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# GENERATION OF DISPLACED SQUEEZED SUPERPOSITIONS OF COHERENT STATES

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We study the method of generation of states that approximate superpositions of large-amplitude coherent states (SCSs) with high fidelity in free-traveling fields. Our approach is based on the representation of an arbitrary single-mode pure state, and SCSs in particular, in terms of displaced number states with an arbitrary displacement amplitude. The proposed optical scheme is based on alternation of photon additions and displacement operators (in the general case, $N$ photon additions and $N-1$ displacements are required) with a seed coherent state to generate both even and odd displaced squeezed SCSs regardless of the parity of the used photon additions. It is shown that the optical scheme studied is sensitive to the seed coherent state if the other parameters are unchanged. Output states can approximate either even squeezed SCS or odd SCS shifted relative to each other by some value. This allows constructing a local rotation operator, in particular, the Hadamard gate, which is a mainframe element for quantum computation with coherent states. We also show that three-photon additions with two intermediate displacement operators are sufficient to generate even displaced squeezed SCS with the amplitude 1.7 and fidelity more than 0.99. The effects deteriorating the quality of output states are considered.

## 1. INTRODUCTION

Laboratory realization of schemes for the generation of specific nonclassical quantum states is one of the most exciting challenges to the researches. It is well known that the range of applications of the nonclassical states of light extends from precision measurements [1] to quantum lithography [2] and quantum information processing [3]. Most optical proposals for quantum information processing require nonclassical states in propagating optical modes that can be easily manipulated by means of linear optics such as beam splitters, phase shifters, and so on. The states generated in cavity experiments are not so useful for the quantum information processing because the field is confined and can be probed only indirectly.

One of such remarkable examples of nonclassical states is given by Schrödinger-cat-like states [4]. We are interested in the states realized in harmonic oscillators and often called superpositions of coherent states (SCSs). The superposition of two coherent (i. e., most classical) states with opposite phases [5] exhibits both some properties similar to those of statistical mixtures and typical interference features. For example, one of the quadrature-component distributions of SCSs shows two peaks that change their mutual distance depending on the amplitude of coherent fields, whereas an oscillatory behavior is observed in another quadrature-component distribution [5]. We note that such behavior mainly occurs only for large amplitudes of coherent states composing SCSs when macroscopically distinguishable outcomes are observed by a homodyne measurement [6]. We also note negative values in the Wigner functions of the SCSs [7], which are manifestation of their nonclassical properties.

In spite of the manifold usefulness of the SCSs, there has not been much progress in the generation of SCSs until recently. Schemes have been proposed to generate such SCSs using strong nonlinearities [8] or photon number resolving detectors [9], which are hardly feasible with the current level of technology. Recently, more realistic schemes have been proposed by differ-

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ent authors [10–12]. For example, the simple observation that an odd SCS with a small amplitude ($\leq 1.2$) is well approximated by squeezed single photon was made in [10]. It was also noted that a squeezed single photon can be obtained by subtracting (or adding) one photon from the pure squeezed vacuum [13]. Theoretical analysis of added/subtracted squeezed vacuum states has been performed in [14]. Single-photon- subtracted squeezed states, which are close to SCSs, were generated in [15]. A squeezed SCS with state size approximately 1.6 was generated and detected in [16]. It may be suited for fundamental tests and quantum information processing despite their squeezing [17]. Subsequent steps were aimed at studying two-photon added/subtracted squeezed vacuum states [18, 19]. A scheme involving time-separated two-photon subtraction to generate large-amplitude SCSs was experimentally demonstrated in [20]. Another remarkable experimental result based on subtracting three photons from a squeezed vacuum was recently presented in [21].

Currently, all the proposals to generate free-traveling Schrödinger-cat-like states are based on use of added/subtracted squeezed vacuum states. Nevertheless, it is interesting to develop a general method of the SCSs generation to apply it to quantum computation with coherent states. It was shown in [22] that an arbitrary single-mode state can be engineered starting from the vacuum by applying a sequence of single-photon additions and displacements. The idea with alternate photon additions and displacements can be adjusted to the SCSs generation [23]. To extend the approach to quantum computation, we propose to use decomposition of the wave functions into series on the displaced number states with arbitrary amplitudes. The decomposition is possible because the set of displaced states is complete and they are orthogonal with respect to an inner product. The use of displaced number states was proposed in dense coding [24] and quantum key distribution [25]. We note that the displaced vacuum is a coherent state, and a displaced single photon was experimentally realized in [26].

## 2. DISPLACED SQUEEZED SCSs IN TERMS OF THE DISPLACED NUMBER STATES

The even and odd regular SCSs are respectively defined as

$$|SCS_{\pm}(\alpha_{SCS})\rangle = \\ = N_{\pm}(\alpha_{SCS})\left(|0,\alpha_{SCS})\rangle \pm |0,-\alpha_{SCS})\rangle\right). \quad (1)$$

Here, $N_{\pm}(\alpha_{SCS}) = 1/\sqrt{2\left[1 \pm \exp(-2|\alpha_{SCS}|^2)\right]}$ are the corresponding normalization factors for the even (+) and odd (−) SCSs and the notation $|0, \pm\alpha_{SCS})\rangle$ for coherent states with amplitudes $\pm\alpha_{SCS}$ is used. We assume $\alpha_{SCS} > 0$ throughout the paper. “Taking $\alpha_{SCS} > 0$ real” means that the field is in phase with the local oscillator that is used for qubit measurement and for making the displacements required for some of the gates. We use the notation $|n,\alpha\rangle = D(\alpha)|n\rangle$ for a displaced $n$-photon state, where $n$ is an arbitrary number and $D(\alpha) = \exp(\alpha a^{+} - \alpha^{*} a)$ is the displacement operator, $a(a^{+})$ is the boson annihilation (creation) operator, and $|n\rangle$ is a number state. In particular, $|0,\alpha_{SCS}\rangle = \hat{D}(\alpha_{SCS})|0\rangle$ is a displaced vacuum state or the same coherent state with the amplitude $\alpha_{SCS}$.

The infinite set of displaced number states $|l,\alpha\rangle$ ($l = 0,1,2,\ldots,\infty$), where $\alpha$ is an arbitrary number, is complete, which allows decomposing any single-mode state with respect to the basic states. We call such a decomposition the $\alpha$-representation of the state. To obtain the $\alpha$-representation of a regular (even or odd) SCSs ($|SCS_{\pm}(\alpha_{SCS},\alpha)\rangle$), we use formulas (A.4) and (A.5) in Appendix A:

$$\begin{aligned}|SCS_{\pm}(\alpha_{SCS},\alpha)\rangle &= N_{\pm}(\alpha_{SCS}) \times \\ &\times \exp\left[-\left(\alpha_{SCS}^2 + |\alpha|^2\right)/2\right] \times \\ &\times \sum_{l=0}^{\infty} \frac{\alpha_{SCS}^{l}}{\sqrt{l!}} \left[\exp(\alpha_{SCS}\alpha^{*})\left(1-\frac{\alpha}{\alpha_{SCS}}\right)^{l} + \right. \\ &\left. + \exp(-\alpha_{SCS}\alpha^{*})(-1)^{l}\left(1+\frac{\alpha}{\alpha_{SCS}}\right)^{l}\right]|l,\alpha\rangle = \\ &= N_{\pm}(\alpha_{SCS})\exp\left[-\left(\alpha_{SCS}^2 + |\alpha|^2\right)/2\right] \times \\ &\times D(\alpha)\sum_{l=0}^{\infty} \frac{\alpha_{SCS}^{l}}{\sqrt{l!}} \left[\exp(\alpha_{SCS}\alpha^{*})\left(1-\frac{\alpha}{\alpha_{SCS}}\right)^{l} \pm \right. \\ &\left. \pm \exp(-\alpha_{SCS}\alpha^{*})(-1)^{l}\left(1+\frac{\alpha}{\alpha_{SCS}}\right)^{l}\right]|l\rangle = \\ &= N_{\pm}(\alpha_{SCS})\exp\left[-\left(\alpha_{SCS}^2 + |\alpha|^2\right)/2\right] \times \\ &\times D(\alpha)\sum_{n=0}^{\infty} a_{\pm n}|l\rangle, \quad (2a)\end{aligned}$$

where $a_{\pm n}$ are the respective amplitudes of the decomposition for even and odd SCSs. Two variables, $\alpha_{SCS}$ and $\alpha$, are used in the notation for an arbitrary $\alpha$-representation of the SCSs $|SCS_{\pm}(\alpha_{SCS},\alpha)\rangle$ in contrast to the direct definition of the SCSs $|SCS_{\pm}(\alpha_{SCS})\rangle$ (Eq. (1)), where $\alpha_{SCS}$ is the SCS amplitude and $\alpha$ is the amplitude of the complete set of displaced number states. In particular, if we take $\alpha = 0$, then we deal with the number state representation (or, equivalently, the 0-representation in our notation) of the SCSs [5]:

$$\begin{gathered}|SCS_+(\alpha_{SCS},0)\rangle = 2N_+(\alpha_{SCS})\times \\ \times\exp\left(-2|\alpha_{SCS}|^2\right)\sum_{n=0}^{\infty}\frac{\alpha_{SCS}^{2n}}{\sqrt{(2n)!}}|2n\rangle, \\ |SCS_-(\alpha_{SCS},0)\rangle = 2N_-(\alpha_{SCS})\times \\ \times\exp\left(-2|\alpha_{SCS}|^2\right)\sum_{n=0}^{\infty}\frac{\alpha_{SCS}^{2n+1}}{\sqrt{(2n+1)!}}|2n+1\rangle.\end{gathered} \tag{2b}$$

Because superpositions (2b) involve either even or odd number states, they are called even and odd SCSs. It is natural to use the same terms for the SCSs in a general $\alpha$-representation with $\alpha \neq 0$ (Eqs. (2a)), although they involve both even and odd displaced number states. Some wave amplitudes $a_{\pm n}$ with $n \geq 1$ are presented in Appendix B.

We next define displaced squeezed (even and odd) SCSs as

$$\begin{gathered}|DSSCS_\pm(\alpha_{SCS},\alpha,r)\rangle = \\ = D(\alpha)S(r)|SCS_\pm(\alpha_{SCS})\rangle = \\ = D(\alpha)S(r)N_\pm(\alpha_{SCS})\times \\ \times\left(|0,\alpha_{SCS}\rangle \pm |0,-\alpha_{SCS}\rangle\right),\end{gathered} \tag{3}$$

where $S(r) = \exp\left[r(a^{+2}-a^2)/2\right]$ is the squeezing operator with $r$ being a squeezing parameter [9–21]. If we take $r = \alpha = 0$, we deal with regular SCSs (Eq. (1)), and if we choose $\alpha = 0$ and $r \neq 0$, then we have squeezed SCSs [10–21]; if we use $r = 0$ and $\alpha \neq 0$, we obtain displaced SCSs. It is well known it is hardly possible to generate regular large-amplitude SCSs with the current level of technology. A natural way to overcome this is to approximate regular (displaced/squeezed) SCSs to any degree of accuracy by some states involving $N+1$ terms

$$|\Psi_{\pm N}\rangle = N_{\pm N}\sum_{n=0}^{N} a_{\pm n}|n\rangle, \tag{4}$$

where $N_{\pm N}$ are the normalization factors for even and odd SCSs and we set $a_{\pm 0} = 1$ for the convenience of calculations. In the general case, there are two main methods for the construction of an arbitrary single-mode finite superposition (4). Both methods are presented in Fig. 1. One is based on alternation of photon additions and displacement operators starting with a seed coherent state, as is shown in Fig. 1*a*. The general description of the method with alternate photon additions and displacement operators and some partial cases of the method are considered in Appendixes C and D. The other approach to the generation of single-mode finite superpositions of number states (4) is presented in Fig. 1*b*. We consider an ideal situation where $m$ photons are added or subtracted from the squeezed coherent states in Fig. 1*b*. At the output we then have the relations

$$\begin{gathered}a^{+m}S(r)|0,\alpha_{In}\rangle = a^{+m}S(r)D(\alpha_{In})|0\rangle = \\ = a^{+m}S^+(-r)D(\alpha_{In})S(-r)S(r)|0\rangle = \\ = a^{+m}D(\alpha')S(r)|0\rangle = \\ = D(\alpha')D^+(\alpha')a^{+m}D(\alpha')S(r)|0\rangle = \\ = D(\alpha')(a^+ + \alpha'^*)^m S(r)|0\rangle = \\ = D(\alpha')S(r)S^+(r)(a^+ + \alpha'^*)^m S(r)|0\rangle = \\ = D(\alpha')S(r)(a^+\operatorname{ch} r + a\operatorname{sh} r + \alpha'^*)^m|0\rangle = \\ = D(\alpha)S(r)\left[(a^+ + \alpha_{In}'^*)\operatorname{ch} r + (a+\alpha_{In})\operatorname{sh} r\right]^m|0\rangle,\end{gathered} \tag{5a}$$

$$\begin{gathered}a^m S(r)|0,\alpha_{In}\rangle = D(\alpha)S(r)\times \\ \times\left[(a+\alpha_{In})\operatorname{ch} r + (a^+ + \alpha_{In}^*)\operatorname{sh} r\right]^m|0\rangle\end{gathered} \tag{5b}$$

with $\alpha = \alpha_{In}\operatorname{ch} r + \alpha_{In}^*\operatorname{sh} r$, where $\alpha_{In}$ is an amplitude of the initial coherent state and we used the relations [27]

$$S^+(r)a^+S(r) = a^+\operatorname{ch} r + a\operatorname{sh} r,$$

$$S^+(r)aS(r) = a\operatorname{ch} r + a^+\operatorname{sh} r.$$

It follows from (5a) and (5b) that $|\Psi_{\pm N}\rangle$ (Eq. (4)) has the form

$$|\Psi_{\pm N}\rangle = \left[(a^+ + \alpha_{In}^*)\operatorname{ch} r + (a+\alpha_{In})\operatorname{sh} r\right]^m|0\rangle \tag{5c}$$

for the $m$-photon added squeezed coherent state and

$$|\Psi_{\pm N}\rangle = \left[(a+\alpha_{In})\operatorname{ch} r + (a^+ + \alpha_{In}^*)\operatorname{sh} r\right]^m|0\rangle \tag{5d}$$

for the $m$-photons subtracted squeezed coherent state. States (5c) and (5d) are not normalized.

We especially focus our attention on the approach based on alternation of the photon additions and displacement operators (Fig. 1*a*) to generate "half-finished products" $|\Psi_{\pm N}\rangle$ (Eq. (4)) for the SCSs in application to coherent quantum computing, leaving the study of the optical scheme shown in Fig. 1*b* to a future investigation. Nevertheless, general features of the method with alternate photon additions and displacement operators are applicable to the $m$-photon added/subtracted squeezed coherent states $a^{+m}S(r)|0,\alpha_{In}\rangle$ and $a^m S(r)|0,\alpha_{In}\rangle$.

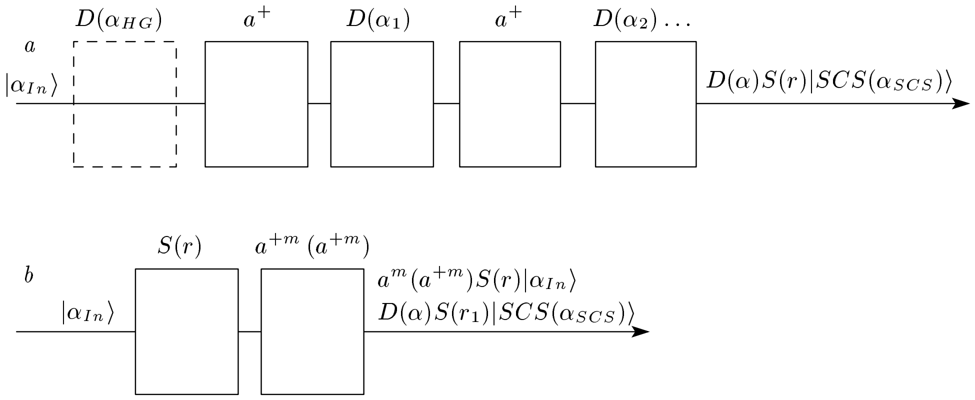


**Fig. 1.** Diagram of the optical scheme for construction of the even and odd displaced squeezed SCSs with high fidelity. ($a$) The optical scheme involves a set of alternate photon additions and displacement operators with the corresponding amplitudes. The output of the scheme is sensitive to the input coherent state $\alpha_{In}$. The displacement operator $D(\beta)$ is used to obtain the corresponding $\alpha_{In}$ if the input is either $|0, \alpha_{HG}\rangle$ or $|0, -\alpha_{HG}\rangle$ where $\alpha_{HG}$ is the Hadamard-gate state amplitude. ($b$) The optical scheme consists of the input squeezed coherent state $S(r)|0, \alpha_{In}\rangle$ subject to either the $m$-photon subtraction $a^m$ or the $m$-photon addition $a^{+m}$

The fidelity between arbitrary states $F = |\langle\varphi_t|\varphi\rangle|$ is a measure of how close a state $|\varphi\rangle$ is to the target state $|\varphi_t\rangle$. It is unity when the two states are identical, and is zero when the two states are orthogonal to each other. In our case, $|\varphi\rangle$ can be $|\Psi_{\pm N}\rangle$ (Eq. (4)) and $|\varphi_t\rangle$ can be either regular SCSs or displaced squeezed SCSs (DSSCSs)

$$F_{\pm N} = |\langle SCS_{\pm N}(\alpha_{SCS})S^+(r)D^+(\alpha)|\Psi_{\pm N}\rangle|^2.$$

The choice of the input conditions may be determined by the aims. The development directions for the generation of SCSs may be as follows. Occasionally, SCSs with a large amplitude $\alpha_{SCS} \geq 2$ have to be generated for macroscopic tests of quantum theory. For quantum information processing, it is important to construct SCSs with higher fidelities $F > 0.99$. The ideal case is to seek optimal conditions to generate SCSs with larger amplitudes and higher fidelities.

It is well known there is no fundamental reason for the restriction to physical systems with two-dimensional Hilbert spaces for encoding. It may be more natural in some contexts to encode logical states as a superposition of a large number of basis states, as is the case with quantum computations involving coherent optical states. We can therefore define a local operation $R(Q)$ as

$$R(Q)|0, \alpha\rangle = \cos Q|0, \alpha\rangle + \sin Q|0, -\alpha\rangle, \tag{6a}$$

$$R(Q)|0, -\alpha\rangle = \sin Q|0, \alpha\rangle - \cos Q|0, -\alpha\rangle, \tag{6b}$$

which is nonunitary due to the nonorthogonality of $|0, \alpha\rangle$ and $|0, -\alpha\rangle$. But $R(Q)$ becomes approximately unitary when the overlap between the two coherent states, $\langle 0, \alpha|0, -\alpha\rangle = \exp(-2|\alpha|^2)$, tends to zero. We note that this overlap rapidly tends to zero as $\alpha$ increases. If we take $Q = \pi/4$, then the local operation $R(Q)$ becomes an Hadamard gate that transforms $|0, \alpha\rangle$ to the even SCS,

$$R(Q = \pi/4) = |0, \alpha\rangle = |0, \alpha\rangle + |0, -\alpha\rangle, \tag{6c}$$

and $|0, -\alpha\rangle$ to the odd SCS,

$$R(Q = \pi/4) = |0, -\alpha\rangle = |0, \alpha\rangle - |0, -\alpha\rangle. \tag{6d}$$

Here, we omit the normalization factor. The Hadamard gate is a mainframe elementary quantum gate used for performance of quantum tasks with coherent states. To achieve an arbitrary 1-bit rotation, we must apply $U(\pi/4)$ and $U(-\pi/4)$, which are respective rotations by $\pi/2$ and $-\pi/2$ around the $x$ axis. The unitary operations $U(\pi/4)$ and $U(-\pi/4)$ can be realized using a Kerr nonlinear interaction [5]. The interaction Hamiltonian of a single-mode Kerr nonlinearity is

$$H_{NL} = \hbar\Omega(a^+a)^2,$$

where $\Omega$ is the Kerr nonlinearity strength. When the interaction time $t$ in the medium is $\pi/\Omega$, coherent states evolve (see Eqs. (6c) and (6d)) up to a relative phase shift by $\pi/2$. An optical fiber is the well-known example of a medium with a Kerr nonlinearity, but only statistical mixing of the states $|0, \alpha\rangle$ and $|0, -\alpha\rangle$ (instead of (6c) and (6d)) occurs at the output of a long fiber due to decoherence effects when optical beams propagate inside the fiber. This may be main drawback for the development of quantum protocols with coherent optical states. In the general case, it is natural to speak about a "rotated" superposition of coherent states (6a) and (6b) instead of using the terms even/odd SCSs, because the states in Eqs. (6c) and (6d) are a particular case of the rotation operator $R(Q)$. In our notation, the $\alpha$-representation of a rotated superposition of coherent states becomes

$$|SCS_Q(\alpha_{SCS},\alpha)\rangle = N_Q(\alpha_{SCS}) \times \\ \times \exp\left[-\left(\alpha_{SCS}^2+|\alpha|^2\right)/2\right] D(\alpha)\sum_{l=0}^{\infty}\frac{\alpha_{SCS}^l}{\sqrt{l!}} \times \\ \times \left(\cos Q\exp(\alpha_{SCS}\alpha^*)\left(1-\frac{\alpha}{\alpha_{SCS}}\right)^l + \right. \\ \left. + \sin Q\exp(-\alpha_{SCS}\alpha^*)(-1)\left(1+\frac{\alpha}{\alpha_{SCS}}\right)^l\right)|l\rangle, \quad (7)$$

where

$$N_Q(\alpha_{SCS}) = \left\{\cos^2 Q + \sin^2 Q + \right. \\ \left. + \cos Q\sin Q\left[1+\exp\left(-2|\alpha_{SCS}|^2\right)\right]\right\}^{-1/2}$$

is a normalization factor and $Q = \pm\pi/4$ respectively corresponds to the even and odd SCSs.

## 3. GENERATION OF SCSs AND APPLICATION OF THE METHOD TO THE CONSTRUCTION OF ELEMENTARY QUANTUM GATES

We analyze all possible cases with $N = 1, 2, 3$ photon additions. An optical scheme with only one photon addition is the simplest as can be seen from Fig. 1*a*, and this scheme allows generating SCSs of moderate amplitudes. Indeed, we have (see Fig. 1*a*)

$$a^+|0,\alpha_{In}\rangle = a^+D(\alpha_{In})|0\rangle = \\ = D(\alpha_{In})D^+(\alpha_{In})a^+D(\alpha_{In})|0\rangle = D(\alpha_{In})(a^+ + \alpha_{In}^*)\times \\ \times|0\rangle = D(\alpha_{In})\left(|1\rangle + \alpha_{In}^*|1\rangle\right), \quad (8a)$$

where we used the relation $D^+(\alpha)a^+D(\alpha) = a^+ + \alpha^*$. The "half-finished product" $|\Psi_{\pm1}\rangle$ (Eq. (4)) for the optical scheme with one photon addition (Fig. 1*a*) is then given by

$$|\Psi_{\pm1}\rangle = \frac{|0\rangle + a_{\pm1}|1\rangle}{\sqrt{1+|a_{\pm1}|^2}} = \frac{|0\rangle + |1\rangle/\alpha_{In}^*}{\sqrt{1+1/|\alpha_{In}|^2}}, \quad (8b)$$

if $\alpha_{In} = 1/a_{\pm1}^*$. Hence, output (8a) is a single-photon added coherent state (SPACS) with the amplitude $\alpha_{In}$,

$$|SPACS(\alpha_{In})\rangle = D(\alpha_{In})\frac{|0\rangle + |1\rangle/\alpha_{In}^*}{\sqrt{1+1/|\alpha_{In}|^2}}, \quad (8c)$$

and is the simplest approximation of the SCSs. Indeed, the fidelity between SPACS and DSSCSs is given by

$$F_{\pm1} = \left|\langle SCS_\pm(\alpha_{SCS})S^+(r)D^+(\alpha)\times \right. \\ \left. \times|SPACS(\alpha_{In})\rangle\right|^2 = \\ = |\langle SCS_\pm(\alpha_{SCS})D(\gamma_{\pm1})S(-r)|SPACS(\alpha_{In})\rangle|^2 = \\ = |\langle SCS_\pm(\alpha_{SCS},\gamma_{\pm1})S(-r)|SPACS(\alpha_{In})\rangle|^2, \quad (9)$$

where $\gamma_{\pm1} = \operatorname{ch} r(\alpha_{In}-\alpha) - \operatorname{sh} r(\alpha_{In}-\alpha)^*$ and $|SCS_\pm(\alpha_{SCS},\gamma_{\pm1})\rangle$ is the $\gamma_{\pm1}$-representation of the SCSs in Eq. (7), where $Q = \pm\pi/4$ is chosen. Here, the parameters $\alpha_{In}$, $\alpha$, and $r$ depend on the rotation angle $Q$ (Eq. (7)), but we omit their subscripts in order to not complicate the notation. It is possible numerically to seek the parameters $a_{\pm1}$, $\gamma_{\pm1}$, and $r$ with which the fidelity in (9) takes the highest possible value. For the even SCS, these values are

$$\alpha_{SCS} = 1, \quad F_{+1} = 0.962444,$$

$$\alpha_{In} = 1.2464i, \quad \alpha = 1.78864i,$$

$$r = -0.445031, \quad \alpha_{SCS} = 1.1,$$

$$F_{+1} = 0.943626, \quad \alpha_{In} = 1.05247i,$$

$$\alpha = 1.6373i, \quad r = -0.491368, \quad \alpha_{SCS} = 1.2,$$

$$F_{+1} = 0.92202, \quad \alpha_{In} = 0.900828i,$$

$$\alpha = 1.52202i, \quad r = -0.537234.$$

Wigner functions of the displaced squeezed SPACS with the corresponding parameters and the regular even SCS with $\alpha_{SCS} = 1$ are presented in Fig. 2. The method of calculation is applicable to finding parameters of the optical scheme in Fig. 1*a* to generate an odd SCS. Our calculations are in total agreement with the results in [10, 13]. For example, we have

$$\alpha_{SCS} = 1, \quad F_{-1} = 0.997109,$$

$$\alpha_{In} \approx 0, \quad \alpha \approx 0, \quad r = -0.31257;$$

$$\alpha_{SCS} = 1.1, \quad F_{-1} = 0.994411,$$

$$\alpha_{In} \approx 0, \quad \alpha \approx 0, \quad r = -0.36893;$$

$$\alpha_{SCS} = 1.2, \quad F_{-1} = 0.99085,$$

$$\alpha_{In} \approx 0, \quad \alpha \approx 0, \quad r = -0.426398.$$

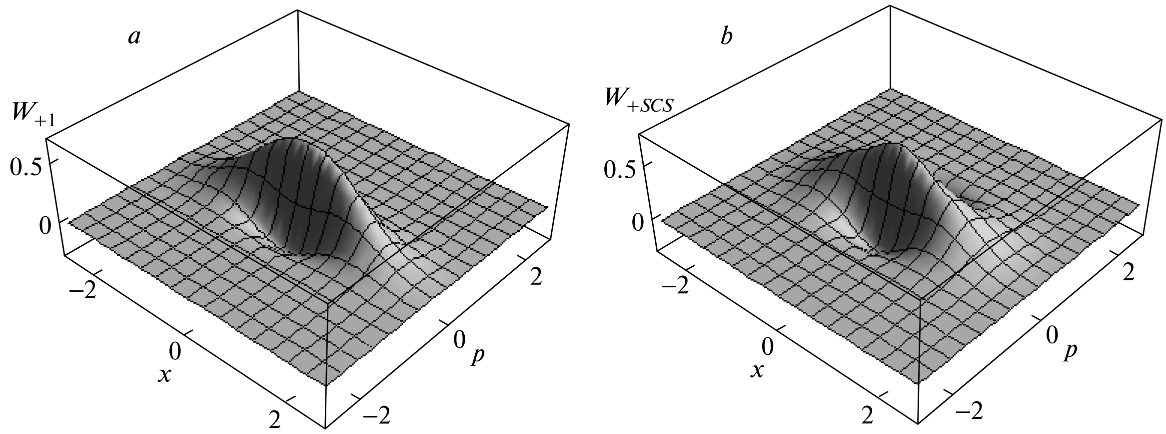


**Fig. 2.** ($a$) The Wigner function $W_{+1}$ of the state $D(\alpha)S(r)\left(|0\rangle + |1\rangle/\alpha_{In}^*\right)/\sqrt{1+1/|\alpha_{In}|^2}$ and ($b$) the Wigner function $W_{+SCS}$ of the regular even SCS with $\alpha_{SCS} = 1$. The fidelity between the states is $F_{+1} = 0.962444$

Because $\alpha_{In} \approx 0$ is taken, this means that the vacuum as an input is used to generate the odd SCS in the optical scheme in Fig. 1$a$. With $\alpha \approx 0$, the output approximates an odd squeezed SCS (not displaced) [10, 13]. Because only one photon creation operator $a^+$ is used to generate SPACS, the method may look attractive due to its simplicity. The SPACSs were experimentally demonstrated in [28]. Comparing the results in [28] with those given above, it is possible to claim the SPACSs generated in [28] do not approximate DSSCSs because the amplitudes of experimental seed coherent states were chosen out of the range needed for generation of DSSCSs.

For a universal gate operation, a CNOT (Control NOT) gate is required besides the 1-bit rotation. It was found that the CNOT operation can be realized using a teleportation protocol. To apply this suggestion to quantum computation with coherent states, we need to use the Hadamard gate (HG), see Eqs. (6c) and (6d). Analysis shows that we can start with the states $|0, \alpha_{HG} = 0.6232i\rangle$ and $|0, -\alpha_{HG} = -0.6232i\rangle$ that form a logical qubit basis in the scheme in Fig. 1$a$. We then apply an additional displacement operator $D(\beta)$ with $\beta = 0.6232i$ (dotted rectangle in Fig. 1$a$) and a single photon addition to obtain the states that approximate

$$|0, \alpha_{HG}\rangle \to D(\alpha_+)S(r)SCS_+(\alpha_{SCS}), \tag{10a}$$

$$|0, -\alpha_{HG}\rangle \to D(\alpha_-)S(r)SCS_-(\alpha_{SCS}) \tag{10b}$$

with fidelities $F_{+1} = 0.962444$ and $F_{-1} = 0.969086$, where $r = -0.445031$ and $\alpha_+ - \alpha_- = 1.78864i$. Hence, the output of this Hadamard gate is squeezed even and odd SCSs shifted relative to each other by $1.78864i$. It is worth noting that $\alpha_{HG} \neq \alpha_{SCS}$.

The same approach to the generation of SCSs of larger amplitudes and construction of the Hadamard gate can be developed in the case $N = 2$, where the "half-finished product" $|\Psi_{\pm2}\rangle$ of the SCSs in Eq. (4) for the optical scheme with two photon additions (Fig. 1$a$) is given by

$$|\Psi_{\pm2}\rangle = \frac{|0\rangle + a_{\pm1}|1\rangle + a_{\pm2}|2\rangle}{\sqrt{1+|a_{\pm1}|^2+|a_{\pm2}|^2}}. \tag{11a}$$

A displaced version of (11a) can be constructed using two photon additions with one intermediate displacement operator shifting by $\alpha_1$ (Fig. 1$a$) as

$$\begin{aligned} &a^+D(\alpha_1)a^+|0,\alpha_{In}\rangle = a^+D(\alpha_1)a^+D(\alpha_{In})|0\rangle = \\ &= e^{i\phi}D(\alpha_1+\alpha_{In})D^+(\alpha_1+\alpha_{In})a^+D(\alpha_1+\alpha_{In}) \times \\ &\qquad \times D^+(\alpha_{In})a^+D(\alpha_{In})|0\rangle \times \\ &\times e^{i\phi}D(\alpha_1+\alpha_{In})\left[a^+ + (\alpha_1+\alpha_{In})^*\right](a^+ + \alpha_{In}^*)|0\rangle = \\ &= e^{i\phi}D(\alpha_1+\alpha_{In})\Big[\alpha_{In}^*(\alpha_1^*+\alpha_{In}^*)|0\rangle + (\alpha_1^*+2\alpha_{In}^*)|1\rangle + \\ &\qquad + \sqrt{2}\,|2\rangle\Big], \end{aligned} \tag{11b}$$

where $\phi$ is some general phase shift and the normalization factor is omitted. Expression (11b) is the wave function $|\Psi_{\pm2}\rangle$ shifted by $\alpha_1 + \alpha_{In}$ if

$$a_{\pm1} = \frac{\alpha_1^* + 2\alpha_{In}^*}{\alpha_{In}^*(\alpha_1^*+\alpha_{In}^*)}, \quad a_{\pm2} = \frac{\sqrt{2}}{\alpha_{In}^*(\alpha_1^*+\alpha_{In}^*)}.$$

The state (11b) can approximate DSSCSs with the fidelity

$$F_{\pm 2} = \left|\langle SCS_{\pm}(\alpha_{SCS}) S^{+}(r_{\pm}) \times \right.$$
$$\left. \times\, D^{+}(\alpha_{\pm}) D(\alpha_1 + \alpha_{In}) |\Psi_{\pm 2}\rangle\right|^2 =$$
$$= |\langle SCS_{\pm}(\alpha_{SCS}) D(\gamma_{\pm 2}) S(-r_{\pm}) |\Psi_{\pm 2}\rangle|^2 =$$
$$= |\langle SCS_{\pm}(\alpha_{SCS}, \gamma_{\pm 2}) S(-r_{\pm}) |\Psi_{\pm 2}\rangle|^2, \quad (12)$$

where $\gamma_{\pm 2} = \operatorname{ch} r_{\pm}(\alpha_{In} + \alpha_1 - \alpha_{\pm}) - \operatorname{sh} r_{\pm}(\alpha_{In} + \alpha_1 - \alpha_{\pm})^*$, with certain parameters.

Following the procedure developed for $N-1$, we can numerically find the parameters $a_{\pm 1}$, $a_{\pm 2}$, $\gamma_{\pm 2}$, and $r_{\pm}$ at which the fidelity in (12) takes the maximum value. This allows estimating the parameters $\alpha_{In}$, $\alpha_1$, and $\alpha_{\pm}$ for the optical scheme in Fig. 1*a* as

$$\alpha_{In} = \pm i|\alpha_{In}| = \pm i\sqrt{\sqrt{2}/a_{+2}}, \quad (13a)$$

$$\alpha_1 = \mp 2i|\alpha_{In}|, \quad (13b)$$

$$\alpha_{+} = \mp i|\alpha_{In}| \quad (13c)$$

for an even SCS ($Q = \pi/4$), where $a_{+2} > 0$ and $a_{+1} = 0$ [16, 18], and

$$\alpha_{In}^* = a_{-1}/\sqrt{2}\, a_{-2} \pm \sqrt{D}/2, \quad (14a)$$

$$\alpha_1^* = \mp\sqrt{D}, \quad (14b)$$

$$D = 2(a_{-1}/a_{-2})^2 - 4\sqrt{2}/a_{-2} \quad (14c)$$

for an odd SCS ($Q = \pi/4$), while the amplitude of the shift $\alpha_{-}$ follows from $\gamma_{-2}$. Knowing $a_{\pm 1}$, $a_{\pm 2}$, and $r_{\pm}$ and using formulas (13) and (14), it is possible to calculate the corresponding parameters of the optical scheme in Fig. 1*a* at which the maximum fidelity is achieved. These parameters are collected in Table 1.

Analysis shows that it is possible to choose the shifting amplitude $\alpha_1$ of the intermediate displacement operator in Fig. 1*a* equal for both even and odd SCSs construction with equal $r_{+} = r_{-} = r$, and only to change the amplitude of the seed coherent state $\alpha_{In}$. Then the output of such a device in Fig. 1*a* is squeezed even and odd SCSs with equal $r_{+} = r_{-} = r$, shifted relative to each other by some value $\alpha_{+} - \alpha_{-}$. This outcome of the device in Fig. 1*a* is described by Eqs. (10a) and (10b). We collect the parameters that can be used for the construction of the Hadamard gate in Table 2.

For example, we consider the case $\alpha_{SCS} = 1.3$. It follows from Table 2 that the amplitude $\alpha_1 = -2.87582i$ of the intermediate displacement operator in Fig. 1*a* is used. Then the output of the optical scheme in Fig. 1*a* can approximate either the even DSSCS

$$D(\alpha = -1.43791i) S(r = -0.351) |SCS_{+}(\alpha_{SCS} = 1.3)\rangle$$

or

$$D(\alpha = 1.43791i) S(r = -0.351) |SCS_{+}(\alpha_{SCS} = 1.3)\rangle$$

with fidelity $F_{+2} = 0.986582$ if the input is a coherent state with the amplitude $\alpha_{In} = 1.43791i$ or $\alpha_{In} = -1.43791i$, or the odd DSSCS

$$D(\alpha = -2.87586i) S(r = -0.351) |SCS_{+}(\alpha_{SCS} = 1.3)\rangle$$

or

$$D(\alpha = 0) S(r = -0.351) |SCS_{+}(\alpha_{SCS} = 1.3)\rangle$$

with fidelity $F_{-2} = 0.986539$ if the input is a coherent state with the amplitude $\alpha_{In} = 0.344349i$ or $\alpha_{In} = -2.53147i$. In the case, the outcome of the optical scheme in Fig. 1*a* depending on $\alpha_{In}$ is given by two states that approximate squeezed even and odd SCSs of $\alpha_{SCS}$ with high fidelity and are shifted relative to each other by approximately $1.43791i$. Because the outcome of the optical scheme in Fig. 1*a* depends on the seed coherent state, we can use an additional displacement operator $D(\beta)$ with either $\beta = 0.89113i$ or $\beta = -1.98469i$ to deal with $|0, \alpha_{HG} = 0.546781i\rangle$ ($|0, \alpha_{HG} = -0.546781i\rangle$) as the input basis of logical zero and one. The same consideration is applicable to the SCSs construction with other values of $\alpha_{SCS}$ presented in Table 2. The Wigner functions of the state $|\Psi_{-2}\rangle$ and odd DSSCS with the amplitude $\alpha_{SCS} = 1.3$ are presented in Fig. 3. The parameters for the plots are taken from Table 1.

We consider the state $|\Psi_{\pm 3}\rangle$ in Eqs. (4) with $N = 3$:

$$|\Psi_{\pm 3}\rangle = \frac{|0\rangle + a_{\pm 1}|1\rangle + a_{\pm 2}|2\rangle + a_{\pm 3}|3\rangle}{\sqrt{1 + |a_{\pm 1}|^2 + |a_{\pm 2}|^2 + |a_{\pm 3}|^2}}, \quad (15)$$

which can be the output of the optical scheme in Fig. 1*a* if at least three single photon additions with two intermediate displacement operators between them are used. This case allows increasing the amplitude of the generated DSSCSs because the squeezing operator acts amplification factor. We only present values of the parameters used, omitting their detailed study for future investigation. We have $F_{+3} = 0.993875$ between $D(\alpha) S(r) |\Psi_{+3}\rangle$ and the regular even SCS with $\alpha_{SCS} = 1.6$ for the following values:

$$a_{+1} = 0.131109, \quad a_{+2} = 0.976048, \quad a_{+3} = -0.509043,$$

$$r = 0.478936, \quad \alpha = 0.253028.$$

The fidelity $F_{+3} = 0.990606$ between $D(\alpha) S(r) |\Psi_{+3}\rangle$ and the regular even SCS with $\alpha_{SCS} = 1.7$ is observed for the following values:

$$a_{+1} = 0.164725, \quad a_{+2} = 1.02245, \quad a_{+3} = -0.57829,$$

$$r = 0.527901, \quad \alpha = 0.264123.$$

**Table 1.** Values of the initial coherent seed $\alpha_{In}$ and the intermediate displacement operator $\alpha_1$ in Fig. 1*a* at which the output approximates either the even DSSCS $D(\alpha_+)S(r_+)|SCS_+(\alpha_{SCS})\rangle$ or the odd DSSCS $D(\alpha_-)S(r_-)|SCS_-(\alpha_{SCS})\rangle$ with maximum fidelity

| $\alpha_{SCS}$ | $Q=\pi/4$ | $Q=-\pi/4$ |
|---|---|---|
| 1.3 | $F_{+2}=0.998728$, $r_+=-0.293054$,<br>a) $\alpha_{In}=1.25598i$, $\alpha_1=-2.251196i$,<br>$\alpha_+=-1.25598i$,<br>b) $\alpha_{In}=-1.25598i$, $\alpha_1=2.51196i$,<br>$\alpha_+=1.25598i$ | $F_{-2}=0.987245$, $r_-=-0.368812$,<br>a) $\alpha_{In}=0.344249i$, $\alpha_1=-2.87582i$,<br>$\alpha_-=-2.87586i$,<br>b) $\alpha_{In}=-2.53147i$, $\alpha_1=2.87582i$,<br>$\alpha_-=0$ |
| 1.4 | $F_{+2}=0.997583$, $r_+=-0.334228$,<br>a) $\alpha_{In}=1.18095i$, $\alpha_1=-2.3619i$,<br>$\alpha_+=-1.18095i$,<br>b) $\alpha_{In}=-1.18095i$, $\alpha_1=2.3619i$,<br>$\alpha_+=1.19095i$ | $F_{-2}=0.981078$, $r_-=-0.407125$,<br>a) $\alpha_{In}=0.373226i$, $\alpha_1=-2.64328i$,<br>$\alpha_-=-2.64334i$,<br>b) $\alpha_{In}=-2.27005i$, $\alpha_1=2.64328i$,<br>$\alpha_-=0$ |
| 1.5 | $F_{+2}=0.995765$, $r_+=-0.376383$,<br>a) $\alpha_{In}=1.11822i$, $\alpha_1=-2.23643i$,<br>$\alpha_+=-1.11822i$,<br>b) $\alpha_{In}=-1.11822i$, $\alpha_1=2.23643i$,<br>$\alpha_+=1.11822i$ | $F_{-2}=0.987245$, $r_-=-0.445339$,<br>a) $\alpha_{In}=0.399473i$, $\alpha_1=-2.45894i$,<br>$\alpha_-=-2.45903i$,<br>b) $\alpha_{In}=-2.05947i$, $\alpha_1=2.45894i$,<br>$\alpha_-=0$ |
| 1.6 | $F_{+2}=0.993085$, $r_+=-0.419055$,<br>a) $\alpha_{In}=1.06794i$, $\alpha_1=-2.13588i$,<br>$\alpha_+=-1.06794i$,<br>b) $\alpha_{In}=-1.06794i$, $\alpha_1=2.13588i$,<br>$\alpha_+=1.06794i$ | $F_{-2}=0.964491$, $r_-=-0.483419$,<br>a) $\alpha_{In}=0.423166i$, $\alpha_1=-2.31033i$,<br>$\alpha_-=-2.31047i$,<br>b) $\alpha_{In}=-1.88716i$, $\alpha_1=2.31033i$,<br>$\alpha_-=0$ |

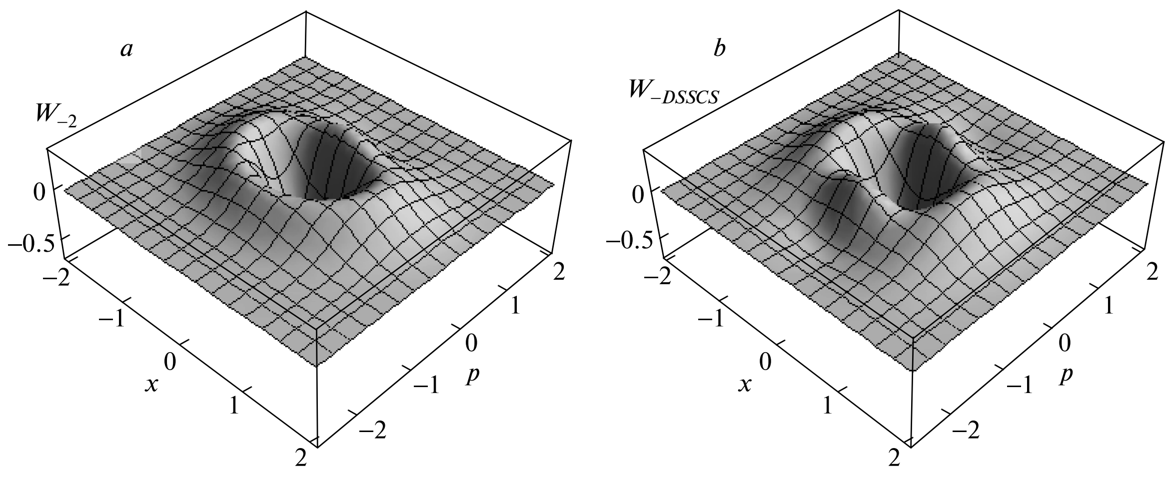


**Fig. 3.** (*a*) The Wigner function $W_{-2}$ of the state $|\Psi_{-2}\rangle$ (11a) and (*b*) the Wigner function $W_{-DSSCS}$ of the odd DSSCS with $\alpha_{SCS}=1.3$. The fidelity between the states is $0.987244$. The corresponding parameters are taken from Table 1

**Table 2.** Values of the initial parameters used in the optical scheme to generate output Eqs. (10a) and (10b)

| $\alpha_{SCS}$, $r$, $\alpha_1$, $\alpha_{HG}$, $\beta$ | $Q=\pi/4$ | $Q=-\pi/4$ |
|---|---|---|
| $\alpha_{SCS}=1.3$, $r=-0.351$, $\alpha_1=-2.87582i$, a) $\alpha_{HG}=0.546781i$, $\beta=0.89113i$, b) $\alpha_{HG}=0.54678i$, $\beta=-1.98469i$ | $F_{+2}=0.986582$, a) $\alpha_{In}=1.43791i$, $\alpha=-1.43791i$, b) $\alpha_{In}=-1.43791i$, $\alpha=1.43791i$ | $F_{-2}=0.986539$, a) $\alpha_{In}=0.344349i$, $\alpha=-2.87586i$, b) $\alpha_{In}=-2.53147i$, $\alpha=0$ |
| $\alpha_{SCS}=1.4$, $r=-0.40712$, $\alpha_1=-2.64328i$, a) $\alpha_{HG}=0.474207i$, $\beta=0.847433i$, b) $\alpha_{HG}=0.474205i$, $\beta=-1.79585i$ | $F_{+2}=0.986162$, a) $\alpha_{In}=1.32164i$, $\alpha=-1.32164i$, b) $\alpha_{In}=-1.32164i$, $\alpha=1.32164i$ | $F_{-2}=0.981078$, a) $\alpha_{In}=0.373226i$, $\alpha=-2.64334i$, b) $\alpha_{In}=-2.27005i$, $\alpha=0$ |
| $\alpha_{SCS}=1.5$, $r=-0.445339$, $\alpha_1=-2.45894i$, a) $\alpha_{HG}=0.414998i$, $\beta=0.8144715i$, b) $\alpha_{HG}=0.415i$, $\beta=-1.64447i$ | $F_{+2}=0.985525$, a) $\alpha_{In}=1.22947i$, $\alpha=-1.22947i$, b) $\alpha_{In}=-1.22947i$, $\alpha=1.22947i$ | $F_{-2}=0.973453$, a) $\alpha_{In}=0.399473i$, $\alpha=-2.45903i$, b) $\alpha_{In}=-2.05947i$, $\alpha=0$ |

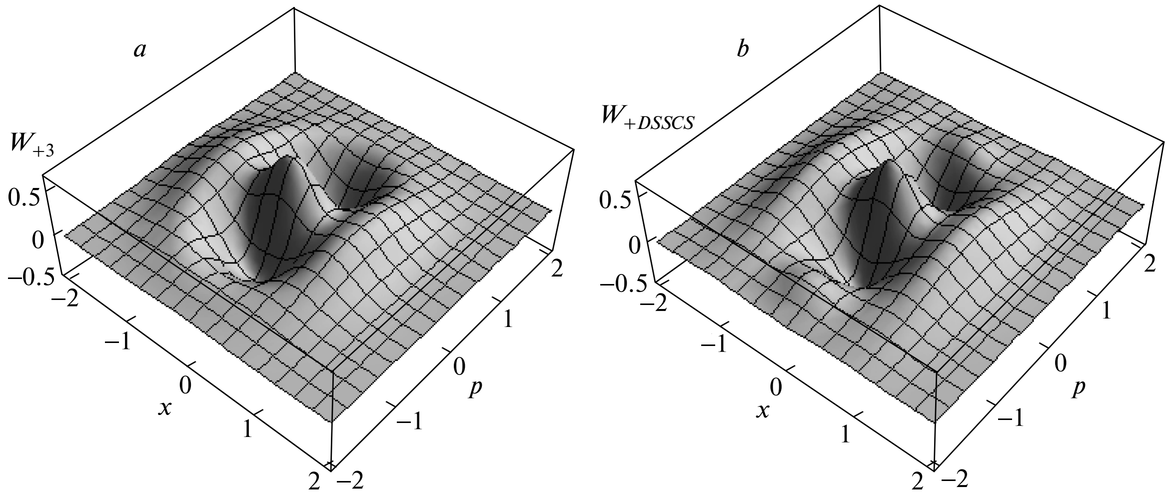


**Fig. 4.** ($a$) The Wigner function $W_{+3}$ of the state $|\Psi_{+3}\rangle$ and ($b$) the Wigner function $W_{+DSSCS}$ of the even DSSCS with $\alpha_{SCS}=1.7$. The fidelity between the states is $0.9906$. The parameters of the states are given in the text

The corresponding Wigner functions of the state $|\Psi_{+3}\rangle$ and the DSSCS are presented in Fig. 4. Interference features of the states manifest in the $p$-distribution, while separated peaks are observed in the $x$-distribution for both $|S\Psi_{+3}\rangle$ and the DSSCS. As regards the odd SCS generation, for example, we have $F_{-3}=0.996303$ between $D(\alpha)S(r)|\Psi_{-3}\rangle$ and the regular odd SCS with $\alpha_{SCS}=1.8$ for the following values:

$$a_{-1} = -20.6595, \quad a_{-2} \approx 0, \quad a_{-3} = -15.1713,$$
$$r = 0.364104, \quad \alpha = -0.012192.$$

Our approach is based on the use of single photon additions. It is well known that a single photon addition can be obtained probabilistically with the help of a parametric down converter. The probability of such an event is low. Nevertheless, SPACSs were experimentally generated in [28] and the probability to register only one photon in the ancillary mode at the output of the down converter prevails over the probabilities to register more than one photon. This can mean that the problem of resolving number states becomes negligible and we can therefore use silicon avalanche photodiodes operating in the visible wavelength having a relatively high efficiency and a small dark count rate. If the dark count rate of a photodetector is negligible, then the output state can be in a mixed state represented as

$$(1-P)W_{SPACS}(\alpha) + PW_0(\alpha),$$

where $W_{SPACS}(\alpha)$ is the Wigner function of the SPACS and $W_0(\alpha)$ is the Wigner function of the vacuum and $P$ is the probability to register an occasional photon. The construction of higher-order states $|\Psi_{\pm 2}\rangle$ requires an intermediate displacement operator and an extra single-photon addition that decreases the success probability of the device in Fig. 1$a$. The displacement operator $D(\beta)$ with the amplitude $\beta$ can be approximated by a beam splitter with high transmittivity $T \to 1$ mixing the input field with the ancillary strong coherent field $|0,\xi\rangle$ ($\xi \gg 1$). Then the output can be evaluated as

$$(1-P)W_{\pm 2}(\alpha) + PW_\alpha(\alpha),$$

where $W_{\pm 2}(\alpha)$ is the Wigner function of $|\Psi_{\pm 2}\rangle$ and $W_\alpha(\alpha)$ is the Wigner function of the coherent state, if we neglect the probability to register two occasional photons. Hence, the fidelity of the generated states in practice depends on the dark count rate and the success probability of the method decreases as $N$ increases.

## 4. CONCLUSION

The ability to investigate the elementary actions of the boson creation operators on a seed coherent state is of interest both as a tool to take a closer look at fundamental events in quantum physics and as a natural extension toward exotic quantum entities, such as SCSs. For this, we proposed a new representation of the SCSs in terms of displaced number states with arbitrary amplitudes ($\alpha$-representation). We were able to show that the type of generated SCSs (even or odd) is independent of the photon parity in the $\alpha$-representation. A photon parity can be defined only for SCSs in the 0-representation. The main motivation to use this representation is to consider problems of generation and rotation (Eqs. (10a) and (10b)) of SCSs in general position, involving different methods of generation and measurements [9–21], and to apply this to quantum computation with coherent states. This allows determining the range of parameters of the optical schemes with which output states can approximate SCSs with high fidelity.

We used a method developed in [22], as is shown in Fig. 1$a$, to construct the states that approximate DSSCSs. Another possible method is to use photon added/subtracted squeezed coherent states $a^{+m}S(r)|0,\alpha_{In}\rangle$ and $a^m S(r)|0,\alpha_{In}\rangle$ (Fig. 1$b$), considering which deserves a separate analysis. Our analysis shows that it is possible to choose the parameters of the optical scheme in Fig. 1$a$ such that the output becomes sensitive to the seed coherent state, which allows constructing local rotations of qubits, in particular the Hadamard gate, consisting of coherent states with high fidelity. We note that these are not rotations because they are defined by expressions (6a) and (6b). The outcomes are the squeezed SCSs shifted relative to each other by some quantity along the $p$-axis (Eqs. (10a) and (10b)). Moreover, the amplitudes of input qubits are not equal to those of output qubits. Nevertheless, we can supply the optical scheme in Fig. 1$a$ additionally by a phase shifter by $\pi/2$ and an absorbing medium (not shown in Fig. 1$a$) to make the amplitude of the initial qubit equal to the amplitude of the output qubit,

$$|0,\alpha_{SCS}\rangle \to |0,i\alpha_{SCS}\rangle \to |0,i\alpha_{SCS}e^{-\Gamma}\rangle = |0,\alpha_{HG}\rangle,$$

$$|0,-\alpha_{SCS}\rangle \to |0,-i\alpha_{SCS}\rangle \to |0,-i\alpha_{SCS}e^{-\Gamma}\rangle = |0,-\alpha_{HG}\rangle,$$

where $\Gamma$ is the absorbing factor of the medium. The Hadamard gate that effects a transformation as in Fig. 1$a$ cannot be unitary. Possible use of the Hadamard gate for quantum computations with coherent states deserves a separate investigation [17]. All parameters needed to construct either even or odd DSSCSs depending on seed coherent states are presented in Tables 1 and 2. It was also shown that the SPACS generated in [28] does not approximate the even DSSCS because the amplitudes of the seed coherent state were chosen outside the required range. An optical scheme with three single photon additions and with two intermediate displacement operators between them allows constructing an even DSSCS with the amplitude 1.7 and fidelity more than 0.99. Consideration of photon added/subtracted squeezed coherent states may be preferable from the practical standpoint, which may become the subject of a future study. In the short term, this approach extends the set of the states that may be used for quantum information processing and adds new methods for manipulations with coherent state qubits.

## APPENDIX A

### Decomposition in terms of displaced number states

We use the coherent state representation

$$
\begin{gathered}
|0,\alpha_{SCS}\rangle = D(\alpha_{SCS})|0\rangle = \exp\left(-|\alpha_{SCS}|^2/2\right)\times \\
\times \exp(\alpha_{SCS}a^+)\exp(\alpha_{SCS}a)|0\rangle = \\
= \exp\left(-|\alpha_{SCS}|^2/2\right)\exp(\alpha_{SCS}a^+)|0\rangle = \\
= \exp\left(-|\alpha_{SCS}|^2/2\right)\exp\left((\alpha+\beta)a^+\right)|0\rangle = \\
= \exp\left(-\left(|\alpha_{SCS}|^2-|\alpha|^2\right)/2\right)\exp(\beta a^+)\times \\
\times \exp\left(-|\alpha|^2/2\right)\exp(\alpha a^+)|0\rangle = \\
= \exp\left(-\left(|\alpha_{SCS}|^2-|\alpha|^2\right)/2\right)\exp(\beta a^+)|0,\alpha\rangle = \\
= \exp\left(-\left(|\alpha_{SCS}|^2-|\alpha|^2\right)/2\right)\times \\
\times \sum_{n=0}^{\infty}\frac{(\beta a^+)^n}{n!}|0,\alpha\rangle, \quad (\text{A.1})
\end{gathered}
$$

where $\alpha_{SCS}=\alpha+\beta$ $(\beta=\alpha_{SCS}-\alpha)$ and $\alpha$ and $\beta$ are arbitrary numbers. We consider $a^{+n}|0,\alpha\rangle$ using the well known formulas [27]. Then

$$
\begin{gathered}
a^{+n}|0,\alpha\rangle = D(\alpha)D^+(\alpha)a^{+n}D(\alpha)|0\rangle = \\
= D(\alpha)(a^+ + \alpha^*)^n|0\rangle = \\
= D(\alpha)\sum_{k=0}^{n}C_n^k\sqrt{k!}\,\alpha^{*\,n-k}|k\rangle = \\
= \sum_{k=0}^{n}\frac{n!\sqrt{k!}\,\alpha^{*\,n-k}}{k!(n-k)!}|k,\alpha\rangle. \quad (\text{A.2})
\end{gathered}
$$

Using (A.2), it is possible to transform $\sum_{n=0}^{\infty}\left((\beta a^+)^n/n!\right)|0,\alpha\rangle$ to

$$
\begin{gathered}
\sum_{n=0}^{\infty}\frac{(\beta a^+)^n}{n!}|0,\alpha\rangle = |0,\alpha\rangle+\beta\left(|1,\alpha\rangle+\alpha^*\right)+(\beta^2/2!)\times \\
\times\left(\sqrt{2}\,|2,\alpha\rangle+2\alpha^*|1,\alpha\rangle++\alpha^{*2}|0,\alpha\rangle\right)+\ldots \\
\ldots+\frac{\beta^n}{n!}\sum_{k=0}^{n}C_n^k\sqrt{k!}\,\alpha^{*\,n-k}|k\rangle+\ldots= \\
=\left(1+\beta\alpha^*+\frac{\beta^2\alpha^{*2}}{2!}+\ldots+\frac{\beta^n\alpha^{*n}}{n!}+\ldots\right)|0,\alpha\rangle+ \\
+\beta(1+\beta\alpha^*+\ldots+C_n^1\beta^{n-1}\alpha^{*\,n-1}/n!+\ldots)\times \\
\times|1,\alpha\rangle+\left(\beta^2/\sqrt{2!}\right)\times \\
\times(1+\beta\alpha^*+\ldots+C_n^2\beta^{n-2}\alpha^{*\,n-2}2!/n!+\ldots)|2,\alpha\rangle+\ldots \\
\ldots+\left(\beta^l/\sqrt{l!}\right)(1+\ldots+C_n^l\beta^{n-l}\alpha^{*\,n-l}/n!+\ldots)\times \\
\times|l,\alpha\rangle+\ldots= \\
=(1+\beta\alpha^*+\beta^2\alpha^{*2}/2!+\ldots+\beta^n\alpha^{*\,n}/n!+\ldots)|0,\alpha\rangle+ \\
+\beta(1+\beta\alpha^*+\ldots+\beta^{n-1}\alpha^{*\,n-1}/(n-1)!+\ldots)\times \\
\times|1,\alpha\rangle+(\beta^2/\sqrt{2!}\,)\times \\
\times(1+\beta\alpha^*+\ldots+\beta^{n-2}\alpha^{*\,n-2}/(n-2)!+\ldots)|2,\alpha\rangle+\ldots \\
\ldots+(\beta^l/\sqrt{l!}\,)(1+\ldots+\beta^{n-l}\alpha^{*\,n-2}/(n-l)!+\ldots)\times \\
\times|l,\alpha\rangle+\ldots=\exp(\beta\alpha^*)\sum_{l=0}^{\infty}\frac{\beta^l}{\sqrt{l!}}|l,\alpha\rangle. \quad (\text{A.3})
\end{gathered}
$$

Finally, we have

$$
\begin{gathered}
|0,\alpha_{SCS}\rangle = \exp\left(-\frac{|\alpha_{SCS}|^2-|\alpha|^2}{2}\right)\times \\
\times\exp(\beta\alpha^*)\sum_{l=0}^{\infty}\frac{\beta^l}{\sqrt{l!}}|l,\alpha\rangle = \\
=\exp\left(-\frac{\alpha_{SCS}^2+|\alpha|^2}{2}\right)\exp(\alpha_{SCS}\alpha^*)\times \\
\times\sum_{l=0}^{\infty}\frac{(\alpha_{SCS}-\alpha)^l}{\sqrt{l!}}|l,\alpha\rangle = \\
=\exp\left(-\frac{\alpha_{SCS}^2+|\alpha|^2}{2}\right)\exp(\alpha_{SCS}\alpha^*)\times \\
\times\sum_{l=0}^{\infty}\frac{\alpha_{SCS}^l}{\sqrt{l!}}\left(1-\frac{\alpha}{\alpha_{SCS}}\right)^l|l,\alpha\rangle. \quad (\text{A.4})
\end{gathered}
$$

The same is applicable to the state $|0,-\alpha_{SCS}\rangle$:

$$
\begin{gathered}
|0,-\alpha_{SCS}\rangle = \\
=\exp\left(-\frac{\alpha_{SCS}^2+|\alpha|^2}{2}\right)\exp(-\alpha_{SCS}\alpha^*)\times \\
\times\sum_{l=0}^{\infty}\frac{\alpha_{SCS}^l}{\sqrt{l!}}(-1)^l\left(1+\frac{\alpha}{\alpha_{SCS}}\right)^l|l,\alpha\rangle. \quad (\text{A.5})
\end{gathered}
$$

Therefore, if we take $\alpha = 0$ in (A.4), then we have the following decomposition of the vacuum state with respect to the basis of displaced number states:

$$|0\rangle = |0, \alpha_{SCS} = 0\rangle = \\ = \exp\left(-\frac{|\alpha|^2}{2}\right) \sum_{l=0}^{\infty} (-1)^l \frac{\alpha^l}{\sqrt{l!}} |l, \alpha\rangle. \quad (A.6)$$

## APPENDIX B

### Wave amplitudes of several first terms of the SCSs

Expressions for the wave amplitudes of the SCSs in an arbitrary $\alpha$-presentation are given by Eqs. (2a). If we take $\alpha_{SCS} = a\alpha_{SCS} + i\varepsilon$, where $a$ and $\varepsilon$ are some real numbers, then the first several wave amplitudes of the even SCS are given by

$$a_{+1}(\alpha_{SCS}, \alpha) = \frac{\alpha_{CSS}}{\sqrt{1!}} A_+(\alpha_{CSS}, \alpha) = \\ = \frac{\alpha_{SCS}}{\sqrt{1!}} \left(\frac{G_2(\alpha_{SCS}, a, \varepsilon)}{G_1(\alpha_{SCS}, a, \varepsilon)} - a - \frac{i\varepsilon}{\alpha_{SCS}}\right), \quad (B.1)$$

$$a_{+2}(\alpha_{SCS}, \alpha) = \frac{\alpha_{SCS}^2}{\sqrt{2!}} B_+(\alpha_{CSS}, \alpha) = \\ = \frac{\alpha_{SCS}^2}{\sqrt{2!}} \left[\left(1 + a^2 - \frac{\varepsilon^2}{\alpha_{SCS}^2} + i\frac{2\varepsilon a}{\alpha_{SCS}}\right) - \right. \\ \left. - \left(2a + i\frac{2\varepsilon}{\alpha_{SCS}}\right) \frac{G_2(\alpha_{SCS}, a, \varepsilon)}{G_1(\alpha_{SCS}, a, \varepsilon)}\right], \quad (B.2)$$

$$a_{+3}(\alpha_{SCS}, \alpha) = \frac{\alpha_{SCS}^3}{\sqrt{3!}} C_+(\alpha_{CSS}, \alpha) = \frac{\alpha_{SCS}^3}{\sqrt{3!}} \times \\ \times \left\{\left(\frac{3a\varepsilon^2}{\alpha_{SCS}^2} - 3a - a^3 + i\left[\frac{\varepsilon^3}{\alpha_{SCS}^3} - \frac{3\varepsilon}{\alpha_{SCS}} - \frac{3a^2\varepsilon}{\alpha_{SCS}}\right]\right) + \right. \\ + \left(1 + 3a^2 - \frac{3\varepsilon^2}{\alpha_{SCS}^2} + i\frac{6a\nabla}{\alpha_{SCS}}\right) \times \\ \left. \times \frac{G_2(\alpha_{SCS}, a, \varepsilon)}{G_1(\alpha_{SCS}, a, \varepsilon)}\right\}, \quad (B.3)$$

where

$$G_1(\alpha_{SCS}, a, \varepsilon) = 2\left[\cos(\alpha_{SCS}\varepsilon)\,\mathrm{ch}(a\alpha_{SCS}^2) - \right. \\ \left. - i\sin(\alpha_{SCS}\varepsilon)\,\mathrm{sh}(a\alpha_{SCS}^2)\right], \quad (B.4)$$

$$G_2(\alpha_{SCS}, a, \varepsilon) = 2\left[\cos(\alpha_{SCS}\varepsilon)\,\mathrm{sh}(a\alpha_{SCS}^2) - \right. \\ \left. - i\sin(\alpha_{SCS}\varepsilon)\,\mathrm{ch}(a\alpha_{SCS}^2)\right]. \quad (B.5)$$

The wave amplitudes of the odd SCS follow from the expressions for the even SCSs if we substitute $G_2(\alpha_{SCS}, a, \varepsilon)/G_1(\alpha_{SCS}, a, \varepsilon)$ onto $G_1(\alpha_{SCS}, a, \varepsilon)/G_2(\alpha_{SCS}, a, \varepsilon)$ in Eqs. (B.1)–(B.3).

## APPENDIX C

### Alternation of photon additions and displacements as a method of generation of an arbitrary single-mode finite superposition of number states

The method of constructing an arbitrary single-mode superposition of number states was proposed in [22]. We briefly recall it. An arbitrary wave function

$$|\Psi\rangle = \sum_{n=0}^{N} \varphi_n |n\rangle = \sum_{n=0}^{N} \frac{\varphi_n}{\sqrt{n!}} a^{+n}|0\rangle, \quad (C.1)$$

can be rewritten as

$$|\Psi\rangle = \frac{\varphi_n}{\sqrt{n!}} (a^+ - \alpha_N^*)(a^+ - \alpha_{N-1}^*) \ldots (a^+ - \alpha_2^*) \times \\ \times (a^+ - \alpha_1^*)|0\rangle,$$

where $\alpha_1^*, \alpha_2^*, \ldots, \alpha_N^*$ are the $N$ complex roots of the characteristic polynomial

$$\sum_{n=0}^{N} \frac{\varphi_n \alpha^{*n}}{\sqrt{n!}} = 0.$$

The relation $a^+ - \alpha_i^*$ can be ensured by applying the creation operator $a^+$ to $|0, -\alpha_i\rangle = D(-\alpha_i)|0\rangle$ [27]:

$$a^+|0, -\alpha_i\rangle = D(-\alpha_i)D^+(-\alpha_i)a^+D(-\alpha_i)|0\rangle = \\ = D(-\alpha_i)(a^+ - \alpha_i^*)|0\rangle.$$

Hence, an arbitrary single-mode superposition of the number states can be obtained by a sequence of alternate single-photon additions and displacements starting with $|\alpha_{In}\rangle$ with the corresponding amplitude $\alpha_{In}$ [22]:

$$|\Psi\rangle = \left(\varphi_n/\sqrt{n!}\right) D^+(-\alpha_N) \times \\ \times a^+ D(-\alpha_N) D^+(-\alpha_{N-1}) a^+ D(-\alpha_{N-1}) \ldots \\ \ldots \times D(-\alpha_3) D^+(-\alpha_2) a^+ D(-\alpha_2) \times \\ \times D^+(\alpha_{In}) a^+ |\alpha_{In}\rangle. \quad (C.2)$$

We can use the relation $D(\alpha)D(\beta) = \exp\left[i\,\mathrm{Im}(\alpha\beta^*)\right] \times D(\alpha + \beta)$ to simplify the expression (C.2):

$$|\Psi\rangle = e^{i\varphi} \left(\varphi_n/\sqrt{n!}\right) D^+(-\alpha_N) \times \\ \times a^+ D(\alpha_{N-1} - \alpha_N) a^+ D(\alpha_{N-2} - \alpha_{N-1}) \ldots \\ \ldots \times D(\alpha_2 - \alpha_3) a^+ D(-\alpha_2 - \alpha_{In}) a^+ |\alpha_{In}\rangle, \quad (C.3)$$

where $\varphi$ is the total phase shift.

In particular, in the 0-representation or the same number state representation (Eqs. (3a) and (3b)), for the even SCSs, we have

$$|SCS_{+N}(\alpha_{SCS},0)\rangle = N_{+N}(\alpha_{SCS},0)(\alpha_{SCS}^{2n}/n!) \times \times (a^{+2}-\alpha_N^{*2})(a^{+2}-\alpha_{N-1}^{*2})\dots(a^{+2}-\alpha_2^{*2}) \times \times (a^{+2}-\alpha_1^{*2})|0\rangle, \quad (C.4)$$

where $\alpha_1^{*2}, \alpha_2^{*2}, \dots, \alpha_N^{*2}$ are the roots of the polynomial

$$\sum_{n=0}^{N} \frac{\alpha_{SCS}^{2n}}{(2n)!}(\alpha^{*2})^n = 0. \quad (C.5)$$

The same is applicable to the generation of the odd SCS if we start with the input state $|1,\alpha_{In}\rangle$

$$|SCS_{-N}(\alpha_{SCS},0)\rangle = = N_{-N}(\alpha_{SCS},0)(\alpha_{SCS}^{2n}/(n+1)!)(a^{+2}-\alpha_N^{*2})\dots \dots\times(a^{+2}-\alpha_2^{*2})(a^{+2}-\alpha_1^{*2})|1\rangle, \quad (C.6)$$

with the roots of the equation

$$\sum_{n=0}^{N} \frac{\alpha_{SCS}^{2n}}{(2n+1)!}(\alpha^{*2})^n = 0. \quad (C.7)$$

## APPENDIX D

### Some particular cases of the use of the method of alternate single-photon additions and displacements

The roots of the characteristic polynomials given in Appendix C can be obtained in the general case only numerically. Nevertheless, some particular solutions can be found analytically. We show this with the examples of constructing the SCSs in the 0-representation. We then have

$$a^+D(-2\alpha_{In})a^+|0,\alpha_{In}\rangle = a^+D(-\alpha_{In}) \times \times (a^+ + \alpha_{In}^*)|0\rangle = = D(-\alpha_{In})D^+(-\alpha_{In})a^+D(-\alpha_{In})(a^+ + \alpha_{In}^*)|0\rangle = = D(-\alpha_{In})(a^+ - \alpha_{In}^*)(a^+ + \alpha_{In}^*)|0\rangle = = D(-\alpha_{In})(a^{+2}-\alpha_{In}^{*2})|0\rangle = = D(-\alpha_{In})\left(\sqrt{2}\,|2\rangle - \alpha_{In}^{*2}|0\rangle\right) = = D(-\alpha_{In})|\alpha_{In}|^2\left(|0\rangle + \alpha_{SCS}^2|2\rangle/\sqrt{2!}\right), \quad (D.1)$$

if $\alpha_{In} = i|\alpha_{In}|$ and $\alpha_{SCS}^2 = 2/|\alpha_{In}|^2$. The output of (D.1) is a state that approximates the even SCS displaced by $\alpha_{In}$ up to a normalization factor. A further extension of (D.1) is given by

$$a^+D(-2\alpha_1)a^+D(\alpha_1+\alpha_{In})a^+D(-2\alpha_{In}^*)a^+|0,\alpha_{In}\rangle =$$

$$= a^+D(-2\alpha_1)a^+D(\alpha_1+\alpha_{In})D(-\alpha_{In})(a^{+2}-\alpha_{In}^{*2})|0\rangle = = D(-\alpha_1)(a^{+2}-\alpha_1^{*2})(a^{+2}-\alpha_{In}^{*2})|0\rangle = = D(-\alpha_1)\left(\sqrt{4!}\,|4\rangle - \sqrt{2}\,(\alpha_1^{*2}+\alpha_{In}^{*2})|2\rangle + \alpha_1^{*2}\alpha_{In}^{*2}|0\rangle\right) = = |\alpha_1|^2|\alpha_{In}|^2 D(-\alpha_1) \times \times \left(|0\rangle + \alpha_{SCS}^2|2\rangle/\sqrt{2!} + \alpha_{SCS}^4|4\rangle/\sqrt{4!}\right), \quad (D.2)$$

where the roots of characteristic polynomial (C.5) are

$$\alpha_{1,2}^{*2} = -2\left(3\pm\sqrt{3}\right)/\alpha_{SCS}^2.$$

Because $\alpha_{SCS} > 0$, it follows that $\alpha_{In} = i|\alpha_{In}|$, $\alpha_1 = = i|\alpha_1|$, where

$$|\alpha_{In}|^2 = \frac{2\left(3+\sqrt{3}\right)}{\alpha_{SCS}^2},$$

$$|\alpha_1|^2 = \frac{2\left(3-\sqrt{3}\right)}{\alpha_{SCS}^2}, \quad \alpha_{SCS}^2 = \frac{2}{|\alpha_{In}|^2}.$$

Finally, the output operator $D(\alpha_1)$ is applied to generate a truncated version of the even SCS with three terms.

For the odd SCS, we have

$$a^+D(-2\alpha_{In})a^+|1,\alpha_{In}\rangle = = D(-\alpha_{In})(a^+ - \alpha_{In}^*)(a^+ + \alpha_{In}^*)|1\rangle = = \alpha_{In}^{*2}D(-\alpha_{In})\left(-|1\rangle + \sqrt{3!}\,|3\rangle/\alpha_{In}^{*2}\right) = = -\alpha_{In}^{*2}D(-\alpha_{In})\left(|1\rangle + \alpha_{SCS}^2|3\rangle/\sqrt{3!}\right) \quad (D.3)$$

if $\alpha_{In} = i|\alpha_1|$ and $\alpha_{SCS}^2 = 6/|\alpha_{In}|^2$. The output of (D.3) is a two-level approximation of the odd SCS displaced by $\alpha_{In}$. If we extend (D.3), then

$$a^+D(-2\alpha_1)a^+D(\alpha_1+\alpha_{In})a^+D(-2\alpha_{In})a^+|1,\alpha_{In}\rangle = = D(-\alpha_2)D^+(-\alpha_2)a^+D(-\alpha_2)D^+(\alpha_2)a^+D(\alpha_2)e^{i\psi} \times \times e^{i\psi}D(-\alpha_1)(a^{+2}-\alpha_{In}^{*2})(a^{+2}-\alpha_{In}^{*2})|1\rangle = = e^{i\psi}\alpha_{In}^{*2}\alpha_1^{*2}D(-\alpha_1)\left[|1\rangle - \frac{3!(\alpha_1^{*2}+\alpha_{In}^{*2})|3\rangle}{\sqrt{3!}\,\alpha_1^{*2}\alpha_{In}^{*2}} + \right. \left. + \frac{5!|5\rangle}{\sqrt{5!}\,\alpha_1^{*2}\alpha_{In}^{*2}}\right] = e^{i\psi}\alpha_1^{*2}\alpha_{In}^{*2}D(-\alpha_1) \times \times \left(|1\rangle + \frac{\alpha_{SCS}^2|3\rangle}{\sqrt{3!}} + \frac{\alpha_{SCS}^4|5\rangle}{\sqrt{5!}}\right), \quad (D.4)$$

where $\alpha_{In}^{*2} = \left(5+i\sqrt{5}\right)/\alpha_{SCS}^2$ and $\alpha_1^{*2} = \left(5-i\sqrt{5}\right)/\alpha_{SCS}^2$. The output state is a three-level approximation of the odd SCS displaced by $-\alpha_1$. Higher-order characteristic polynomials (C.5) and (C.7) can be solved numerically. Numerical values of the fidelities are

$$F_{+1}(\alpha_{SCS+1}^{0.99} = 0.861557) = F_{-1}(\alpha_{SCS-1}^{0.99} = 1.04403) = F_{+2}(\alpha_{SCS+2}^{0.99} = 1.27247) = F_{-2}(\alpha_{SCS+2}^{0.99} = 1.45741) = 0.99,$$

where $\alpha_{SCS+N}^{0.99}$ is the size of the SCS for which the fidelity takes the value 0.99 ($F_{\pm N}(\alpha_{SCS\pm N}^{0.99}) = 0.99$). Comparing the fidelities, we see that $\alpha_{SCS-1}^{0.99} = 1.04403 > \alpha_{SCS+1}^{0.99} = 0.861557$ and $\alpha_{SCS-2}^{0.99} = 1.45741 > \alpha_{SCS+2}^{0.99} = 1.27247$.

## APPENDIX E

### Wigner functions

We have considered the presentation of SCSs and their approximations on the phase plane. The Wigner functions of even/odd SCSs can be expressed as

$$W_{\pm SCS}(\alpha) = N_{\pm}(\alpha_{SCS}) \times [W_0(\alpha) + W_{-0}(\alpha) \pm 2X_{\alpha_{SCS}}(\alpha)], \quad (E.1)$$

where $\alpha = x + ip$, $\alpha_{SCS} = x_{SCS} + ip_{SCS}$ and

$$W_0(\alpha) = \frac{2}{\pi} \times \exp\left[-2(x - x_{SCS})^2 - 2(p - p_{SCS})^2\right], \quad (E.2)$$

$$W_{-0}(\alpha) = \frac{2}{\pi} \times \exp\left[-2(x + x_{SCS})^2 - 2(p + p_{SCS})^2\right], \quad (E.3)$$

$$X_{\alpha_{SCS}}(\alpha) = \frac{2}{\pi} \times \exp(-2x^2 - 2p^2)\cos\left[4(xp_{SCS} - px_{SCS})\right]. \quad (E.4)$$

The Wigner function of a two-level superposition ($N = 1$, Eq. (8b)), being the simplest approximation of the DSSCSs, is given by

$$W_{\pm 1}(\alpha) = \frac{1}{1 + |a_{\pm 1}|^2} \times \left(W_0(\alpha) + |a_{\pm 1}|^2 W_1(\alpha) + X_{\pm 01}(\alpha)\right), \quad (E.5)$$

$$W_0(\alpha) = Y(\alpha), \quad (E.6)$$

$$W_1(\alpha) = Y(\alpha)(4x^2 + 4p^2 - 1), \quad (E.7)$$

$$X_{\pm 01}(\alpha) = 2Y(\alpha)\left[a_{\pm 1}^*(x + ip) + a_{\pm 1}(x - ip)\right], \quad (E.8)$$

$$Y(\alpha) = \frac{2}{\pi}\exp(-2x^2 - 2p^2). \quad (E.9)$$

The Wigner function of a three-level superposition ($N = 2$, Eq. (11a)), being the next approximation of the SCSs, is given by

$$W_{\pm 2}(\alpha) = \frac{1}{1 + |a_{\pm 1}|^2 + |a_{\pm 2}|^2} \times \left[W_0(\alpha) + |a_{\pm 1}|^2 W_1(\alpha) + |a_{\pm 2}|^2 W_2(\alpha) + X_{\pm 01}(\alpha) + X_{\pm 02}(\alpha) + X_{\pm 12}(\alpha)\right], \quad (E.10)$$

where

$$W_2(\alpha) = Y(\alpha)\left[1 + 4(2x^2 + 2p^2 - 1) + 3 + 4(2x^2 - 3)x^2 + 4(2p^2 - 3)p^2 + (1 - 4x^2)(1 - 4p^2)\right], \quad (E.11)$$

$$X_{02}(\alpha) = 2\sqrt{2}\,Y(\alpha)\left[a_{\pm 2}(x - ip)^2 + \text{c.c.}\right], \quad (E.12)$$

$$X_{12}(\alpha) = 2\sqrt{2}\,Y(\alpha) \times \left\{a_{\pm 1}a_{\pm 2}^*\left[(x + ip)(2x^2 + 2p^2 - 1)\right] + \text{c.c.}\right\}. \quad (E.13)$$

Higher-order Wigner functions $W_{\pm N}(\alpha)$ with $N > 2$ can be calculated only numerically because of their complexity. Marginal distributions for the momentum and position are given by $\int W(x,p)\,dx = \langle p|\rho|p\rangle$ and $\int W(x,p)\,dp = \langle x|\rho|x\rangle$, where $\rho$ is a density matrix.

If the Wigner functions of the SCSs are given by $W_{\pm SCS}(\alpha)$, then it is possible to show that the Wigner function transforms as

$$W_{\pm SCS}(\alpha) \to W_{\pm DSSCS} \times [\operatorname{ch} r(\alpha - \beta) - \operatorname{sh} r(\alpha^* - \beta^*)] \quad (E.14)$$

for the DSSCSs, where $r$ is the squeezing parameter and $\beta$ is the displacement amplitude. Then, the Wigner functions of the even/odd DSSCSs can be expressed as

$$W_{\pm DSSCS}(\alpha) = N_{\pm}(\alpha_{SCS}) \times [W_0(\alpha) + W_{-0}(\alpha) \pm 2X_{\alpha_{DSSCS}}(\alpha)], \quad (E.15)$$

where $\alpha = x + ip$, $\alpha_{SCS} = x_{SCS} + ip_{SCS}$, $\beta = x_\beta + ip_\beta$ and

$$W_0(\alpha) = \frac{2}{\pi}\exp\left[-2\left(\frac{x - x_\beta}{e^r} - x_{SCS}\right)^2 - 2\left(\frac{p - p_\beta}{e^{-r}} - p_{SCS}\right)^2\right], \quad (E.16)$$

$$W_{-0}(\alpha) = \frac{2}{\pi}\exp\left[-2\left(\frac{x-x_\beta}{e^r}+x_{SCS}\right)^2 - \right.$$
$$\left. - 2\left(\frac{p-p_\beta}{e^{-r}}+p_{SCS}\right)^2\right], \quad \text{(E.17)}$$

$$X_{\alpha_{DSSCS}}(\alpha) = \frac{2}{\pi}\exp\left[-2\frac{(x-x_\beta)^2}{e^{2r}} - 2\frac{(p-p_\beta)^2}{e^{-2r}}\right]\times$$
$$\times\cos\left[4\left(p_{SCS}\frac{x-x_\beta}{e^r} - x_{SCS}\frac{p-p_\beta}{e^{-r}}\right)\right]. \quad \text{(E.18)}$$

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