and the statistics of initial magnetic fluctuations. The $2n$th moment of the magnetic field induction is then written as

$$\left\langle \mathbf{B}^{2n}(t) \right\rangle = \int d\rho\, P(\rho) \left\lfloor \mathbf{B}^{2n}(t) \right\rfloor. \tag{3.1}$$

In our approximation, $W_1 \sim e^{\rho}$ and $W_2 \sim 1$, and therefore $B(t) \sim e^{\rho}\mathfrak{B}_0$ in the diffusionless regime, whereas $\lfloor |\mathbf{B}|^2 \rfloor \sim \mathfrak{B}_0^2 (l/r_d) e^{\rho}$ in the diffusion regime, as follows from Eq. (2.13). Substituting the expressions in Eq. (3.1) and integrating over ρ (in the saddle-point approximation), we find $\gamma_n = \lambda_{2n}$ for the diffusionless regime and $\gamma_n = \lambda_n$ for the diffusion regime. Thus, we have related the dynamo growth rates introduced in Eq. (2.2) to the statistical properties of the flow. Our results can be summarized in terms of the estimations

$$\langle |\mathbf{B}(t)|^{2n} \rangle \sim \\ \sim \begin{cases} \exp(\lambda_{2n} t) \mathfrak{B}_0^{2n}, & t < \lambda^{-1} \ln \dfrac{l}{r_d}, \\ \left(\dfrac{l}{r_d}\right)^n \exp(\lambda_n t) \mathfrak{B}_0^{2n}, & t > \lambda^{-1} \ln \dfrac{l}{r_d}. \end{cases} \tag{3.2}$$

The main contribution to the moments $\left\langle \mathbf{B}^{2n}(t) \right\rangle$ is associated with the component B_x of the magnetic induction directed along the velocity of the shear flow (see Fig. 1). We turn to moments of the component B_y directed along the gradient of the shear flow, $\langle B_y^{2n} \rangle$. The moments are much smaller than the moments $\left\langle \mathbf{B}^{2n}(t) \right\rangle$, the smallness being caused by the strong shear flow. It follows from Eqs. (2.16) and (2.20) that $\lfloor B_x^2(t) \rfloor = \chi^2 \lfloor B_y^2(t) \rfloor$. Hence, the variable χ is a measure of the magnetic field anisotropy, χ^{-1} determines the tilt angle ϕ of the magnetic blobs to the shear velocity (see Fig. 1). Because the variable χ has a statistics homogeneous in time, the factor χ^{-2} does not produce a difference in the growth rates, and hence both moments $\langle B_x^{2n} \rangle$ and $\langle B_y^{2n} \rangle$ are proportional to the same exponential $\exp(\gamma_n t)$. But the prefactors at the exponentials are different. To find the difference in the prefactors, is not enough to know statistical properties of ρ that determine the exponentials. Generally, the mutual probability distribution of $\rho(t)$ and $\chi(t)$ must be known, which is quite a complicated object depending on the details of the flow dynamics. However, we can establish an estimation for typical fluctuations $\chi \sim s/\lambda$ that follows from Eqs. (2.19) and (2.21). Therefore, e.g., $\lfloor B_x^2(t) \rfloor \sim (s^2/\lambda^2) \lfloor B_y^2(t) \rfloor$.

There is a question concerning moments of the third component of the magnetic induction, $\langle B_z^{2n} \rangle$. Analyzing their behavior requires taking the components of the matrix $\hat{T}_L$ into account, which we ignored in investigating B_x and B_y. We then conclude that the time dependence of $\langle B_z^{2n}\rangle$ is characterized by the same exponentials $\exp(\gamma_n t)$ both at the diffusionless and diffusion stages. As regards the prefactors, they depend on the details of the flow statistics.

3.1. Pair correlations

We consider the one-time magnetic field pair correlation function

$$F_{ij}(t,\mathbf{r}) = \langle B_i(t,\mathbf{r}_1+\mathbf{r})B_j(t,\mathbf{r}_1)\rangle. \quad (3.3)$$

Here, as previously, angular brackets mean averaging over space (that is, integration over $\mathbf{r}_1$ with the inverse volume as a factor). We assume statistical homogeneity in space of both the velocity and the initial magnetic field fluctuations; that is why spatial average (3.3) characterizes the magnetic field correlations in the whole volume. We consider the case $r \ll \eta$, which allows using the smooth flow approximation.

Again, we start from representation (2.7). Then, analogously to the second moment, pair correlation function (3.3) can be written as

$$F_{ij}(t) = \left\langle \lfloor W_{ik}\mathfrak{B}_k[\mathbf{R}(0)]W'_{jl}\mathfrak{B}_l[\mathbf{R}'(0)]\rfloor \right\rangle,$$

where the trajectories $\mathbf{R}$ and $\mathbf{R}'$ terminate at the respective points $\mathbf{r}_1+\mathbf{r}$ and $\mathbf{r}_1$, at a time t. We then obtain

$$F_{ij}(t) = \langle \lfloor W_{ik}(t)W_{jl}(t)\mathcal{F}_{kl}[\Delta\mathbf{R}(0)]\rfloor\rangle, \quad (3.4)$$

where $\mathcal{F}_{ij}$ is the initial (at $t = 0$) pair correlation function of the magnetic field fluctuations and $\Delta\mathbf{R} = \mathbf{R}-\mathbf{R}'$. The correlation length l of $\mathcal{F}$ is smaller than η, and we can therefore consider $|\Delta\mathbf{R}| < \eta$. Then both evolution matrices in (3.4) can be taken at the same point $\mathbf{R}$. Averaging in Eq. (3.4) can be treated as averaging over the velocity statistics.

The difference $\Delta\mathbf{R}$ satisfies the same equation (2.10) if $|\Delta\mathbf{R}| \ll \eta$. However, we are now interested in the solution with the final condition $\Delta\mathbf{R} = \mathbf{r}$. This solution is written as

$$\Delta\mathbf{R}(t') = \hat{W}(t')\hat{W}^{-1}(t)\mathbf{r} - \hat{W}(t')\int_{t'}^{t} dt_1 \hat{W}^{-1}(t_1)[\boldsymbol{\xi}(t_1)-\boldsymbol{\xi}'(t_1)] \quad (3.5)$$

instead of Eq. (2.11). We immediately conclude from Eq. (3.5) that the pair correlation function coincides with the second moment if $r \lesssim r_d$. In what follows, we therefore examine the case $r \gg r_d$, where the second term in Eq. (3.5) is negligible and we find

$$\Delta\mathbf{R}(0) = \hat{W}^{-1}(t)\mathbf{r}. \quad (3.6)$$

To be more precise, expression (3.6) is correct if $y \sim r \gg r_d$, that is understood below.

There are two different regimes for the pair correlation function. If $t < \lambda^{-1}\ln(l/r)$, then $|\Delta\mathbf{R}(0)|$ is typically less than l; this regime exists if $r \ll l$. In this case, two Lagrangian trajectories $\mathbf{R}$ and $\mathbf{R}'$ remains typically within the correlation radius l at $t = 0$ and the behavior of expression (3.4) is insensitive to the separation r. Therefore, the pair correlation function F_{ij} virtually coincides with the single-point average $\langle B_iB_j\rangle$ in this regime and, consequently, its time dependence is determined by the growth rate $\gamma = \lambda_2$.

If $t > \lambda^{-1}\ln(l/r)$, then $|\Delta\mathbf{R}(0)|$ is typically larger than l and only rare events where $|\Delta\mathbf{R}(0)| < l$ contribute to the correlation function. Using the representation $\hat{W} = \hat{T}_L\hat{\Delta}\hat{T}_R$, we obtain from Eq. (3.6) that $|\Delta\mathbf{R}(0)| \approx e^{\rho}(r_x - \chi r_y)$, where r_x and r_y are coordinates of the separation $\mathbf{r}$. The probability that the quantity is less than or of the order of l is estimated as $e^{-\rho}l/r$ (if $r_x \sim r_y \sim r$), which is an interval of values of χ where $\Delta R(0) < l$. Therefore, $B_iB_j \sim \mathfrak{B}_0^2 e^{\rho}l/r$ and, consequently, $F(t) \sim \mathfrak{B}_0^2\exp(\lambda_1 t)l/r$.

We collect the obtained results:

$$F(t) \sim \begin{cases} \mathfrak{B}_0^2\exp(\lambda_2 t), & t < \dfrac{1}{\lambda}\ln\dfrac{l}{r}, \\ \mathfrak{B}_0^2\exp(\lambda_1 t)\dfrac{l}{r}, & t > \dfrac{1}{\lambda}\ln\dfrac{l}{r}, \end{cases} \quad (3.7)$$

where the inequality $r_d \ll r \ll l$ is assumed. Therefore, the pair correlation function is governed by the same exponentials as the second moment. In addition, we find the r-dependence of the pair correlation function. We note that expression (3.7) turns into expression (3.2) for the second moment at $r \sim r_d$, as it should.

Returning to expression (3.4), we conclude that a difference between the pair correlation function F_{ij} and the moments $\langle B_iB_j\rangle$ is solely in the behavior of $\Delta\mathbf{R}$. Therefore, relations between the components of F_{ij} controlled by the evolution matrices in Eq. (3.4) are the same as for the moments $\langle B_iB_j\rangle$, e.g., $F_{yy} \sim (\lambda/s)^2F_{xx}$.

3.2. Mellin transform

It is instructive to examine the Mellin transform of the pair correlation function. This analysis reveals its scaling properties. We define the Mellin transform as

$$\tilde{F}(t,k) = \int_0^\infty \frac{dr}{r}\left(\frac{r}{l}\right)^{-ik} F(t,r), \qquad (3.8)$$

where the direction of the radius vector $\mathbf{r}$ is assumed to be fixed. Because the velocity field is smooth, different harmonics $\tilde{F}(t,k)$ evolve independently, being represented as a sum of exponentials characterizing different structures of $\tilde{F}_{ij}$. At times $t \gg \lambda^{-1}$, only the leading exponential survives, that is, $\tilde{F}(k,\varphi) \propto \exp[\gamma(k)t]$.

To return to the real space, we should perform the inverse Mellin transform

$$F(t,r) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp\left[-ik\ln\frac{l}{r} + \gamma(k)t\right]\tilde{\mathcal{F}}(k). \qquad (3.9)$$

The quantity $\tilde{\mathcal{F}}(k)$ is determined by the initial magnetic field fluctuations, correlated on the scale l. That is why we incorporated this quantity into relations (3.8) and (3.9).

A remark about analytic properties of $\tilde{\mathcal{F}}(k)$ is in order. We assume that at $r > l$, the initial pair correlation function $\mathcal{F}(r)$ rapidly decreases as r increases. Integral (3.8) then, converges if $\mathrm{Im}\, k > 0$. Therefore, $\tilde{\mathcal{F}}(k)$ is analytic in the upper k-halfplane. Besides, the integral diverges (at small r) as $k \to 0$. Therefore, singularities of $\tilde{\mathcal{F}}(k)$ lie in the lower k-halfplane, starting from the point $k = 0$. The character of the singularities depends on analytic properties of the initial function $\mathcal{F}(r)$. If it is analytic in r, then we expect $\tilde{\mathcal{F}}(k)$ to have a series of poles along the lower imaginary semiaxis, with the first one at $k = 0$. We note that in accordance with general rules, the integration contour in Eq. (3.9) should run above the first singular point $k = 0$.

We can draw some general conclusions taking into account that $\gamma(k) \sim \lambda$. If $\ln(l/r) > \lambda t$, then integral (3.9) is determined by a narrow vicinity of the point $k = 0$. Then $F(t,r) \propto \exp[\gamma(0)t]$, and we identify $\gamma(0)$ and λ_2. If $\ln(l/r) < \lambda t$, then integral (3.9) can be calculated in the saddle-point approximation. To find the saddle point, we should shift the integration contour into the upper halfplane to reach the saddle point $k = iq_\star$, where $q_\star$ determines the minimal value of $\gamma(iq)$ for $q > 0$. Indeed, the growth rate $\gamma(iq)$ is real, and therefore the point $k = iq_\star$ is a solution of the equation $d\gamma/dk = 0$ giving an extremum of the exponential in Eq. (3.9). Then

$$F(t,r) \propto (l/r)^{q_\star} \exp[\gamma(iq_\star)t],$$

and we identify $\gamma(iq_\star)$ with λ_1 (see Eq. (3.7)). We note that in accordance with asymptotic law (3.7), $q_\star$ should be equal to unity, $q_\star = 1$.

3.3. Higher-order correlations

Here, we consider higher-order correlation functions of magnetic field (2.8). We obtain expressions like Eq. (3.5) for separations $\Delta\mathbf{R}$ between the points $\mathbf{R}_1(0), \ldots, \mathbf{R}_{2n}(0)$, which are needed to calculate F_{2n} in accordance with Eq. (2.7). If $\lambda t < \ln(l/|\Delta\mathbf{r}|)$ for all separations between the points $\mathbf{r}_1, \ldots, \mathbf{r}_{2n}$, then all separations $\Delta\mathbf{R}(0)$ are less than l. In this situation, we arrive at the same expression $F_{2n} \sim \mathfrak{B}_0^{2n}\langle e^{2n\rho}\rangle$ as for the $2n$th moment, and we conclude that $F_{2n} \propto \exp(\lambda_{2n}t)$, see Eq. (3.2).

We now turn to the case where $\lambda t > \ln(l/r)$, with all separations $\Delta\mathbf{r}$ assumed to be of the same order. We first consider the geometry where all the points $\mathbf{r}_1, \ldots, \mathbf{r}_{2n}$ lie on a line, that is, all vectors $\mathbf{r}_\alpha - \mathbf{r}_\beta$ have the same directions, and we can write $\Delta\mathbf{r} \sim \mathbf{r}$, where $\mathbf{r}$ is one of the separations. We then arrive at the estimation $|\Delta\mathbf{R}| \sim e^\rho(r_x - \chi r_y)$, similar to that for the pair correlation function. We thus obtain the same probability $\sim e^{-\rho} l/r$ that the separations $|\Delta\mathbf{R}(0)| \lesssim l$. Then $F_{2n} \sim \mathfrak{B}_0^{2n}\langle e^{(2n-1)\rho}\rangle l/r$, where the factor $e^{2n\rho}$ originates from the product of the matrices $\hat{W}$, appearing in accordance with expression (2.7). Averaging this expression, we obtain $F_{2n}(t) \sim \mathfrak{B}_0^{2n}\exp(\lambda_{2n-1}t)l/r$. We stress that the growth rates are here different from those of the corresponding moments.

However, the above expression is correct only if $t < \lambda^{-1}\ln(l/r_d)$. For larger t, the diffusion contributions to the differences ΔR_α become relevant (see Eq. (3.5)). Then, by manipulating with χ, only one difference among the $\Delta\mathbf{R}(0)$ can be made less than l. After that, all the other differences typically acquire values of the order of $r_d e^\rho$, and the probability that a difference is smaller than l is estimated as $(l/r_d)e^{-\rho}$. We therefore conclude that

$$F_{2n} \sim \left\langle \frac{l}{r}\left(\frac{l}{r_d}\right)^{n-1} e^{n\rho}\right\rangle \propto \exp(\lambda_n t).$$

The same results (up to combinatorial factors) hold for the collinear geometry, where the set $\mathbf{r}_1, \ldots, \mathbf{r}_{2n}$ is separated into n pairs with parallel vectors $\mathbf{r}_\alpha - \mathbf{r}_\beta$ characterizing the pairs. Then the corresponding differences $\Delta\mathbf{R}$ behave as previously and the same arguments apply. We summarize our results for the collinear geometry:

$$F_{2n}(t) \sim \begin{cases} \mathfrak{B}_0^{2n} \exp(\lambda_{2n} t), & t < \dfrac{1}{\lambda} \ln \dfrac{l}{r}, \\ \mathfrak{B}_0^{2n} \exp(\lambda_{2n-1} t) \dfrac{l}{r}, & \dfrac{1}{\lambda} \ln \dfrac{l}{r} < t < \dfrac{1}{\lambda} \ln \dfrac{l}{r_d}, \\ \mathfrak{B}_0^{2n} \dfrac{l}{r} \left(\dfrac{l}{r_d}\right)^{n-1} \exp(\lambda_n t), & t > \dfrac{1}{\lambda} \ln \dfrac{l}{r_d}. \end{cases} \quad (3.10)$$

It is a generalization of expressions (3.7) for the pair correlation function. We note that in the collinear geometry, similarly to the pair correlation function, the estimate $B_y \sim (\lambda/s) B_x$ determines relations between different components of F_{2n}, whereas the z-components require a separate investigation.

If the collinear geometry is destroyed, then it is impossible to put all the separations $|\Delta\mathbf{R}(0)|$ inside the scale l at any χ, if $t > \lambda^{-1} \ln(l/r)$. In this situation, the behavior of the correlation function F_{2n} is nonuniversal, being sensitive to the details of the initial spatial distribution of $\mathfrak{B}$. In any case, the value of F_{2n} in the noncollinear geometry is much less than in the collinear one. The situation resembles the one realized for a randomly advected passive scalar on scales larger than the pumping length [34]. We conclude that at $t > \lambda^{-1} \ln(l/r)$, correlations of the magnetic field are concentrated near collinear geometries, decaying away from the geometries. The decaying length is estimated as $le^{-\lambda t}$ at $t < \lambda^{-1} \ln(l/r_d)$ and as r_d at $t > \lambda^{-1} \ln(l/r_d)$.

4. SHORT-CORRELATED FLOW

Here, we consider a strong steady shear flow complemented by a random component short-correlated in time. This case admits an analytical solution and can therefore be used to verify our general assertions and predictions. In addition, the case is naturally realized because the strong shear destroys correlations of the random component, and we therefore expect that the short-correlated case is frequently encountered in real flows.

In the short-correlated case, the matrix of the velocity gradients $\hat{\sigma}$ describing the random component of the flow has to be treated as white noise, that is, a variable δ-correlated in time. In the isotropic case, we obtain the tensorial structure

$$\langle \sigma_{ik}(t_1) \sigma_{jn}(t_2) \rangle = = D(4\delta_{ij}\delta_{kn} - \delta_{ik}\delta_{jn} - \delta_{in}\delta_{jk}) \delta(t_1 - t_2), \quad (4.1)$$

where the factor D characterizes the random flow strength and the numerical factor is introduced as in Ref. [35]. But as we have argued, the only relevant component of the random velocity gradient matrix in the case $\lambda \ll s$ (which is a manifestation of the random flow weakness) is $\sigma \equiv \sigma_{yx}$. We characterize its statistical properties by the expression

$$\langle \sigma(t_1) \sigma(t_2) \rangle = 4D\delta(t_1 - t_2), \quad (4.2)$$

formally coinciding with Eq. (4.1) for the yx-component. Other components of $\hat{\sigma}$ can have correlation functions different from (4.1). The random component can be considered to be weaker than the steady shear flow if $D \ll s$.

Statistical properties of the separation $\Delta\mathbf{R}(t)$ between close Lagrangian trajectories in a random smooth flow with strong shear component in the short-correlated case are investigated in Ref. [36] (in the context of polymer dynamics). We here present the results obtained in that paper without derivation. We note that our variable χ is related to the tilt angle ϕ in Ref. [36] as $\chi = \operatorname{ctg}\phi$, or $\chi \approx \phi^{-1}$ in the case of small tilt angles in which we are interested. The expression for the Lyapunov exponent found in Ref. [36] is

$$\lambda = \frac{3^{1/3}\sqrt{\pi}}{\Gamma(1/6)} D^{1/3} s^{2/3}. \quad (4.3)$$

Therefore, the condition $s \gg D$ does guarantee the inequality $\lambda \ll s$. We also note that $\lambda \gg D$ and that $\lambda \to 0$ as $D \to 0$. The last property is a natural consequence of the vanishing Lyapunov exponent for a purely shear flow.

For the short-correlated case, it is possible to find the exponents λ_n characterizing the growth rates of the moments of $\Delta\mathbf{R}(t)$ (see Eq. (2.23)) if $n \gg 1$. Then a saddle-point (instanton) approximation in the functional space [37] can be used, which leads to [36]

$$\gamma_n = \frac{3}{2^{5/3}} n^{4/3} D^{1/3} s^{2/3} \sim \lambda n^{4/3}. \quad (4.4)$$

The nonlinear dependence of λ_n on n, $\lambda_n \propto n^{4/3}$, signals a strong intermittency of the flow. We note that in our anisotropic case, the growth rates λ_n increase as n increases slower than in the isotropic case, where $\lambda_n \propto n^2$ for the short-correlated flow (see Ref. [15]).

4.1. Pair correlation function

We next examine the two-point one-time correlation function (3.3) for the velocity field short-correlated in time. In this case, it is possible to derive a closed equation for the correlation function (see, e.g., Ref. [12]). We here study the function at scales much larger than the diffusion scale r_d (but much smaller than the velocity correlation length η). It is then possible to neglect diffusion effects, and we omit all terms with the noise $\boldsymbol{\xi}$ in subsequent relations.

We briefly explain the derivation of the equation. First, it follows from definition (2.6) that

$$\hat{W}(t) = \mathrm{T}\exp(\hat{\Xi})\hat{W}(t-\tau), \quad \hat{\Xi} = \int\limits_{t-\tau}^{t} dt'\ \hat{\Sigma}, \qquad (4.5)$$

where τ is an arbitrary time (less than t). We choose τ to be much smaller than λ^{-1}, but much larger than the velocity correlation time (this is possible for a short-correlated flow). Then $\hat{\Xi}$ is a small factor, although the two factors in Eq. (4.5) can be treated as statistically independent. Substituting expression (4.5) in Eq. (3.4), expanding the result into a series in $\hat{\Xi}$ (up to the second order) and averaging the result inside the time interval $(t-\tau, t)$ in accordance with Eq. (4.2), we obtain a variation of F_{ij} under passing from $t-\tau$ to t. Because the variation is small, it can be rewritten in terms of a differential equation.

Assuming the isotropic correlation function of fluctuations in (4.1), we obtain the equation

$$\begin{aligned}\partial_t F_{ij} = {} & -2sy\partial_x F_{ij} + sF_{iy}\delta_{jx} + sF_{yj}\delta_{ix} + \\ & + 4D\left[\delta_{ij}F_{kk} - \frac{1}{2}F_{ij} - r_k\partial_j F_{ik} - r_k\partial_i F_{kj} + \right. \\ & \left. + \frac{1}{2}(\mathbf{r}\nabla)F_{ij} + \frac{1}{2}r^2\nabla^2 F_{ij} - \frac{1}{4}r_m r_n \partial_m\partial_n F_{ij}\right]. \qquad (4.6)\end{aligned}$$

In the absence of shear (at $s = 0$), system of equations (4.6) leads to a closed equation for the trace of the correlation function $H = F_{kk}$:

$$\partial_t H = D\left(10H + 6r\partial_r H + r^2\partial_r^2 H\right).$$

The equation coincides with one presented in Ref. [14] (for the scaling exponent $\xi = 2$ and zero forcing).

We now eliminate irrelevant terms in Eq. (4.6) using the following properties: s is much larger than D, the characteristic value of x is much larger than that of y, and, accordingly, $\partial_y \gg \partial_x$. We can then keep solely the terms originating from σ_{yx} in Eq. (4.6). The resulting equation leads to a closed system of equations for the three components F_{xx}, F_{xy}, and F_{yy} of the pair correlation function:

$$\begin{aligned}\partial_t F_{xx} &= -2sy\partial_x F_{xx} + 2sF_{xy} + 4Dx^2\partial_y^2 F_{xx}, \\ \partial_t F_{xy} &= -2sy\partial_x F_{xy} + sF_{yy} + \\ &\quad + 2Dx^2\partial_y^2 F_{xy} - 4Dx\partial_y F_{xx}, \qquad (4.7) \\ \partial_t F_{yy} &= -2sy\partial_x F_{yy} + 2Dx^2\partial_y^2 F_{yy} - \\ &\quad - 8Dx\partial_y F_{xy} + DF_{xx}.\end{aligned}$$

Further, we use the dimensionless time $T = (8Ds^2)^{1/3}t$ and introduce the notation

$$f = F_{xx}, \quad g = (s/D)^{1/3}F_{xy}, \quad h = (s/D)^{2/3}F_{yy}.$$

We investigate a special case of coinciding points. At $\mathbf{r} = 0$, all terms with derivatives drop from Eqs. (4.7), and they take the form

$$\partial_T \begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1/2 \\ 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} f \\ g \\ h \end{pmatrix}. \qquad (4.8)$$

An increasing solution of the equation is

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix}_{r=0} \propto \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} e^T. \qquad (4.9)$$

This behavior corresponds to the growth rate $\gamma = (8Ds^2)^{1/3}$ of the magnetic field second moment. On the other hand, the case corresponds to small r, that is, to the condition $\ln(l/r) > \lambda t$. Therefore, $\gamma = \lambda_2$, and we conclude that $\lambda_2 = (8Ds^2)^{1/3}$ in our case.

It is convenient to pass to the "polar coordinates" ϱ and φ in the shear plane: $x = \varrho\cos\varphi$, $y = (D/s)^{1/3}\varrho\sin\varphi$. We then perform the Mellin transform $f, g, h \to \tilde{f}, \tilde{g}, \tilde{h}$ in terms of ϱ and derive the equations for $\tilde{f}, \tilde{g}$, and $\tilde{h}$ from system (4.7). In terms of the quantity $q = -ik$, the equations are written as

$$\begin{aligned}
&\partial_T \tilde f = \tilde f'' \cos^4\varphi + \\
&+ \tilde f' \left[\sin^2\varphi - 2(q+1)\sin\varphi\cos^3\varphi\right] + \\
&+ \tilde f \left[q\sin\varphi\cos\varphi - q\cos^4\varphi + \right.\\
&+ \left.(q^2+q)\sin^2\varphi\cos^2\varphi\right] + \tilde g, \\
&\partial_t \tilde g = \tilde g'' \cos^4\varphi + \\
&+ \tilde g' \left[\sin^2\varphi - 2(q+1)\sin\varphi\cos^3\varphi\right] + \\
&+ \tilde g \left[q\sin\varphi\cos\varphi - q\cos^4\varphi + \right.\\
&+ \left.(q^2+q)\sin^2\varphi\cos^2\varphi\right] - \\
&- 2\tilde f' \cos^2\varphi + 2q\tilde f \sin\varphi\cos\varphi + \tilde h/2, \\
&\partial_t \tilde h = \tilde h'' \cos^4\varphi + \\
&+ \tilde h' \left[\sin^2\varphi - 2(q+1)\sin\varphi\cos^3\varphi\right] + \\
&+ \tilde h \left[q\sin\varphi\cos\varphi - q\cos^4\varphi + \right.\\
&+ \left.(q^2+q)\sin^2\varphi\cos^2\varphi\right] - \\
&- 4\tilde g' \cos^2\varphi + 4q\tilde g \sin\varphi\cos\varphi + 2\tilde f,
\end{aligned} \tag{4.10}$$

where the prime denotes the derivative over the angle φ.

4.2. Numerics

Next, we study the time evolution of system (4.10) numerically for different real values of q (with imaginary $k = iq$) using the implicit difference scheme on the interval $(-\pi/2, \pi/2)$ for φ with periodic boundary conditions. We have chosen as initial conditions for f, g, and h as the same Gaussian functions centered near $\varphi = 0$ and with a width of the order of unity. Then we extract the leading growth rate $\gamma(iq) = (8Ds^2)^{1/3}c(q)$ dominating the behavior of the system at $T \gg 1$. The dimensionless quantity c was extracted as

$$c = \frac{1}{T}\ln\frac{f(T_0+T)}{f(T_0)},$$

where $T_0 + T$ is chosen to be large enough (near 30) and T_0 is introduced to exclude the influence of an initial transient process (we have chosen $T_0 = T$).

The quantity c is plotted as a function of q in Fig. 2. It turned out to be positive everywhere, with a minimum at $q = 1$, $c(1) = 0.435$. The value $q = 1$ is in accordance with Eq. (3.7) and the general analysis in Sec. 3. As we argued there, the minimum value of $\gamma(iq)$ determines λ_1, that is, $\lambda_1 = c(q_\star)(8Ds^2)^{1/3}$. The value of c at $q = 0$ is $c = 1$, in accordance with Eq. (4.9) and the general arguments given in Sec. 3. Therefore, the obtained results confirm our general assertions.

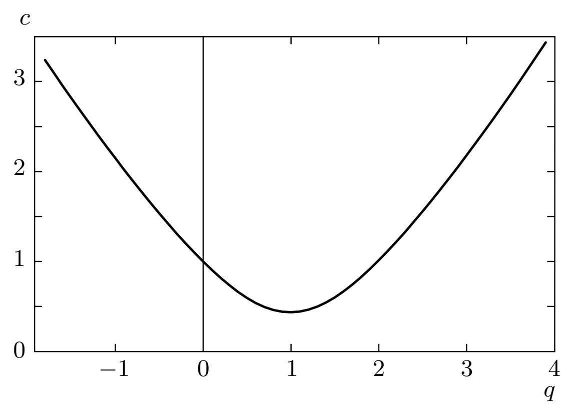

Fig. 2. Growth rate of the Mellin transform of the pair correlation function on the imaginary axis

5. DISCUSSION

We have analyzed the kinematic dynamo stage when small-scale fluctuations of the magnetic field grow in a steady shear flow complemented by relatively weak random velocity fluctuations. The weakness is characterized by the inequality $s \gg \lambda$, where s is the shear rate and λ is the Lyapunov exponent of the flow. The universal features we have established are revealed at times $t \gg \lambda^{-1}$. The shear makes the flow strongly anisotropic, which, paradoxically, simplifies the analysis of the dynamo phenomenon because a single component of the random velocity gradient appears to be relevant. We analyzed the situation where the correlation length l of the initial magnetic field fluctuations is less than the velocity correlation length η (i.e., the Kolmogorov length for developed turbulence). Probably, the smallness of l is not crucial for our scheme because small scales of the magnetic field distribution in space are inevitably produced by the hydrodynamic motion.

We stress that in the leading approximation in λ/s, our problem is reduced to a purely two-dimensional velocity field (with components along the shear velocity and along its gradient). We have proved the existence of the dynamo in this case (that is, the exponential growth of the magnetic field moments). The result obviously contradicts the statement in Refs. [38–40] (Zeldovich theorem) that there cannot be a magnetic dynamo in two-dimensional flows. We assert that this statement is wrong and the error is in ignoring the third component B_3 of the magnetic induction (perpendicular to the velocity plane). The third component satisfies the passive scalar equation and, consequently, decays exponentially. But B_3 cannot be ignored in the divergence-free condition $\nabla\mathbf{B} = 0$ because the characteristic scale of the magnetic field along the direction of its growth increases faster than the magnetic field itself. It can be checked that all the terms in

$\nabla\mathbf{B} = \partial_x B_x + \partial_y B_y + \partial_z B_z$ decay with the same exponent, and therefore the condition $\nabla\mathbf{B} = 0$ leads to an effectively divergent in-plane magnetic field. The dynamo effect is not forbidden for such a field. A detailed analysis of the discrepancy will be published elsewhere. The existence of the dynamo effect for two-dimensional flows is a subject of numerical verification.

For our general conclusions, we do not specify statistical properties of the random flow, exploring only its smoothness at scales less than the velocity correlation length η. It is then possible to relate the kinematic growth rates γ_n of the magnetic field (see Eq. (2.2)) to intrinsic characteristics of the flow characterizing the divergence of close Lagrangian trajectories (see Eq. (2.23)). We find that $\gamma_n = \lambda_{2n}$ in the diffusionless regime and $\gamma_n = \lambda_n$ in the diffusion regime. We also related the anisotropy degree of the magnetic field to the same intrinsic characteristics of the flow. Therefore, the main features of the magnetic field statistics (including its intermittency) are dictated by the flow statistics. We note that our general scheme can be applied without essential modifications to the statistically isotropic flows or to random flows with other types of anisotropy.

We established the principal features of the magnetic field correlation functions. The pair correlation function behaves like the second moment at small separations r, and increases with the growth rate characteristic of the diffusion regime; at larger r, it is proportional to $1/r$. As regards higher-order correlation functions, the situation is more complicated. At small times t, they behave like the corresponding moments. But at larger time $t > \lambda^{-1}\ln(l/r)$, correlations are peaked near the collinear geometry (where $2n$ points are separated into n pairs with parallel separations) and there is an intermediate asymptotic regime when the correlation functions grow with the rates that do not coincide with the growth rates of the moments. Then, at times $t > \lambda^{-1}\ln(l/r_d)$, the correlation function grows with the same exponent as the corresponding moment in the diffusion regime. The scaling behavior of the correlation functions in the collinear regime is $\propto 1/r$. The correlations decay rapidly with the deviation from the collinear geometry. This reflects a complicated spacial structure of the magnetic field that is strongly correlated for special geometries produced by affine geometric transformations from the initial magnetic fluctuations.

Our general assertions can be verified by solving the model with the fluctuating component short-correlated in time. This model admits several analytic results. The nonlinear n-dependence of the growth rates γ_n, $\gamma_n \propto n^{4/3}$, at large n signals a strong intermittency of the magnetic field. Therefore, only rare events concentrated in a restricted part of space contribute to high moments of the magnetic field. We analyzed the pair correlation function of the magnetic field in detail, and the analysis confirms all our general assertions, including scaling behavior in different regimes.

The ideology and the analytic approach developed in this paper can be tended to the dynamics of polymer solutions possessing elasticity that is described similarly to the magnetic field. Along these lines, we hope to clarify some aspects of the so-called elastic turbulence [22–24] that are still not explained.

We thank A. I. Chernykh, M. V. Chertkov, G. E. Falkovich, and S. S. Vergeles for the helpful discussions and remarks. The work was supported in part by the RFBR (grant No. 09-02-01346-a) and by the RF Ministry of Education and Science under the Federal Target Program "Kadry".

REFERENCES

1. A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics*, Vol. 1 and 2, MIT Press, Cambridge Mass. (1975).

2. U. Frisch, *Turbulence: the Legacy of A. N. Kolmogorov*, Cambridge Univ. Press, New York (1995).

3. S. Vainshtein and Ya. Zeldovich, Uspekhi Fiz. Nauk **106**, 431 (1972).

4. H. K. Moffatt, *Magnetic Field Generation in Electrically Conducting Fluids*, Cambridge Univ. Press (1978).

5. E. N. Parker, *Cosmic Magnetic Fields, Their Origin and Activity*, Clarendon Press, Oxford (1979).

6. Ya. B. Zeldovich, A. A. Ruzmaikin, and D. D. Sokolov, *The Almighty Chance*, World Sci., Singapore (1990).

7. S. Childress and A. Gilbert, *Stretch Twist Fold: The Fast Dynamo*, Springer-Verlag, Berlin (1995).

8. G. Rüdiger and R. Hollerbach, *The Magnetic Universe: Geophysical and Astrophysical Dynamo Theory*, Wiley, New York (2004).

9. *Fluid Dynamics and Dynamos in Astrophysics and Geophysics*, ed. by A. M. Soward, Ch. A. Jones, D. W. Hughes, and N. O. Weiss, CRC Press (2005).

10. E. T. Vishniac, A. Lazarian, and J. Cho, *Problems and Progress in Astrophysical Dynamos*, Springer, Berlin (2008).

11. A. A. Schekochihin, S. C. Cowley, S. F. Taylor, L. Maron, and J. C. McWilliams, Astrophys. J. **612**, 276 (2004).

12. G. Falkovich, K. Gawedzki, and M. Vergassola, Rev. Mod. Phys. **73**, 913 (2001).

13. R. Kraichnan and S. Nagarajan, Phys. Fluids **10**, 859 (1967).

14. A. P. Kazantsev, Zh. Eksp. Teor. Fiz. **53**, 1806 (1967).

15. M. Chertkov, G. Falkovich, I. Kolokolov, and M. Vergassola, Phys. Rev. Lett. **83**, 4065 (1999).

16. V. Urpin, J. Geophys. Astrophys. Fluid Dynam. **95**, 269 (2001).

17. R. B. Bird, C. F. Curtiss, R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric Liquids*, Wiley, New York (1987).

18. M. Doi and S. F. Edwards, *The Theory of Polymer Dynamics*, Oxford Univ. Press (1988).

19. J. L. Lumley, Symp. Math. **9**, 315 (1972).

20. E. Balkovsky, A. Fouxon, and V. Lebedev, Phys. Rev. Lett. **84**, 4765 (2000).

21. M. Chertkov, I. Kolokolov, V. Lebedev, and K. Turitsyn, J. Fluid Mech. **531**, 251 (2005).

22. A. Groisman and V. Steinberg, Nature **405**, 53 (2000).

23. A. Groisman and V. Steinberg, Phys. Rev. Lett. **86**, 934 (2001).

24. A. Groisman and V. Steinberg, Nature **410**, 905 (2001).

25. V. R. Kogan, I. V. Kolokolov, and V. V. Lebedev, J. Phys. A **43**, 182001 (2010).

26. L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics, Electrodynamics of Continuous Media*, Pergamon Press (1984).

27. M. Chertkov, I. Kolokolov, and M. Vergassola, Phys. Rev. E **56**, 5483 (1997).

28. S. B. Pope, Ann. Rev. Fluid Mech. **26**, 23 (1994).

29. H. Furstenberg, Trans. Amer. Math. Soc. **108**, 377 (1963).

30. I. Goldhirsch, P.-L. Sulem, and S. A. Orszag, Physica D **27**, 311 (1987).

31. E. Balkovsky and A. Fouxon, Phys. Rev. E **60**, 4164 (1999).

32. R. Ellis, *Entropy Large Deviations and Statistical Mechanics*, Springer-Verlag, New York (1985).

33. L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics, Statistical Physics*, pt. 1, Pergamon Press (1980).

34. E. Balkovsky, G. Falkovich, V. Lebedev, and M. Lysiansky, Phys. Fluids **11**, 2269 (1999).

35. M. Chertkov, G. Falkovich, V. Kolokolov, and V. Lebedev, Phys. Rev. E **52**, 4924 (1995).

36. K. S. Turitsyn, Zh. Eksp. Teor. Fiz. **132**, 746 (2007).

37. G. Falkovich, I. Kolokolov, V. Lebedev, and A. Migdal, Phys. Rev. E **54**, 4896 (1996).

38. Ya. B. Zeldovich, Zh. Eksp. Teor. Fiz. **31**, 154 (1956).

39. Ya. B. Zeldovich and A. A. Ruzmaikin, Zh. Eksp. Teor. Fiz. **78**, 980 (1980).

40. Ya. B. Zeldovich, A. A. Ruzmaikin, S. A. Molchanov, and D. D. Sokolov, J. Fluid Mech. **144**, 1 (1984).