ЖЭТФ, 2012, том 141, вып. 2, стр. 355–365

THERMODYNAMICS OF AN INTERACTING FERMI SYSTEM IN THE STATIC FLUCTUATION APPROXIMATION

R. R. Nigmatullin, A. A. Khamzin*[*], I. I. Popov

Theoretical Physics Department, Kazan (Volga Region) Federal University
420008, Kazan, Russia

Received April 19, 2011

We suggest a new method of calculation of the equilibrium correlation functions of an arbitrary order for the interacting Fermi-gas model in the framework of the static fluctuation approximation method. This method based only on a single and controllable approximation allows obtaining the so-called far-distance equations. These equations connecting the quantum states of a Fermi particle with variables of the local field operator contain all necessary information related to the calculation of the desired correlation functions and basic thermodynamic parameters of the many-body system. The basic expressions for the mean energy and heat capacity for the electron gas at low temperatures in the high-density limit were obtained. All expressions are given in the units of r_s, where r_s determines the ratio of a mean distance between electrons to the Bohr radius a_0. In these expressions, we calculate terms of the respective order r_s and r_s^2. It is also shown that the static fluctuation approximation allows finding the terms related to higher orders of the decomposition with respect to the parameter r_s.

1. INTRODUCTION TO THE STATIC FLUCTUATION APPROXIMATION METHOD

The calculation of the correlation energy of the ground state in the absence of external fields for the strongly degenerated electron gas constitutes the central problem in solid state physics, in particular, the physics of metals. In the pioneering works of Gell-Mann and Brueckner [1] and Wigner [2], the first analytic results for the high-density and low-density electron gas were obtained. In those papers, an expression for the ground state energy was obtained in the terms of the dimensionless parameter $r_s = r_0/a_0$, where $r_0 = \sqrt[3]{3/4\pi n}$ is the mean distance between electrons, n is the electron gas density, and a_0 determines the conventional Bohr radius. For the high-density limit ($r_s < 1$), the ground state energy has the form [2]

$$\frac{E_0}{N} = \left(\frac{2.21}{r_s^2} - \frac{0.916}{r_s} + 0.0622\ln r_s - 0.094\right)\mathrm{Ry}\,. \quad (1)$$

The first term corresponds to the kinetic energy and the second term determines the contribution of the exchange energy. The last two terms in Eq. (1) describe the correlation energy E_{corr}. We note that another numerical coefficient before the logarithmic term was obtained in [3]:

$$\frac{E_{corr}}{N} = (0.0570\ln r_s - 0.094)\,\mathrm{Ry}\,. \quad (2)$$

Besides this observation, we note papers [4–6], where the different numerical values for the constant term in Eq. (2) are presented. Therefore, the problem of the correct evaluation of the expression for the correlation energy remains open.

The second task is related to the problem of the accurate (error controllable) calculations of the corrections to the correlation energy having higher degrees of r_s. In [1], the authors suggested only a way to find the desired corrections respectively proportional to r_s and $r_s \ln r_s$. But the necessary calculations were not performed. In Ref. [7], in the framework of the random-phase approximation of the Rayleigh–Schrödinger perturbation theory, similar calculations were performed, but it was shown that the numerical coefficient before r_s can be evaluated only at $r_s = 1$. The desired terms having higher orders of r_s were not found.

To summarize the foregoing, we conclude that the conventional calculation (in the framework of the diagram summation or the Green's function method with

---

[*] E-mail: airat.khamzin@rambler.ru

uncontrollable decoupling) of any higher-order term entering the correlation energy is a complex and laborious work in and of itself. It is therefore necessary to develop a method that would allow realizing these calculations for a wide class of interactions and temperatures having only one (and error controllable) approximation.

One of us (R. R. N.) suggested the desired method, which was called the static fluctuation approximation (SFA). The SFA generalizes the well-known mean field method and allows calculating the desired equilibrium correlation functions with necessary thermodynamic values of the many-body system considered. The basic idea of the SFA can be formulated as follows. Instead of replacing the components of the local field operator by its corresponding mean value, it is possible to preserve its operator structure and find the spectrum of the remaining local field operator self-consistently. We need to clarify the basic idea of the SFA using a simple example. We consider some Hamiltonian H describing the interacting Fermi gas. The simplest equation of motion for the Fermi creation operator $a_k^\dagger(\tau)$, $\tau = it$, takes the form

$$\frac{da_k^\dagger}{d\tau} = [H, a_k^\dagger] = E_k a_k^\dagger, \tag{3}$$

where $k = (\mathbf{k}, s)$ determines the state of a Fermi particle. The local field operator E_k, as a projection on states of a single Fermi particle, is easily calculated from the relation

$$E_k = \Big[[H, a_k^\dagger], a_k \Big]_+, \quad [H, E_k] \approx 0, \tag{4}$$

where $[A, B]_+ = AB + BA$ is an anticommutator. We suppose that this local field operator commutes with the Hamiltonian of the system. Simple calculations lead to the equation

$$\langle n_k A \rangle = \left\langle \frac{A}{1 + \exp(\beta E_k)} \right\rangle, \quad \beta = \frac{1}{T}, \tag{5}$$

where $\langle \ldots \rangle = \mathrm{Sp}[\ldots \exp(-\beta H)] / \mathrm{Sp}[\exp(-\beta H)]$ is an averaging over an equilibrium statistical ensemble. The operator A in Eq. (5) determines an arbitrary combination of Fermi operators that commutes at least with operator $n_k = a_k^\dagger a_k$ pertaining to the individual state k.

The next step is to replace the eigen-values of the spectrum of E_k with their approximate values, which, in turn, are calculated self-consistently. We suppose that the spectrum of the local field operator E_k is degenerate and finite. In this case, it can be presented as

$$\prod_{m=1}^{n} (E_k - \alpha_m) = 0, \tag{6}$$

where α_m are the eigenvalues of the local field operator E_k. Using the Cayley–Hamilton theorem [8], we can express any operator function $F(E_k)$ as a polynomial,

$$F(E_k) = \sum_{p=0}^{n-1} \eta_p (E_k)^p. \tag{7}$$

The unknown coefficients η_p involved in (7) are found from the system of linear equations

$$F(\alpha_m) = \sum_{p=0}^{n-1} \eta_p (\alpha_m)^p, \quad m = 1, 2, \ldots, n. \tag{8}$$

We limit ourselves to considering only the linear term with respect to the difference operator $\Delta E_k = E_k - \langle E_k \rangle$. We also suppose that this operator approximately satisfies the relation

$$\begin{gathered} (\Delta E_k)^2 = b_k \, \Delta E_k + c_k, \\ b_k = \frac{\langle (\Delta E_k)^3 \rangle}{\langle (\Delta E_k)^2 \rangle}, \quad c_k = \langle (\Delta E_k)^2 \rangle. \end{gathered} \tag{9}$$

These expressions contain the basic approximation of the SFA. The unknown parameters b_k and c_k in (9) are calculated self-consistently, because they can be expressed as a combination of the operators $n_{k'}$, $k' \neq k$, involved in b_k and c_k and entering the operator A in Eq. (5). The details of calculations (with $b_k = 0$) are given below.

Using the Cayley–Hamilton theorem, we find an approximate relation connecting the operator n_k with ΔE_k. This equation has the form

$$\langle n_k A \rangle = \eta_0(T) \langle A \rangle + \eta_1(T) \langle \Delta E_k A \rangle \tag{10}$$

and is called the far-distance equation E [9]. This equation allows closing all the relation for the desired equilibrium correlation functions and thermodynamic values that need to be calculated. We do not give the values of the temperature constants $\eta_0(T)$ and $\eta_1(T)$ here. They are easily obtained from system (7) and Eq. (5). Other necessary details are given in the next section.

The SFA was successfully used in the analysis of equilibrium properties of the Ising model in an arbitrary dimension [10–12], thermodynamics of the interacting Bose gas [13] and Hubbard model [14], and nanosystems in the Hubbard model [15–17]. This new method was also used in studying the thermodynamic properties of different Bose systems [18–20].

The SFA essentially improves the mean field approximation and allows considering the thermodynamics of strongly interacting systems and a wide interval of temperatures and fields. From the mathematical standpoint, we obtain a nonlinear system of difference equations (lattice models) or a closed system of nonlinear integral equations (Fermi and Bose systems), and hence further information can only be obtained by numerical methods or in the form of approximate decompositions with the use of some small decomposition parameter. In this paper, using the SFA allows calculating the decomposition coefficients for the correlation energy with r_s (the parameter defined above) as a small parameter for a detailed analysis of a high-density electron gas.

2. THE BASIC EQUATIONS

The full Hamiltonian of the interacting homogeneous electron gas in the secondary quantization representation can be written as

$$H = \sum_{\mathbf{k}s} \varepsilon_{\mathbf{k}} n_{\mathbf{k}s} + \frac{1}{2\Omega} \sum_{\mathbf{q}\neq 0} V(\mathbf{q})(\rho_{\mathbf{q}}\rho_{-\mathbf{q}} - N), \tag{11}$$

$$\rho_{\mathbf{q}} = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}s} a^{\dagger}_{\mathbf{q}+\mathbf{k}s} a_{\mathbf{k}s}, \tag{12}$$

$$V(\mathbf{q}) = \frac{4\pi}{q^2 + q_0^2}, \tag{13}$$

where

$$\varepsilon_{\mathbf{k}} = \frac{k^2}{2} - \mu, \quad q_0 = \sqrt{\frac{6\pi n}{\varepsilon_F^0}} = \sqrt[3]{\frac{9}{\pi^2}}\,\frac{1}{a_0 r_s}$$

is the inverse Thomas–Fermi screening radius, μ is a chemical potential. For convenience in what follows, we realize the transformation of the given coordinate system as $\zeta = (3/4\pi)^{1/3}\mathbf{r}/r_0$. Then the Fermi system of a given volume Ω is transformed into the system with the volume N. The mean particle density in this space is equal to unity. The Hamiltonian of the electron gas in the secondary quantization representation then becomes

$$H' = \sum_{\mathbf{k}s} \varepsilon_{\mathbf{k}} n_{\mathbf{k}s} + \frac{\lambda}{2} \sum_{\mathbf{q}\neq 0} V(\mathbf{q})(\rho_{\mathbf{q}}\rho_{-\mathbf{q}} - 1), \tag{14}$$

where $\lambda = (4\pi/3)^{1/3} r_s$ defines the bond constant characterizing the interaction strength; after this transformation, energy is expressed in double rydbergs. The relation to the initial Hamiltonian H is given by

$$H = \frac{1}{\lambda^2} H'. \tag{15}$$

We also note that for the modified system, the inverse screening radius is equal to $q_0 = \sqrt{4k_F\lambda/\pi}$ and the Fermi vector becomes dimensionless and equal to $k_F = (3\pi^2)^{1/3}$.

The equation of motion for the operator $a^{\dagger}_{\mathbf{k}s}$ in the Heisenberg representation becomes

$$\frac{da^{\dagger}_{\mathbf{k}s}}{d\tau} = [H', a^{\dagger}_{\mathbf{k}s}] = \varepsilon_k a^{\dagger}_{\mathbf{k}s} + \frac{\lambda}{2\sqrt{N}} \times \\ \times \sum_{\mathbf{q}} V(\mathbf{q}) \left(\rho_{\mathbf{q}} a^{\dagger}_{\mathbf{k}-\mathbf{q}s} + a^{\dagger}_{\mathbf{k}+\mathbf{q}s}\rho_{-\mathbf{q}}\right). \tag{16}$$

As before, $\tau = it$. We suppose that the Hamiltonian of the system can be presented in the form

$$H' = \sum_{\mathbf{k}s} E_{\mathbf{k}s} n_{\mathbf{k}s}, \tag{17}$$

where $E_{\mathbf{k}s}$ determines a local field operator, and it follows from Eq. (4) that

$$\frac{da^{\dagger}_{\mathbf{k}s}}{d\tau} = E_{\mathbf{k}s} a^{\dagger}_{\mathbf{k}s} = \varepsilon_k a^{\dagger}_{\mathbf{k}s} + \\ + \frac{\lambda}{2\sqrt{N}} \sum_{\mathbf{q}} V(\mathbf{q}) \left(\rho_{\mathbf{q}} a^{\dagger}_{\mathbf{k}-\mathbf{q}s} + a^{\dagger}_{\mathbf{k}+\mathbf{q}s}\rho_{-\mathbf{q}}\right), \tag{18}$$

$$E_{\mathbf{k}s} = \left[\frac{da^{\dagger}_{\mathbf{k}s}}{d\tau}, a_{\mathbf{k}s}\right]_{+} = \varepsilon_k - \frac{\lambda}{N} \sum_{\mathbf{q}\neq\mathbf{k}} V(\mathbf{k}-\mathbf{q}) n_{\mathbf{q}s}. \tag{19}$$

Expression (19) determines the so-called Hartree–Fock operator, which satisfies the commutation relations

$$[E_{\mathbf{k}s}, a_{\mathbf{k}s}] = 0, \quad [H', E_{\mathbf{k}s}] = 0. \tag{20}$$

In the mean field approximation, the operator $E_{\mathbf{k}s}$ is replaced by its mean value, which signifies that the fluctuations of the local field operator are not taken into account (the Hartree–Fock approximation). Here, we want to show how to take the fluctuations of the local filed operator into account. For this, we replace the square of the local field deviation operator $\Delta E_{\mathbf{k}s} = E_{\mathbf{k}s} - \langle E_{\mathbf{k}s}\rangle$ by its mean value. In accordance with expressions (9), we then consider only the simplest case where

$$(\Delta E_{\mathbf{k}s})^2 \approx \langle(\Delta E_{\mathbf{k}s})^2\rangle \equiv \varphi^2_{\mathbf{k}s}. \tag{21}$$

Approximate equality (21) is the key point of the SFA. The physical meaning of the SFA is that alongside the mean value of the local field operator, its quadratic fluctuations are taken into account.

We next apply this basic approximation to the calculation of the desired characteristics of the interacting Fermi gas. For this, we find an expression for the operator $a^{\dagger}_{\mathbf{k}s}(\tau)$ in the Heisenberg representation

$$a^{\dagger}_{\mathbf{k}s}(\tau) = a^{\dagger}_{\mathbf{k}s} \exp(E_{\mathbf{k}s}\tau). \tag{22}$$

We then find the correlation function

$$\langle a^\dagger_{\mathbf{k}s}(\tau) a_{\mathbf{k}s} A\rangle = \langle a^\dagger_{\mathbf{k}s} a_{\mathbf{k}s} \exp(E_{\mathbf{k}s}\tau)\rangle. \tag{23}$$

Setting $\tau = \beta$ here and using the quantum identity for a pair of noncommuting operators,

$$\langle O_1(\beta) O_2\rangle = \langle O_2 O_1\rangle, \tag{24}$$

where $O_1(\tau)$ is an arbitrary operator in the Heisenberg representation and O_2 is another arbitrary operator at $\tau = 0$, we obtain the relation

$$\langle a^\dagger_{\mathbf{k}s}(\beta) a_{\mathbf{k}s} A\rangle = \langle a_{\mathbf{k}s} a^\dagger_{\mathbf{k}s} A\rangle = \\ = \langle a^\dagger_{\mathbf{k}s} a_{\mathbf{k}s} \exp(\beta E_{\mathbf{k}s}) A\rangle. \tag{25}$$

Here, A is an arbitrary combination of Fermi operators commuting with $n_{\mathbf{k}s}$ and $E_{\mathbf{k}s}$. Using the commutation rules for Fermi operators, we can represent Eq. (25) in the form

$$\langle n_{\mathbf{k}s} [1 + \exp(\beta E_{\mathbf{k}s})] A\rangle = \langle A\rangle. \tag{26}$$

Replacing $A \to [1 + \exp(\beta E_{\mathbf{k}s})]^{-1} A$, we can rewrite Eq. (26) as

$$\langle n_{\mathbf{k}s} A\rangle = \left\langle \frac{1}{1 + \exp(\beta E_{\mathbf{k}s})} A \right\rangle = \langle f(E_{\mathbf{k}s}) A\rangle, \tag{27}$$

where $f(x)$ is the standard Fermi–Dirac function. We say that Eq. (27), which separates the Fermi operators $n_{\mathbf{k}s}$ describing the state of a single particle from the components of the local field operator, is the far-distance equation [9]. It is shown in what follows that the FDE allows establishing the desired relation between equilibrium correlation functions of any order and thereby closing the system of nonlinear and self-consistent equations for the given many-body system considered. Using the Cayley–Hamilton theorem and taking relations (7) and (8) into account, we obtain

$$f(E_{\mathbf{k}s}) = \frac{1}{1 + \exp(\beta E_{\mathbf{k}s})} = \\ = \frac{1}{1 + \exp\left[\beta\left(\langle E_{\mathbf{k}s}\rangle + \Delta E_{\mathbf{k}s}\right)\right]} = \\ = \eta_0(\mathbf{k}s) + \eta_1(\mathbf{k}s) \Delta E_{\mathbf{k}s}, \tag{28}$$

where

$$\eta_0(\mathbf{k}s) = \frac{1}{2}\left(f\left(\langle E_{\mathbf{k}s}\rangle + \varphi_{\mathbf{k}s}\right) + f\left(\langle E_{\mathbf{k}s}\rangle - \varphi_{\mathbf{k}s}\right)\right), \tag{29}$$

$$\eta_1(\mathbf{k}s) = \frac{1}{2\varphi_{\mathbf{k}s}}\left(f\left(\langle E_{\mathbf{k}s}\rangle + \varphi_{\mathbf{k}s}\right) - \\ - f\left(\langle E_{\mathbf{k}s}\rangle - \varphi_{\mathbf{k}s}\right)\right). \tag{30}$$

As a result of this decomposition, far-distance equation (28) becomes more convenient for further analysis:

$$\langle n_{\mathbf{k}s} A\rangle = \eta_0(\mathbf{k}s)\langle A\rangle + \eta_1(\mathbf{k}s)\langle \Delta E_{\mathbf{k}s} A\rangle. \tag{31}$$

Here, we introduce the operator

$$\Delta E_{\mathbf{k}s} = -\frac{\lambda}{N}\sum_{\mathbf{q}\neq\mathbf{k}} V(\mathbf{k}-\mathbf{q}) \Delta n_{\mathbf{q}s}. \tag{32}$$

Equation (31) allows obtaining a closed nonlinear system of integral equations for the calculation of the desired thermodynamic values of the Fermi system considered. Setting $A = 1$ in (31), we obtain an equation for $\langle n_{\mathbf{k}s}\rangle$:

$$\langle n_{\mathbf{k}s}\rangle = \eta_0(\mathbf{k}s). \tag{33}$$

Using (33), we can rewrite Eq. (31) in an elegant form that it is more convenient for further calculations:

$$\langle \Delta n_{\mathbf{k}s} A\rangle = \eta_1(\mathbf{k}s)\langle \Delta E_{\mathbf{k}s} A\rangle. \tag{34}$$

To calculate pair correlation functions, we set $A = \Delta n_{\mathbf{q}s'}$ in (34) and take into account that the wave vectors $\mathbf{q} \neq \mathbf{k}$ cannot coincide with each other in (34). As a result of this substitution, we obtain the equation

$$\langle \Delta n_{\mathbf{k}s} \Delta n_{\mathbf{q}s'}\rangle = \Delta_{\mathbf{k}s}\delta_{\mathbf{k},\mathbf{q}}\delta_{s,s'} - \frac{\lambda\eta_1(\mathbf{k}s)}{N} \times \\ \times \sum_{\mathbf{q}'} V(\mathbf{k}-\mathbf{q}')\langle \Delta n_{\mathbf{q}'s}\Delta n_{\mathbf{q}s'}\rangle\left(1 - \delta_{\mathbf{k},\mathbf{q}}\delta_{s,s'}\right), \tag{35}$$

where we introduce the notation $\Delta_{\mathbf{k}s} = \langle n_{\mathbf{k}s}\rangle(1 - \langle n_{\mathbf{k}s}\rangle)$ and use expression (32) for $\Delta E_{\mathbf{k}s}$. In addition, we use the kinematic identity $\langle(\Delta n_{\mathbf{k}s})^2\rangle = \Delta_{\mathbf{k}s}$ that is valid for any Fermi system.

Equation (35) written as an integral equation serves for the calculation of binary correlation functions for the Fermi system considered. Next, it is necessary to obtain an equation for the local field quadratic fluctuations. For this, we set $A = \Delta E_{\mathbf{k}s}$ in Eq. (34) and use definition (21) to obtain

$$\varphi^2_{\mathbf{k}s} = -\frac{\lambda}{\eta_1(\mathbf{k}s) N}\sum_{\mathbf{q}\neq\mathbf{k}} V(\mathbf{k}-\mathbf{q})\langle \Delta n_{\mathbf{k}s}\Delta n_{\mathbf{q}s}\rangle. \tag{36}$$

The set of Eqs. (19), (33), (35), and (36) is a closed system of nonlinear integral equations for the desired values $\langle E_{\mathbf{k}s}\rangle$, $\langle n_{\mathbf{k}s}\rangle$, $\langle n_{\mathbf{k}s}\Delta n_{\mathbf{q}s'}\rangle$, and $\varphi_{\mathbf{k}s}$.

In the framework of the SFA, it is easy to obtain expressions for the mean energy and partition function Z:

$$\langle H\rangle = \frac{2}{\lambda^2}\langle H'\rangle = \frac{2}{\lambda^2}\sum_{\mathbf{k}s}\langle n_{\mathbf{k}s} E_{\mathbf{k}s}\rangle = \\ = \frac{2}{\lambda^2}\sum_{\mathbf{k}s}\left(\eta_0(\mathbf{k}s)\langle E_{\mathbf{k}s}\rangle + \eta_1(\mathbf{k}s)\varphi^2_{\mathbf{k}s}\right), \tag{37}$$

$$\begin{aligned}\ln Z &= \ln \operatorname{Sp} \exp(-\beta H) = \\ &= \ln \operatorname{Sp} \exp\left(-\beta \frac{2}{\lambda^2} \sum_{\mathbf{k}s} E_{\mathbf{k}s} n_{\mathbf{k}s}\right) = \\ &= \ln \prod_{\mathbf{k}s} \sum_{n_{\mathbf{k}s}\Delta E_{\mathbf{k}s}} \exp\left(-\frac{2\beta}{\lambda^2} E_{\mathbf{k}s} n_{\mathbf{k}s}\right) = N \ln 2 + \\ &+ \sum_{\mathbf{k}s} \ln\left[1 + \exp\left(-\frac{2\beta}{\lambda^2}\langle E_{\mathbf{k}s}\rangle\right) \operatorname{ch}\left(\frac{2\beta}{\lambda^2}\varphi_{\mathbf{k}s}\right)\right] \qquad (38)\end{aligned}$$

(the mean energy is here measured in rydbergs).

We emphasize once again that the closed system of equations is based only on a single approximation (21), and the Fermi system can be described in a wide range of temperatures and potentials within this approximation. A possible generalization of these equations that takes the effect of the asymmetry parameter a_k into account can be considered elsewhere.

3. SOLUTION OF INTEGRAL EQUATIONS FOR A HIGH-DENSITY ELECTRON GAS

To be able to obtain analytic solutions, we consider the interacting electron gas with a high density. It is easy to see that the high-density electron gas ($r_s \to 0$) corresponds to the case where the interaction in (14) is weak in comparison with the kinetic energy. This means that the fluctuations of the local field are small in comparison with this field mean value, $\varphi_{\mathbf{k}s} \ll \langle E_{\mathbf{k}s}\rangle$. In this case, an approximate expression for the coefficient $\eta_1(\mathbf{k}s)$ in (30) can be derived in the form

$$\begin{aligned}\eta_1(\mathbf{k}s) &\approx f'\left(\langle E_{\mathbf{k}s}\rangle_0\right) = \\ &= -\beta f\left(\langle E_{\mathbf{k}s}\rangle_0\right) f\left(-\langle E_{\mathbf{k}s}\rangle_0\right), \qquad (39)\end{aligned}$$

where $\langle E_{\mathbf{k}s}\rangle_0 = \langle E_{\mathbf{k}s}\rangle_{\varphi_{\mathbf{k}s}=0}$ is independent of the value of $\varphi_{\mathbf{k}s}$ and hence of the influence of the binary correlation function $\langle \Delta n_{\mathbf{k}s} \Delta n_{\mathbf{q}s'}\rangle$. In this case, we can apply the step-by-step method for solution of Eq. (35) and then insert the result into (36) to finally obtain the desired decomposition

$$\begin{aligned}\varphi^2_{\mathbf{k}s} &= \frac{1}{N} \sum_{\mathbf{q}\neq\mathbf{k}} \Delta_{\mathbf{q}s} V(\mathbf{k}-\mathbf{q}) \sum_{n=1}^{\infty} \left(\frac{\lambda}{N}\right)^{n+1} \times \\ &\times \beta^{n-1} \left(\prod_{i=1}^{n-1} f\left(\langle E_{\mathbf{k}s}\rangle_0\right) f\left(-\langle E_{\mathbf{k}s}\rangle_0\right)\right) \times \\ &\times V(\mathbf{k}-\mathbf{k}_1) V(\mathbf{k}_1-\mathbf{k}_2) \ldots V(\mathbf{k}_{n-1}-\mathbf{q}). \qquad (40)\end{aligned}$$

Because $\varphi_{\mathbf{k}s}$ is small, we can decompose Eq. (33) and represent it in the from of an infinite series containing integer degrees of $\varphi_{\mathbf{k}s}$:

$$\langle n_{\mathbf{k}s}\rangle = f\left(\langle E_{\mathbf{k}s}\rangle_0\right) + \frac{\beta^2}{2} f_3\left(\langle E_{\mathbf{k}s}\rangle_0\right) \varphi^2_{\mathbf{k}s} + \ldots, \qquad (41)$$

where $f_n(x) = (-1)^{n-1} d^n f(x)/dx^n$ and $f(x)$ is the standard Fermi–Dirac function. Equations (40) and (41) constitute a closed system of equations for the mean value $\langle n_{\mathbf{k}s}\rangle$ and $\varphi_{\mathbf{k}s}$. This system can also be solved by the step-by-step method with a given accuracy in the parameter λ. Keeping only the terms of the order $\varphi^2_{\mathbf{k}s}$, we approximately obtain

$$\begin{aligned}\langle H\rangle = \frac{1}{\lambda^2} \sum_{\mathbf{k}s} \Bigg\{ &\langle n_{\mathbf{k}s}\rangle_0 \langle E_{\mathbf{k}s}\rangle_0 + \\ &+ \left[\frac{\beta^2}{2}\langle E_{\mathbf{k}s}\rangle_0 f_3\left(\langle E_{\mathbf{k}s}\rangle_0\right) - \beta f_2\left(\langle E_{\mathbf{k}s}\rangle_0\right)\right] \varphi^2_{\mathbf{k}s} - \\ &- \lambda\beta^2 \langle n_{\mathbf{k}s}\rangle_0 \frac{1}{2N} \sum_{\mathbf{q}\neq\mathbf{k}} V(\mathbf{k}-\mathbf{q}) f_3\left(\langle E_{\mathbf{q}s}\rangle_0\right) \varphi^2_{\mathbf{q}s} \Bigg\}, \qquad (42)\end{aligned}$$

where $\langle n_{\mathbf{k}s}\rangle_0 = f\left(\langle E_{\mathbf{k}s}\rangle_0\right)$.

4. THE GROUND-STATE ENERGY AND HEAT CAPACITY OF THE HIGH-DENSITY ELECTRON GAS AT ZERO TEMPERATURE

In the general expressions obtained for the high-density electron gas ($r_s \to 0$), we keep the terms up to the order λ^2. Using decomposition (40), we present the function $\varphi^2_{\mathbf{k}s}$ in the approximate form

$$\varphi^2_{\mathbf{k}s} \approx \left(\frac{\lambda}{N}\right)^2 \sum_{\mathbf{q}\neq\mathbf{k}} V^2(\mathbf{k}-\mathbf{q}) f_2(\varepsilon_{\mathbf{q}}). \qquad (43)$$

We also note that the expression

$$\frac{1}{N}\sum_{\mathbf{k}} \ldots = \frac{1}{(2\pi)^3} \int \ldots d^3k \qquad (44)$$

implies that the values of the quadratic fluctuations $\varphi^2_{\mathbf{k}s}$ are proportional to $1/N$ and become negligible, especially in the three-dimensional case. This is an expected result because we are solving a nonlinear system of integral equations in the high-density limit when $\varphi_{\mathbf{k}s} \ll \langle E_{\mathbf{k}s}\rangle$. But the closed system of equations obtained in Sec. 2 is in principle correct for any relation between $\varphi_{\mathbf{k}s}$ and $\langle E_{\mathbf{k}s}\rangle$. For example, in the low-density case ($r_s \to \infty$), when the potential energy exceeds the kinetic energy, the values of fluctuations $\varphi_{\mathbf{k}s}$ become comparable with the mean value of the local field $\langle E_{\mathbf{k}s}\rangle$.

We also note that in Eq. (17), in projecting the given Hamiltonian H' to the number of states $n_{\mathbf{k}s}$, some

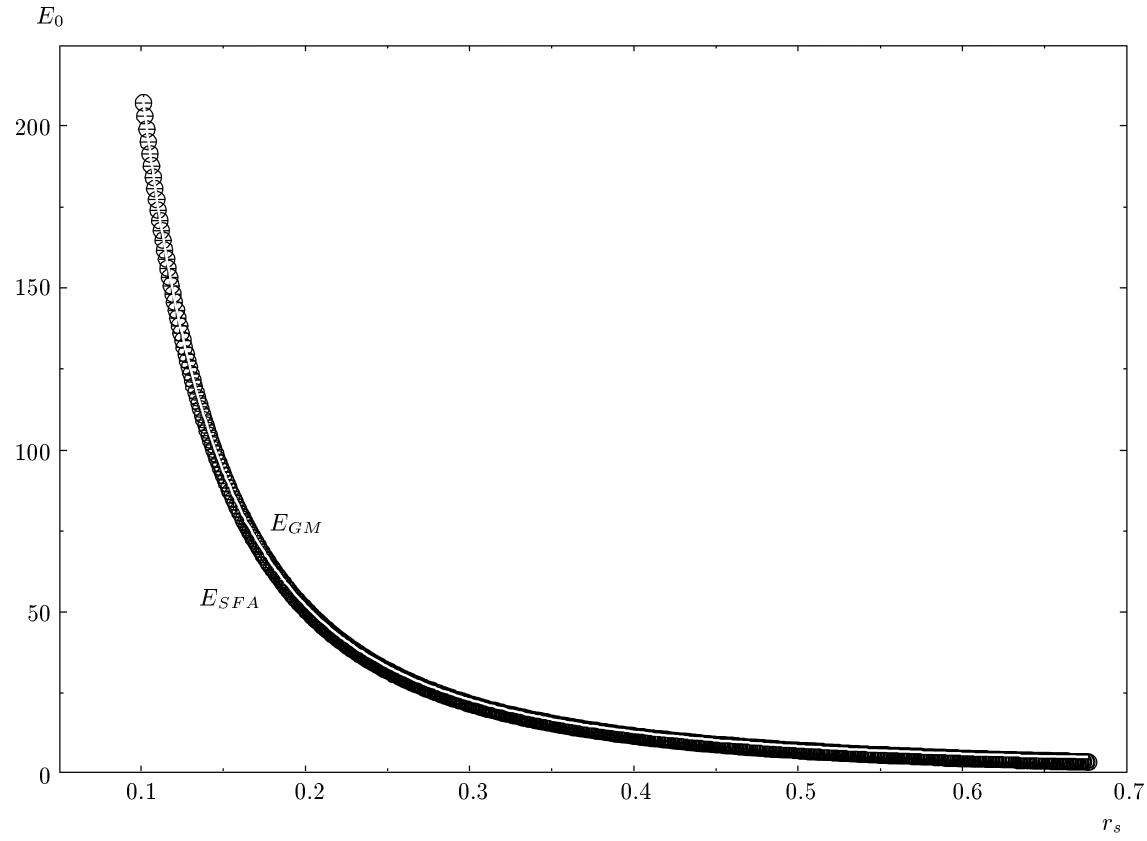

Fig. 1. Comparison of the ground-state energy E_{GM} obtained by Gell-Mann Eq. (1) with E_{SFA} obtained in the SFA framework (Eq. (46)). The data are seen to practically coincide with each other. The relative error (defined as the ratio of the standard deviation taken from the difference $E_{SFA} - E_{GM}$ to the E_{SFA} mean value) does not exceed $0.5\,\%$

terms in the expression for the mean energy are not taken into account. These terms arise because some terms do not commute with the Fermi operators $a^{\dagger}_{\mathbf{k}s}$ ($a_{\mathbf{k}s}$). In our case, these terms are the exchange (or background) energy,

$$\frac{E_{ex}}{N} = -\frac{2}{\lambda N}\sum_{\mathbf{q}} V(\mathbf{q}-\mathbf{q}_0). \tag{45}$$

Following the results in [21], it can be shown that this term is proportional to $-0.916/r_s$.

Taking the above remarks into account and decomposing the mean value $\langle n_{\mathbf{k}s}\rangle_0$ in (41) up to terms of the order λ^2, we can obtain the desired expression for the ground-state energy. The calculation scheme and the evaluation of some terms are presented in Appendix A. The final expression for the ground-state energy (at $T=0$) is

$$\frac{E_0}{N} = \left(\frac{2.21}{r_s^2} - \frac{0.916}{r_s} + \frac{0.781}{\sqrt{r_s}} + 1.556\sqrt{r_s} + {} \right. \\ \left. {} + 0.304\ln r_s - 1.002 + O(r_s)\right)\,\mathrm{Ry}\,. \tag{46}$$

Here we did not take the temperature dependence of the chemical potential $\mu(T) \approx \mu(0)$ into account. Comparing Eq. (1) with Eq. (46), we note that formally they are strongly different. But when plotted on the same figure (Fig. 1), they practically coincide, with the relative fitting error not exceeding $0.5\,\%$. This testifies in favor of a very good coincidence of these expressions if we take into account that different values for the constant and the coefficient before the logarithmic term were chosen in [3–6]. In Fig. 2, we show the relative differences between expressions obtained by other authors [1, 3, 6] and the SFA expression. We compare the previous expressions with (46) for only one reason. The expressions obtained by other authors contain uncontrollable errors (because of a separate summation of a

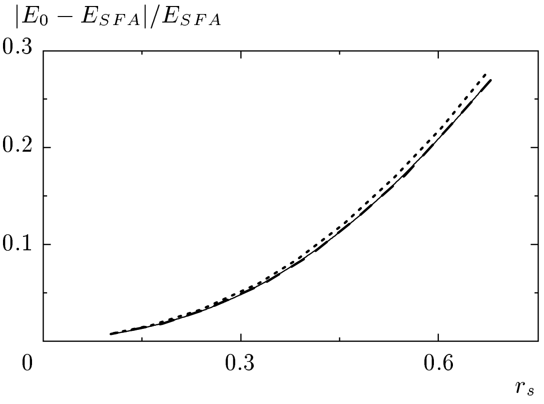

Fig. 2. The relative expressions $|E_0 - E_{SFA}|/E_{SFA}$ for the ground-state energy obtained by Gell-Mann [1], Talman [3], and Porter [6] (solid, dashed, and dotted curves, respectively). The relative deviations of data obtained by the first two authors practically coincide with each other. Visible deviations at relatively large r_s are observed for Porter's results. All relative differences increase with increasing r_s

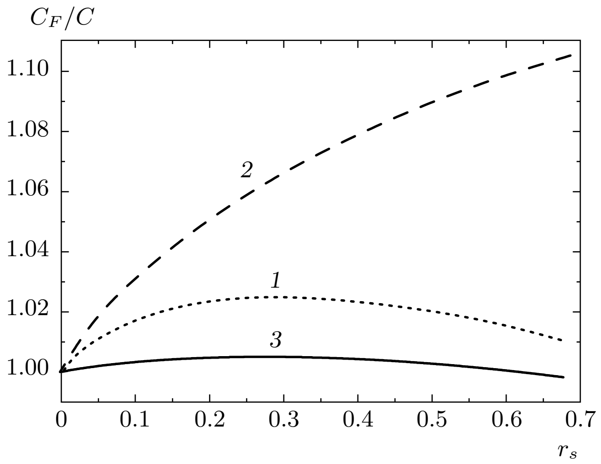

Fig. 3. The relative heat capacities C_F/C obtained by Gell-Mann [1] and Pines [4] (curves *1* and *2*, respectively) compared with the data obtained in the SFA framework (curve *3*). The deviations are noticeable. In the SFA method, we have the minimal deviations for all range of r_s compared with the result for ideal Fermi gas

certain class of diagrams). In our case, we made only one assumption (21) and this supposition is controllable, with the ratio $\varphi_{\mathbf{k}s}/\langle E_{\mathbf{k}s}\rangle \ll 1$.

Taking the derivative of the mean energy with respect to temperature yields the heat capacity at low temperatures ($T = 0$). For comparison, it is convenient to represent this result in the form

$$\frac{C_F}{C} = 1 + 0.055 r_s - 0.070 r_s^{3/2} + O(r_s^2), \qquad (47)$$

where C_F is the heat capacity of the noninteracting Fermi gas. We compare our result with similar results of other authors [19, 20] in Fig. 3. We observe more essential discrepancies than in Fig. 1. But we again consider our result more accurate because it contains the minimal value of the error expressed by Eq. (21). Within the SFA method, it is rather easy to find the terms of higher orders in r_s. For this, it suffices to keep not only quadratic terms proportional to λ^2 in the decomposition of the mean energy containing $\langle n_{\mathbf{k}s}\rangle_0$. Keeping the terms proportional to λ^3 in this λ-decomposition, we can find terms of the respective order r_s and r_s^2 in the expressions for the mean energy and heat capacity. Additional terms appearing in the decomposition for the mean energy are given in Appendix A. The expressions for the ground-state energy and heat capacity then become

$$\frac{E_0}{N} = \left[\frac{2.21}{r_s^2} - \frac{0.916}{r_s} + \frac{0.781}{\sqrt{r_s}} + 1.556\sqrt{r_s} + \right.$$
$$+ 0.304 \ln r_s - 1.002 + (-1.644 + 0.538 \ln r_s) r_s +$$
$$\left. + O(r_s^{3/2})\right] \mathrm{Ry}, \qquad (48)$$

$$\frac{C_F}{C} = 1 + 0.055 r_s - 0.070 r_s^{3/2} + [-0.026 +$$
$$+ 0.012 \ln r_s - 0.007 \ln^2 r_s]\, r_s^2 + O(r_s^2). \qquad (49)$$

The contributions of the third-order terms with respect to expression (46) are shown in Fig. 4. It can be seen that the third-order terms are becoming more essential as the parameter r_s increases.

5. RESULTS AND DISCUSSION

We have demonstrated the calculation of the correlation energy in the framework of the SFA method. The calculation scheme is very simple but leads finally to the solution of a nonlinear system of integral equations. This system of equations, based on only one assumption (21), allows easily calculating the desired correlation functions and obtaining the expressions for the ground state energy and heat capacity that are in accordance with results obtained by other authors. It is interesting to note that the simplest decoupling scheme (21) admits some generalization given in (9). The more

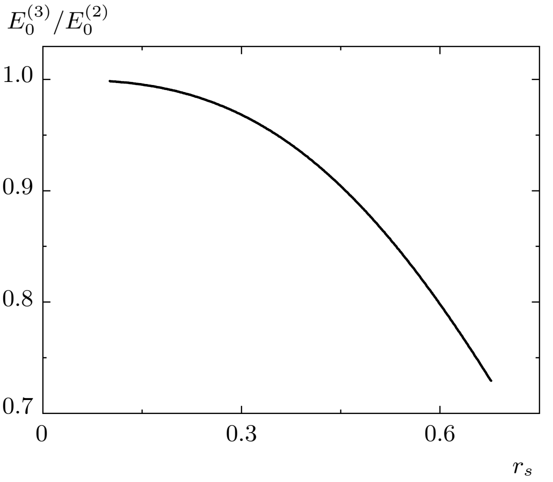

Fig. 4. The influence of third-order corrections involved in Eq. (48) with respect to Eq. (46). The contributions of the third-order terms are seen to become essential as the parameter r_s increases

general scheme in (9) allows considering an asymmetric spectrum of the local field operator and opens new possibilities in obtaining more general results. This generalization for the Fermi gas and other many-body systems with a strong interaction between particles merits a separate research.

APPENDIX A

To find an analytic expression for the mean energy, it is necessary to realize the decomposition of the expression for $\langle n_{\mathbf{k}s}\rangle_0$ in (42) with respect to λ and to take into account that the fluctuation terms $\varphi_{\mathbf{k}s}$ are proportional to $1/N$ in the case of high density. In this case, keeping the terms proportional to λ^3 in the corresponding decomposition, we write the expression for $\langle n_{\mathbf{k}s}\rangle_0$ as

$$\langle n_{\mathbf{k}s}\rangle_0 = f(\varepsilon_{\mathbf{k}}) + \frac{\lambda}{N}\beta f_2(\varepsilon_{\mathbf{k}}) \sum_{\mathbf{q}\neq\mathbf{k}} V(\mathbf{k}-\mathbf{q}) f(\varepsilon_{\mathbf{q}}) +$$

$$+ \left(\frac{\lambda}{N}\right)^2 \beta^2 f_2(\varepsilon_{\mathbf{k}}) \Bigg[\left(\frac{1}{2} - f(\varepsilon_{\mathbf{k}})\right) \times$$

$$\times \left(\sum_{\mathbf{q}\neq\mathbf{k}} V(\mathbf{k}-\mathbf{q}) f(\varepsilon_{\mathbf{q}})\right)^2 + \sum_{\mathbf{q}\neq\mathbf{k}} V(\mathbf{k}-\mathbf{q}) f_2(\varepsilon_{\mathbf{q}}) \times$$

$$\times \sum_{\mathbf{q}_1\neq\mathbf{q}} V(\mathbf{q}-\mathbf{q}_1) f(\varepsilon_{\mathbf{q}_1}) \Bigg] +$$

$$+ \left(\frac{\lambda}{N}\right)^3 \beta^3 \Bigg[\frac{1}{6} f_4(\varepsilon_{\mathbf{k}}) \left(\sum_{\mathbf{q}\neq\mathbf{k}} V(\mathbf{k}-\mathbf{q}) f(\varepsilon_{\mathbf{q}})\right)^3 +$$

$$+ f_3(\varepsilon_{\mathbf{k}}) \sum_{\mathbf{q}\neq\mathbf{k}} V(\mathbf{k}-\mathbf{q}) f(\varepsilon_{\mathbf{q}}) \sum_{\mathbf{q}\neq\mathbf{k}} V(\mathbf{k}-\mathbf{q}) f_2(\varepsilon_{\mathbf{q}}) \times$$

$$\times \sum_{\mathbf{q}\neq\mathbf{q}_1} V(\mathbf{q}-\mathbf{q}_1) f(\varepsilon_{\mathbf{q}_1}) + \frac{1}{2} f_2(\varepsilon_{\mathbf{k}}) \times$$

$$\times \sum_{\mathbf{q}\neq\mathbf{k}} V(\mathbf{k}-\mathbf{q}) f_3(\varepsilon_{\mathbf{q}}) \left(\sum_{\mathbf{q}\neq\mathbf{q}_1} V(\mathbf{q}-\mathbf{q}_1) f(\varepsilon_{\mathbf{q}_1})\right)^2 +$$

$$+ f_2(\varepsilon_{\mathbf{k}}) \sum_{\mathbf{q}\neq\mathbf{k}} V(\mathbf{k}-\mathbf{q}) f_2(\varepsilon_{\mathbf{q}}) \sum_{\mathbf{q}\neq\mathbf{q}_1} V(\mathbf{q}-\mathbf{q}_1) f_2(\varepsilon_{\mathbf{q}_1}) \times$$

$$\times \sum_{\mathbf{q}_2\neq\mathbf{q}_1} V(\mathbf{q}_1-\mathbf{q}_2) f(\varepsilon_{\mathbf{q}_2}) \Bigg]. \qquad (A.1)$$

Then the mean energy can be represented in the form

$$\frac{E}{N} = \frac{\langle H\rangle}{N} = \frac{2\langle H'\rangle}{\lambda^2 N} = \frac{2}{\lambda^2}(E_0+E_1+E_2+E_3), \qquad (A.2)$$

where E_0 contains the zeroth-order terms with respect to λ,

$$E_0 = \frac{2}{N}\sum_{\mathbf{k}} \varepsilon_k f(\varepsilon_k). \qquad (A.3)$$

The term E_1 contains only a pair of first-order terms with respect to λ,

$$E_1 = E_1^{(1)} - E_1^{(2)}, \qquad (A.4)$$

$$E_1^{(1)} = \frac{2\lambda\beta}{N^2}\sum_{\mathbf{k}} \varepsilon_k f_2(\varepsilon_k) \sum_{\mathbf{q}\neq\mathbf{k}} V(\mathbf{k}-\mathbf{q}) f(\varepsilon_q), \qquad (A.5)$$

$$E_1^{(2)} = \frac{2\lambda}{N^2}\sum_{\mathbf{k}} f(\varepsilon_k) \sum_{\mathbf{q}\neq\mathbf{k}} V(\mathbf{k}-\mathbf{q}) f(\varepsilon_q). \qquad (A.6)$$

The expression for E_2 entering (A.2) contains three terms of the second order with respect to λ:

$$E_2 = E_2^{(1)} + E_2^{(2)} - E_2^{(3)}, \qquad (A.7)$$

$$E_2^{(1)} = \frac{\lambda^2\beta^2}{N^3} \times$$

$$\times \sum_{k} \varepsilon_k f_3(\varepsilon_k) \left(\sum_{\mathbf{q}\neq\mathbf{k}} V(\mathbf{k}-\mathbf{q}) f(\varepsilon_q)\right)^2, \qquad (A.8)$$

$$E_2^{(2)} = \frac{2\lambda^2\beta^2}{N^3}\sum_k \varepsilon_k f_2(\varepsilon_k)\sum_{\mathbf{q}\neq\mathbf{k}} V(\mathbf{k}-\mathbf{q}) f_2(\varepsilon_q)\times$$

$$\times\sum_{\mathbf{q}_1\neq\mathbf{q}} V(\mathbf{q}-\mathbf{q}_1) f(\varepsilon_{q_1}), \quad (\text{A.9})$$

$$E_2^{(3)} = \frac{4\lambda^2\beta}{N^3}\sum_k f(\varepsilon_k)\sum_{\mathbf{q}\neq\mathbf{k}} V(\mathbf{k}-\mathbf{q}) f_2(\varepsilon_q)\times$$

$$\times\sum_{\mathbf{q}_1\neq\mathbf{q}} V(\mathbf{q}-\mathbf{q}_1) f(\varepsilon_{q_1}). \quad (\text{A.10})$$

Expression for E_3 in (A.2) already contains six third-order terms:

$$E_3 = E_3^{(1)}+E_3^{(2)}+E_3^{(3)}+E_3^{(4)}-E_3^{(5)}-E_3^{(6)}, \quad (\text{A.11})$$

$$E_3^{(1)} = \frac{\lambda^3\beta^3}{3N^4}\times$$

$$\times\sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} f_4(\varepsilon_{\mathbf{k}})\left(\sum_{\mathbf{q}\neq\mathbf{k}} V(\mathbf{k}-\mathbf{q}) f(\varepsilon_{\mathbf{q}})\right)^3, \quad (\text{A.12})$$

$$E_3^{(2)} = \frac{2\lambda^3\beta^3}{N^4}\sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} f_3(\varepsilon_{\mathbf{k}})\sum_{\mathbf{q}'\neq\mathbf{k}} V(\mathbf{k}-\mathbf{q}') f(\varepsilon_{\mathbf{q}'})\times$$

$$\times\sum_{\mathbf{q}\neq\mathbf{k}} V(\mathbf{k}-\mathbf{q}) f_2(\varepsilon_{\mathbf{q}})\sum_{\mathbf{q}\neq\mathbf{q}_1} V(\mathbf{q}-\mathbf{q}_1) f(\varepsilon_{\mathbf{q}_1}), \quad (\text{A.13})$$

$$E_3^{(3)} = \frac{\lambda^3\beta^3}{N^4}\sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} f_2(\varepsilon_{\mathbf{k}})\sum_{\mathbf{q}\neq\mathbf{k}} V(\mathbf{k}-\mathbf{q}) f_3(\varepsilon_{\mathbf{q}})\times$$

$$\times\left(\sum_{\mathbf{q}\neq\mathbf{q}_1} V(\mathbf{q}-\mathbf{q}_1) f(\varepsilon_{\mathbf{q}_1})\right)^2, \quad (\text{A.14})$$

$$E_3^{(4)} = \frac{2\lambda^3\beta^3}{N^4}\sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} f_2(\varepsilon_{\mathbf{k}})\sum_{\mathbf{q}\neq\mathbf{k}} V(\mathbf{k}-\mathbf{q}) f_2(\varepsilon_{\mathbf{q}})\times$$

$$\times\sum_{\mathbf{q}_1\neq\mathbf{q}} V(\mathbf{q}-\mathbf{q}_1) f_2(\varepsilon_{\mathbf{q}_1})\times$$

$$\times\sum_{\mathbf{q}_2\neq\mathbf{q}_1} V(\mathbf{q}_1-\mathbf{q}_2) f(\varepsilon_{\mathbf{q}_2}), \quad (\text{A.15})$$

$$E_3^{(5)} = \frac{\lambda^3\beta^2}{N^4}\sum_{\mathbf{k}} f(\varepsilon_{\mathbf{k}})\sum_{\mathbf{q}\neq\mathbf{k}} V(\mathbf{k}-\mathbf{q}) f_3(\varepsilon_{\mathbf{q}})\times$$

$$\times\left(\sum_{\mathbf{q}_1\neq\mathbf{q}} V(\mathbf{q}-\mathbf{q}_1) f(\varepsilon_{\mathbf{q}_1})\right)^2, \quad (\text{A.16})$$

$$E_3^{(6)} = \frac{2\lambda^3\beta^2}{N^4}\sum_{\mathbf{k}} f(\varepsilon_{\mathbf{k}})\sum_{\mathbf{q}\neq\mathbf{k}} V(\mathbf{k}-\mathbf{q}) f_2(\varepsilon_{\mathbf{q}})\times$$

$$\times\sum_{\mathbf{q}_1\neq\mathbf{q}} V(\mathbf{q}-\mathbf{q}_1) f_2(\varepsilon_{\mathbf{q}_1})\times$$

$$\times\sum_{\mathbf{q}_2\neq\mathbf{q}_1} V(\mathbf{q}_1-\mathbf{q}_2) f(\varepsilon_{\mathbf{q}_2}). \quad (\text{A.17})$$

To clarify the basic calculation scheme, we consider the calculation of the term $E_2^{(2)}$ (determined by expression (A.9)) in detail. We first replace the summation by integration in accordance with (44), also using definition (13) for the potential $V(\mathbf{k}-\mathbf{q})$:

$$E_2^{(2)} = \frac{2\lambda^2\beta^2}{(2\pi)^9}\int \varepsilon_k f_2(\varepsilon_k)\, d^3k \int V(\mathbf{k}-\mathbf{q}) f_2(\varepsilon_q)\, d^3q\times$$

$$\times\int V(\mathbf{q}-\mathbf{q}_1) f(\varepsilon_{q_1})\, d^3q_1 =$$

$$= \frac{\lambda^2\beta^2}{\pi^4}\int_0^\infty \varepsilon_k f_2(\varepsilon_k)\, k^2 dk \int_{-1}^{1} d\nu_1 \times$$

$$\times\int_0^\infty \frac{f_2(\varepsilon_q) q^2 dq}{k^2+q^2-2kq\nu_1+q_0^2}\int_{-1}^{1} d\nu_2 \times$$

$$\times\int_0^\infty \frac{f(\varepsilon_{q_1}) q_1^2 dq_1}{q_1^2+q^2-2q_1 q\nu_2+q_0^2} = \frac{\lambda^2\beta^2}{4\pi^4}\times$$

$$\times\int_0^\infty \varepsilon_k f_2(\varepsilon_k) k\, dk \int_0^\infty f_2(\varepsilon_q)\ln\left|\frac{(k+q)^2+q_0^2}{(k-q)^2+q_0^2}\right| dq \times$$

$$\times\int_0^\infty f(\varepsilon_{q_1})\ln\left|\frac{(q_1+q)^2+q_0^2}{(q_1-q)^2+q_0^2}\right| q_1 dq_1. \quad (\text{A.18})$$

Changing the variables as

$$\frac{k^2}{2} = x\frac{q^2}{2} = y\frac{q_1^2}{2} = z,$$

we obtain

$$E_2^{(2)} = \frac{\lambda^2\beta^2}{4\sqrt{2}\,\pi^4}\int_0^\infty f_2\left(\frac{x-\mu}{T}\right) dx \int_0^\infty f_2\left(\frac{y-\mu}{T}\right) dy\times$$

$$\times\int_0^\infty f_1\left(\frac{z-\mu}{T}\right) h(x,y,z)\, dz, \quad (\text{A.19})$$

where

$$h(x,y,z) = \frac{x}{\sqrt{y}} \ln\left(\frac{\left(\sqrt{x}+\sqrt{y}\right)^2+\varepsilon_0}{\left(\sqrt{x}-\sqrt{y}\right)^2+\varepsilon_0}\right) \times \\ \times \ln\left(\frac{\left(\sqrt{y}+\sqrt{z}\right)^2+\varepsilon_0}{\left(\sqrt{y}-\sqrt{z}\right)^2+\varepsilon_0}\right) \qquad \text{(A.20)}$$

with $\varepsilon_0 = q_0^2/2$. Using relations (B.2) and (B.3) in Appendix B, we obtain the expression

$$E_2^{(2)} = \frac{\lambda^2\beta^2}{4\sqrt{2}\,\pi^4} \int\limits_0^\infty f_2\left(\frac{x-\mu}{T}\right) dx \int\limits_0^\infty f_2\left(\frac{y-\mu}{T}\right) dy \times \\ \times \left\{ \int\limits_0^\mu h(x,y,z)\,dz + \frac{\pi^2}{6}\, h'_z(x,y,\mu) T^2 \right\} = \\ = \frac{\lambda^2\beta}{4\sqrt{2}\,\pi^4} \int\limits_0^\infty f_2\left(\frac{x-\mu}{T}\right) dx \times \\ \times \left\{ \int\limits_0^\mu h(x,\mu,z)\,dz + \frac{\pi^2}{6} \left(h'_z(x,\mu,\mu) \; + \right.\right. \\ \left.\left. + \int\limits_0^\mu h''_{yy}(x,\mu,z)\,dz \right) T^2 \right\} = \frac{\lambda^2}{4\sqrt{2}\,\pi^4} \times \\ \times \left\{ \int\limits_0^\mu h(\mu,\mu,z)\,dz + \frac{\pi^2}{6} \left[h'_z(\mu,\mu,\mu) \; + \right.\right. \\ \left.\left. + \int\limits_0^\mu \left(h''_{xx}(\mu,\mu,z) + h''_{yy}(\mu,\mu,z) \right) dz \right] T^2 \right\}. \qquad \text{(A.21)}$$

Performing simple but cumbersome calculations, we finally obtain

$$E_2^{(2)} = \frac{\lambda^2}{4\sqrt{2}\,\pi^4} \times \\ \times \left\{ \frac{p\varepsilon_0^{3/2}}{2} \left[p + \ln(1+p^2) - 2p \operatorname{arctg} p \right] \ln(1+p^2) \; + \right. \\ + \frac{\pi^2}{6\sqrt{\varepsilon_0}} \left[-\frac{2p(p^4-p^2-4)}{(1+p^2)^2} + \frac{4(p^2+2)(p^2-1)}{(1+p^2)^2} \operatorname{arctg} p \; - \right. \\ - \frac{2(4p^4+9p^2+3)}{p(1+p^2)^2} \ln(1+p^2) + \frac{2(p^2+3)}{p^3} \times \\ \left.\left. \times \ln^2(1+p^2) \right] T^2 + \ldots \right\}, \qquad \text{(A.22)}$$

where $p = 2\sqrt{\mu/\varepsilon_0}$. Other terms involved in the basic expression (A.2) are calculated similarly.

We recall that all mean values for the sought thermodynamic functions were obtained using the Hamiltonian H'. To pass to H, it is necessary to use the relation

$$\langle \hat{A}(\beta) \rangle = Q^{-1} \operatorname{Sp} \left[\exp\left(-H\beta\right) \hat{A} \right] = \\ = Q^{-1} \operatorname{Sp} \left[\exp\left(-H' 2\beta/\lambda^2\right) \hat{A} \right] = \\ = \left\langle \hat{A}\left(\frac{2}{\lambda^2}\beta\right) \right\rangle \qquad \text{(A.23)}$$

and the replacements

$$\beta \to \tilde{\beta} = \frac{2}{\lambda^2}\beta, \quad T \to \tilde{T} = \frac{\lambda^2}{2} T. \qquad \text{(A.24)}$$

APPENDIX B

We consider the integrals

$$I_i(T,\mu) = \int\limits_0^\infty g(x) f_i\left(\frac{x-\mu}{T}\right) dx, \\ i = 1,2,3,4, \qquad \text{(B.1)}$$

where $g(x)$ defines a smooth function, $f_i(x) = (-1)^{i-1} d^{i-1} f(x)/dx^{i-1}$, and $f(x)$ is the standard Fermi–Dirac function. In the case $i = 1$, at low temperatures, we have the well-known low-temperature decomposition

$$I_1(T,\mu) = \int\limits_0^\mu g(x)\,dx + \frac{\pi^2}{6} g'(\mu) T^2 + \ldots \qquad \text{(B.2)}$$

Differentiating both parts of this expression with respect to the chemical potential μ, we obtain the useful expressions

$$I_2(T,\mu) = g(\mu)T + \frac{\pi^2}{6} g''(\mu) T^3 + \ldots, \qquad \text{(B.3)}$$

$$I_3(T,\mu) = g'(\mu)T^2 + \frac{\pi^2}{6} g'''(\mu) T^4 + \ldots, \qquad \text{(B.4)}$$

$$I_4(T,\mu) = g''(\mu)T^3 + \frac{\pi^2}{6} g^{(\mathrm{IV})}(\mu) T^5 + \ldots, \qquad \text{(B.5)}$$

which were used in evaluating expression (A.21).

REFERENCES

1. M. Gell-Mann and K. A. Brueckner, Phys. Rev. **106**, 364 (1957).

2. E. P. Wigner, Phys. Rev. **46**, 1002 (1934).

3. J. D. Talman, Phys. Rev. A **10**, 1333 (1974).

4. D. Pines, Phys. Rev. **92**, 626 (1953).

5. W. Macke, Z. Naturforsch. **5a**, 192 (1950).

6. J. C. Porter, Phys. Rev. **80**, 205102 (2009).

7. W. J. Carr, Jr. and A. A. Maradudin, Phys. Rev. **133**, A371 (1964).

8. R. Bellman, *Introduction to Matrix Analysis*, McGraw-Hill, New York (1960).

9. M. P. Zhelifonov, Teor. Mat. Fiz. **8**, 401 (1971).

10. R. R. Nigmatullin, A. A. Khamzin, and H. B. Ghassib, Int. J. Theor. Phys. **39**, 405 (2000).

11. R. R. Nigmatullin, A. A. Khamzin, and H. B. Ghassib, Phys. Rev. E **61**, 3441 (2000).

12. R. R. Nigmatullin, A. A. Khamzin, and H. B. Ghassib, Sol. St. Comm. **113**, 257 (2000).

13. M. K. Al-Sugheir, H. B. Ghassib, and R. R. Nigmatullin, Int. J. Theor. Phys. **40**, 1033 (2001).

14. V. V. Loskutov, G. I. Mironov, and R. R. Nigmatullin, Fiz. Nizkikh Temp. **22**, 282 (1996); G. I. Mironov, Fiz. Tverdogo Tela **41**, 951 (1999).

15. G. I. Mironov, Fiz. Tverdogo Tela **48**, 1299 (2006).

16. G. I. Mironov, Fiz. Tverdogo Tela **49**, 527 (2007).

17. A. I. Murzashev, Zh. Eksp. Teor. Fiz. **135**, 122 (2009).

18. M. K. Al-Sugheir and H. B. Ghassib, Int. J. Theor. Phys. **31**, 705 (2002).

19. A. S. Sandouqa, M. K. Al-Sugheir, and H. B. Ghassib, Int. J. Theor. Phys. **45**, 159 (2006).

20. N. M. Ghulam, M. K. Al-Sugheir, and H. B. Ghassib, Int. J. Theor. Phys. **47**, 2326 (2008).

21. M. N. March, W. H. Young, and S. Sampanthar, *The Many-Body Problem in Quantum Mechanics*, Cambridge University Press (1967).

22. M. Gell-Mann, Phys. Rev. **106**, 369 (1957).

23. D. Pines, Sol. St. Phys. **1**, 367 (1955).