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SCALE-INVARIANT STREAMLINE EQUATIONS AND STRINGS OF SINGULAR VORTICITY FOR PERTURBED ANISOTROPIC SOLUTIONS OF THE NAVIER–STOKES EQUATION

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A linear combination of a pair of dual anisotropic decaying Beltrami flows with spatially constant amplitudes (the Trkal solutions) with the same eigenvalue of the curl operator and of a constant velocity orthogonal vector to the Beltrami pair yields a triplet solution of the force-free Navier–Stokes equation. The amplitudes slightly variable in space (large scale perturbations) yield the emergence of a time-dependent phase between the dual Beltrami flows and of the upward velocity, which are unstable at large values of the Reynolds number. They also lead to the formation of large-scale curved prisms of streamlines with edges being the strings of singular vorticity.

1. INTRODUCTION

The importance of coherent structures in a turbulent flow is undoubted. Nevertheless, the process of their appearance remains unclear. In fact, the theory of turbulence evolved in the opposite direction during many years. Due to the seminal 1941 paper by Kolmogorov, the emphasis was on statistical concepts of "chaotization" of turbulent flows, while the initial $3D$ Navier–Stokes equations remained in the shadows. The concept that Beltrami-type flows are predominant in a developed turbulent flow was initially proposed by Levich and coauthors [1–4] and by Moffatt [5–7]. Although Beltrami flows (anisotropic vector eigenfunctions of the curl operator) are stationary solutions of the Euler equations containing no viscosity terms, the modified Beltrami flows with their decay in a viscous fluid, in the absence of external forces (solutions of the "force-free Navier–Stokes equations"), were found in [8] as early as 1918.

The emergence of domains of helical Beltrami-type flows characterized by the collinearity of the velocity and vorticity vectors is currently firmly established in geophysical observations and numerical simulations of the Navier–Stokes equations [9–19]. The fact of formation of large-scale Beltrami-like helical structures in tornadoes, tropical storms, cloud streets, etc., is firmly established by climate observations [20]. Therefore, we here assume the existence of Beltrami-type fluctuations. To build a mathematical model of the emergence of the coherent structures, we use the idea and technique proposed by Sivashinsky [21], who regarded a large-scale structures as a manifestation of a long-wavelength instability of spatially periodic solutions of the Navier–Stokes equations. This approach was used in [22–25] to study mostly the instability of the linearized $2D$ Navier–Stokes equations, sometimes using Galerkin approximations, which could possibly decrease the validity of the obtained analytic result. The $2D$ nonlinear equations have been investigated for finite values of the Reynolds numbers R, when the major phenomena are related to the trespassing of the so-called critical value R_0 of the Reynolds number [21], whereas coherent structures appear only at large values of R. In Ref. [26], the linear stability of the nonstationary Trkal solution [8] for the force-free $2D$ Navier–Stokes equations and the stationary Beltrami solution for the forced equations for large values of the Reynolds number R was studied. Although the linearized forced $2D$ Navier–Stokes equations with the Beltrami external force are unstable under perturbations with the wavelength L proportional to R, it was established that nonstationary solutions of the linearized force-free Navier–Stokes equations can be unstable under perturbations with an intermediate large wavelength L that is less than R; in this case, the order of the quantity L remains unexplained.

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In Ref. [27], a nonlinear asymptotic analysis of long-wavelength perturbations of the Trkal solution for the force-free $3D$ Navier–Stokes equation at large Reynolds numbers R was performed and an asymptotic solution consisting of Beltrami-type flows and terms associated with them was effectively constructed. It turned out that the asymptotic procedure can be implemented in that case only for a single value of the scaling parameter equal to $R^{1/2}$, just as the reduction of the coupling constant in the nonlinear term in renormalized perturbation expansions by $R^{1/2}$, as done by Levich. This reduction stems from the assumption that Beltrami fluctuations dominate in a developed turbulent flow [3]. This allows writing equations for plane streamlines that in the quasistationary case turn out to be gradient lines of a function of two variables determined by the initial conditions (actually, the energy density); due to the emerging upward flow, the resulting $3D$ streamlines, as well as the vorticity lines, form $3D$ tubes invariant under flow that can be regarded as large-scale structures. Moffatt envisaged such pictures for the solutions of the Euler equations [28].

As is explicitly demonstrated here, the crucial point of the whole analysis is the coupling of two large-scale amplitude-modulated anisotropic Beltrami flows with the same eigenvalue of the curl operator ("dual anisotropic Beltrami flows"). Together with a constant vector orthogonal to the pair of dual Beltrami flows, linear combinations of the three vectors form solutions of the force-free Navier–Stokes equation, which we call triplets. No other finite linear combination of linearly independent anisotropic Beltrami flows yields a solution of the Euler equations. The same is true for the corresponding Trkal solutions of the force-free Navier–Stokes equations. The coupling of two dual "plane" anisotropic Beltrami flows with constant amplitudes yields stationary geometry of streamlines. Spatially variable amplitudes yield the emergence of a time depending phase between them, which is unstable at times of the order of $R^{1/2}$. This time-dependent phase yields an upward velocity and brings the formation of triplets transformed at large Reynolds numbers under long-wavelength amplitude perturbation into large-scale streamline tubes. At the initial stage at least, the vorticity and the velocity fields are collinear inside these tubes. These streamline tubes are vortex tubes as well. As a result, large-scale streamline–vortex tubes are stable at times of the order of $R^{1/2}$ and vanish at times $t \propto R^2$, and might therefore be regarded as metastable coherent structures.

The punch of the present endeavor lies in the conviction that the fundamental phenomena of flows of the incompressible liquid in $3D$ space are tied to the interplay between the explicit time dependence and the inner stream geometry. This is exemplified by building a large-scale flow model that stems exclusively from the $3D$ Navier–Stokes equations and relates to the accumulated observation data as well as to the results of computer simulations. Many of them demonstrate the emergence in a flow, right from the onset, of regular helical structures characterized by an almost complete alignment of the velocity and vorticity vectors as well as a built-in singularity of the vorticity inside streamline-vortex tubes. The proposed model demonstrates that although the flow velocity along the tubes is unstable at times of the order of $R^{1/2}$, the streamline $3D$ geometry remains stable almost permanently. For this purpose, we apply Sivashinsky's method of multiscaling analysis [21] to the long wavelength perturbations of the so-called Trkal flows at large Reynolds numbers.

2. EXPLICIT ANISOTROPIC SOLUTIONS OF THE FORCE-FREE NAVIER–STOKES EQUATIONS

The Navier–Stokes equations for homogeneous incompressible viscid fluids are

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\Delta\mathbf{u} + \mathbf{f},$$

where $\mathbf{u}$ is the flow velocity, p is the pressure, ρ is the density, assumed to be constant, ν is the kinematic viscosity, and $\mathbf{f}$ is a body force.

The Euler equations for the ideal liquid ($\nu = 0$) are

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{f}. \tag{2.1}$$

The so-called Beltrami flow

$$\mathbf{e}_0(z) = u_0 \begin{pmatrix} \sin\dfrac{z}{d} \\ \cos\dfrac{z}{d} \\ 0 \end{pmatrix}$$

is a solution of Euler equation (2.1), while

$$\mathbf{u}_0 = \exp\left(-\frac{\nu t}{d^2}\right)\cdot\mathbf{e}_0 = u_0\exp\left(-\frac{\nu t}{d^2}\right)\begin{pmatrix} \sin\dfrac{z}{d} \\ \cos\dfrac{z}{d} \\ 0 \end{pmatrix}$$

is a solution of the force-free ($f = 0$) Navier–Stokes equation, where $2\pi d$ is the characteristic spatial period of the flow and u_0 is the typical velocity [8].

Navier–Stokes equation (2.1) is usually considered in the regularized dimensionless form

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla p + \frac{1}{R}\Delta\mathbf{u} + \frac{1}{R}\mathbf{f}, \qquad \operatorname{div}\mathbf{u} = 0, \tag{2.2}$$

where $R = u_0 d/\nu$ is the so-called Reynolds number. For our purposes, we prefer to apply the curl operator to both sides of the last equation. Due to the well-known formula

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = [\operatorname{rot} u \times u] + \frac{1}{2}\operatorname{grad}|\mathbf{u}|^2,$$

we obtain the so-called vorticity equation

$$\frac{\partial(\operatorname{rot}\mathbf{u})}{\partial t} + \operatorname{rot}[\operatorname{rot}\mathbf{u}\times\mathbf{u}] = \frac{1}{R}\Delta(\operatorname{rot}\mathbf{u}) + \frac{1}{R}\operatorname{rot}\mathbf{f}.$$

Because we intend to investigate the solutions of the force-free Navier–Stokes equation, the last equation is reduced to

$$\frac{\partial(\operatorname{rot}\mathbf{u})}{\partial t} + \operatorname{rot}[\operatorname{rot}\mathbf{u}\times\mathbf{u}] = \frac{1}{R}\Delta(\operatorname{rot}\mathbf{u}), \qquad \operatorname{div}\mathbf{u} = 0. \tag{2.3}$$

In the dimensionless form, the Beltrami flow

$$\mathbf{e}_1(z) = \begin{pmatrix} \sin z \\ \cos z \\ 0 \end{pmatrix}$$

is a solution for the Euler equation

$$\frac{\partial(\operatorname{rot}\mathbf{u})}{\partial t} - \operatorname{rot}[\mathbf{u}\times\operatorname{rot}\mathbf{u}] = 0, \qquad \operatorname{div}\mathbf{u} = 0 \tag{2.4}$$

because, obviously,

$$\operatorname{rot}\,\mathbf{e}_1(z) = \mathbf{e}_1(z),$$

i. e., $\mathbf{e}_1(z)$ is the eigenvector of the curl operator with the eigenvalue 1. It is also obvious that

$$\mathbf{g}_1(z,t) = Ae^{-t/R}\,\mathbf{e}_1(z)$$

is a solution of force-free Navier–Stokes equation (2.3) (the so-called Trkal solution).

In fact,

$$\mathbf{e}_m(z) = \begin{pmatrix} \sin mz \\ \cos mz \\ 0 \end{pmatrix}, \quad m = 0, \pm1, \pm2 \ldots,$$

is an eigenvector of the curl operator with the eigenvalue m, and

$$\mathbf{g}_m(z,t) = A\exp\left(-\frac{m^2 t}{R}\right)\mathbf{e}_m(z)$$

is a solution of force-free Navier–Stokes equation (2.3). On the other hand, it can be easily seen that the same holds for the vectors $\mathbf{h}_m$ defined as

$$\mathbf{h}_m(z) = \begin{pmatrix} \cos mz \\ -\sin mz \\ 0 \end{pmatrix}, \quad m = 0, \pm1, \pm2 \ldots,$$

i. e.,

$$\operatorname{rot}\mathbf{h}_m(z) = m\mathbf{h}_m(z),$$

and $\boldsymbol{\nu}_m = A\exp(-m^2 t/R)\mathbf{h}_m(z)$ is a solution of Eq. (2.3). Obviously,

$$[\mathbf{e}_m(z)\times\mathbf{e}_n(z)] = \begin{pmatrix} 0 \\ 0 \\ \sin(m-n)z \end{pmatrix},$$
$$[\mathbf{h}_m(z)\times\mathbf{h}_n(z)] = \begin{pmatrix} 0 \\ 0 \\ \sin(m-n)z \end{pmatrix}, \tag{2.5}$$
$$[\mathbf{h}_m(z)\times\mathbf{e}_n(z)] = \begin{pmatrix} 0 \\ 0 \\ \cos(m-n)z \end{pmatrix}.$$

Clearly, $\mathbf{e}_m(z)$ and $\mathbf{h}_m(z)$ are orthogonal. Following Ref. [29], we call them the dual Beltrami flows. There are no other linearly independent eigenvectors of the curl operator with the eigenvalue m, which are anisotropic in z.

We consider 2π-periodic three-dimensional vector fields with zero divergence, which are anisotropic in z (i. e., depend only on z). Then the vectors

$$\frac{1}{\sqrt{2\pi}}\{\mathbf{e}_m(z)\},\ \frac{1}{\sqrt{2\pi}}\{\mathbf{h}_m(z)\}, \text{and } \frac{1}{\sqrt{2\pi}}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

form an orthogonal basis in the space of square-integrable vector functions of z on $[0, 2\pi]$. If we seek an anisotropic solution of (2.3) as a finite linear combination of $\{\mathbf{e}_m(z)\}$ and $\{\mathbf{h}_m(z)\}$, then, due to Eq. (2.2), the only possible finite linear combinations are

$$\mathbf{u} = \gamma_0\mathbf{e}_m(z) + \gamma_1\mathbf{h}_m(z) + \begin{pmatrix} 0 \\ 0 \\ \delta \end{pmatrix}, \tag{2.6}$$

where γ_0, γ_1, and δ are some functions of time t. This follows from the fact that for a given m, the vectors $\mathbf{e}_m(z)$, $\mathbf{h}_m(z)$, and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ form a closed set with respect to the cross-product operation:

$$[\mathbf{e}_m \times \mathbf{h}_m] = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \left[\mathbf{e}_m(z) \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right] = \mathbf{h}_m,$$

$$\left[\mathbf{h}_m \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right] = -e_m.$$

Indeed, if $\mathbf{u} = \alpha\,\mathbf{e}_m + \beta\mathbf{e}_n$, then because

$$(\mathbf{g}\cdot\nabla)\mathbf{f} = \frac{1}{2}\{\operatorname{rot}[\mathbf{g}\times\mathbf{f}] + \operatorname{grad}(\mathbf{g}\,,\mathbf{f}) - \mathbf{f}\operatorname{div}\mathbf{g} + \\ + \mathbf{g}\operatorname{div}\mathbf{f} - [\mathbf{f}\times\operatorname{rot}\mathbf{g}] - [\mathbf{g}\times\operatorname{rot}\mathbf{f}]\},$$

where $(\mathbf{g},\mathbf{f})$ is the scalar product, it follows from Eqs. (2.5) that

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = (n-m)\alpha\beta[\mathbf{e}_m \times \mathbf{e}_n] = \\ = (n-m)\alpha\beta\begin{pmatrix} 0 \\ 0 \\ \sin(n-m)z \end{pmatrix}.$$

Hence, new terms with new space frequencies appear in Eq. (2.2) and $\mathbf{u}$ cannot be a solution of (2.2) if $n \neq m$. We call (2.6) the Beltrami triplet. Because the discussion in what follows is entirely valid for any integer m, we set $m = 1$ from now on. Substitution of (2.6) into (2.3) yields

$$\frac{\partial\gamma_1}{\partial t}\mathbf{h}_1 + \frac{\partial\gamma_0}{\partial t}\mathbf{e}_1(z) + \delta\gamma_0\mathbf{h}_1(z) - \delta\gamma_1\mathbf{e}_1(z) = \\ = -\frac{\gamma_0}{R}\mathbf{e}_1(z) - \frac{\gamma_1}{R}\mathbf{h}_1(z).$$

Hence,

$$\frac{\partial\gamma_0}{\partial t} = \delta\gamma_1 - \frac{\gamma_0}{R}, \\ \frac{\partial\gamma_1}{\partial t} = -\delta\gamma_0 - \frac{\gamma_1}{R}. \tag{2.7}$$

It then follows that

$$\frac{1}{2}\frac{\partial}{\partial t}(\gamma_0^2 + \gamma_1^2) = -\frac{\gamma_0^2+\gamma_1^2}{R}$$

or

$$\gamma_0^2 + \gamma_1^2 = C_0^2 e^{-2t/R}, \quad C_0^2 = \gamma_0^2(0) + \gamma_1^2(0),$$

and therefore

$$\gamma_0 = C_0 e^{-t/R}\cos\phi(t), \\ \gamma_1 = C_0 e^{-t/R}\sin\phi(t). \tag{2.8}$$

Substitution of (2.8) into (2.7) yields

$$-\sin\phi\frac{\partial\phi}{\partial t} = \delta\sin\phi - \frac{1}{R}\cos\phi, \\ \cos\phi\frac{\partial\phi}{\partial t} = -\delta\cos\phi - \frac{1}{R}\sin\phi.$$

Multiplying the first equation by $\sin\phi$ and the second by $-\cos\phi$ and adding them, we obtain

$$-\frac{\partial\phi}{\partial t} = \delta.$$

Hence, the Beltrami triplet can be presented as

$$u_0 = C_0 e^{-t/R}\cos\phi(t)\begin{pmatrix} \sin z \\ \cos z \\ 0 \end{pmatrix} + \\ + C_0 e^{-t/R}\sin\phi(t)\begin{pmatrix} \cos z \\ \sin z \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\partial\phi/\partial t \end{pmatrix} = \\ = \begin{pmatrix} C_0 e^{-t/R}\sin(z+\phi(t)) \\ C_0 e^{-t/R}\cos(z+\phi(t)) \\ -\partial\phi/\partial t \end{pmatrix}. \tag{2.9}$$

The streamline equations are

$$\dot{x} = C_0 e^{-t/R}\sin(z+\phi(t)), \\ \dot{y} = C_0 e^{-t/R}\cos(z+\phi(t)), \\ \frac{d(z+\phi)}{dt} = 0. \tag{2.10}$$

Thus, if the interaction of two dual plane Beltrami flows yields the time-dependent phase $\phi(t)$, it also yields the upward flow, depending only on $\phi(t)$. In other words, the flow becomes three-dimensional only if the coefficients in (2.6) are not constant. The vectors $\mathbf{u}_0$ and rot $\mathbf{u}_0$ are not collinear and the angle between them is not small. However, $\phi(t)$ is an arbitrary function of time because of the axial symmetry. To eliminate this indefinite state, we have to break the symmetry.

3. THE SCALING PROCEDURE

We deviate slightly from the strict anisotropy. We suppose that C_0, ϕ, and δ (and hence $\mathbf{u}$) in (2.9) “slowly” depend on x and y, and we extend this scaling to time:

$$\phi(t) \to \phi(\varepsilon x, \varepsilon y, \varepsilon t), \quad \frac{1}{R} < \varepsilon < 1,$$

where ε is a small parameter; we “quench”

$$e^{-t/R} = e^{-\tau/\varepsilon R} = b_0 \approx 1$$

for times under consideration ($t \sim 1/\varepsilon$). Similarly, we regard γ_0 and γ_1 as functions of εx, εy, and εt. Hence,

$$C_0^2 = \gamma_0^2(0) + \gamma_1^2(0) \to \gamma_0^2(\varepsilon x, \varepsilon y, 0) + \gamma_1^2(\varepsilon x, \varepsilon y, 0) = C_0^2(\varepsilon x, \varepsilon y).$$

We assume that all the functions of the new “slow” variables

$$\xi = \varepsilon x \quad \left(\frac{\partial}{\partial x} \to \varepsilon \frac{\partial}{\partial \xi}\right),$$

$$\eta = \varepsilon y \quad \left(\frac{\partial}{\partial y} \to \varepsilon \frac{\partial}{\partial \eta}\right),$$

$$\tau = \varepsilon t \quad \left(\frac{\partial}{\partial t} \to \varepsilon \frac{\partial}{\partial \tau}\right),$$

are periodic in space. We are therefore dealing with “long-wave perturbations” of a finite-amplitude Trkal fluctuation

$$\mathbf{g}_1(z,t) = A e^{-t/R}\, \mathbf{e}_1(z),$$

which is a solution for the force-free Navier–Stokes equations.

We seek the perturbed solutions in form (2.9). We supplement γ_0 and γ_1 as

$$\gamma_0(t) \to A + \gamma_0(\varepsilon x, \varepsilon y, \varepsilon t) = A + \gamma_0(\xi, \eta, \tau),$$

$$\gamma_1(t) \to \gamma_1(\varepsilon x, \varepsilon y, \varepsilon t) = \gamma_1(\xi, \eta, \tau),$$

$$C_0^2(0) \to C_0^2(\xi, \eta) = (\gamma_0(\xi, \eta, 0) + A)^2 + \gamma_1^2(\xi, \eta, 0).$$

The coefficients $\gamma_0(t)$ and $\gamma_1(t)$ in (2.6) then become long-wavelength amplitude modulation factors (in x and in y) for $\mathbf{e}_1(z)$ and $\mathbf{h}_1(z)$:

$$\mathbf{u}_0 \to \mathbf{u}_0 + \varepsilon \boldsymbol{\delta}_1 = (A + \gamma_0(\xi, \eta, \tau))\mathbf{e}_1(z) + \gamma_1(\xi, \eta, \tau)\mathbf{h}_1(z) + \varepsilon \begin{pmatrix} 0 \\ 0 \\ -\dfrac{\partial \phi(\xi, \eta, \tau)}{\partial \tau} \end{pmatrix}. \quad (3.1)$$

In fact, we have already obtained another form of (2.9) for the scaled Beltrami triplet:

$$\mathbf{u}_0 + \varepsilon \boldsymbol{\delta}_1 = \begin{pmatrix} C_0(\xi, \eta) \sin(z + \phi(\xi, \eta, \tau)) \\ C_0(\xi, \eta) \cos(z + \phi(\xi, \eta, \tau)) \\ 0 \end{pmatrix} + \varepsilon \begin{pmatrix} 0 \\ 0 \\ -\dfrac{\partial \phi}{\partial \tau} \end{pmatrix}, \quad (3.2)$$

where $C_0^2(\xi, \eta) = (A + \gamma_0(\xi, \eta, 0))^2 + \gamma_1^2(\xi, \eta, 0)$ is determined by the initial conditions for γ_0 and γ_1, i. e., by the initial small long-wavelength (“noise”) modulations of $\mathbf{e}_1(z)$ and $\mathbf{h}_1(z)$ amplitudes. Because the angle between $\mathbf{u}_0$ and rot $\mathbf{u}_0$ is small and almost proportional to ε, these vectors are nearly collinear. The rescaled equation (2.3)

$$\varepsilon \frac{\partial}{\partial \tau} \operatorname{rot}\, \mathbf{u}(\xi, \eta, \tau) + \operatorname{rot}[\operatorname{rot} \mathbf{u} \times \mathbf{u}] = \frac{1}{R} \Delta(\operatorname{rot}\, \mathbf{u}),$$

where

$$\operatorname{rot} \mathbf{u}(\xi, \eta, z, \tau) = \operatorname{rot}_z\, \mathbf{u} + \varepsilon \operatorname{rot}_{\xi\eta} \mathbf{u},$$

$$\Delta \mathbf{u} = \frac{\partial^2 \mathbf{u}}{\partial z^2} + \varepsilon^2 \left(\frac{\partial^2 \mathbf{u}}{\partial \xi^2} + \frac{\partial^2 \mathbf{u}}{\partial \eta^2}\right),$$

i. e., the equation

$$\begin{aligned} &\operatorname{rot}_z[\operatorname{rot}_z u \times u] + \\ &+\varepsilon \left\{ \frac{\partial(\operatorname{rot}_z \mathbf{u})}{\partial \tau} + \operatorname{rot}_{\xi\eta}[\operatorname{rot}_z \mathbf{u} \times \mathbf{u}] + \operatorname{rot}_z[\operatorname{rot}_{\xi\eta} \mathbf{u} \times \mathbf{u}] - \right. \\ &\left. - \frac{1}{R} \frac{\partial^2(\operatorname{rot}_{\xi\eta} \mathbf{u})}{\partial z^2} \right\} + \varepsilon^2 \left\{ \frac{\partial(\operatorname{rot}_{\xi\eta} \mathbf{u})}{\partial \tau} - \frac{1}{R} \Delta_{\xi\eta}(\operatorname{rot}_z \mathbf{u}) \right\} - \\ &- \frac{\varepsilon^3}{R} \Delta_{\xi\eta}(\operatorname{rot}_{\xi\eta} \mathbf{u}) = \frac{1}{R} \frac{\partial^2(\operatorname{rot}_{\xi\eta} \mathbf{u})}{\partial z^2} \end{aligned} \quad (3.3)$$

was investigated in Ref. [27] via an asymptotic expansion of the solution in powers of ε,

$$\mathbf{u}(\xi, \eta, z, \tau) = \mathbf{u}_0(\xi, \eta, z, \tau) + \varepsilon \mathbf{u}_1(\xi, \eta, z, \tau) + \varepsilon^2 \mathbf{u}_2(\xi, \eta, z, \tau) + \ldots$$

Since the term $(1/R)\partial^2(\operatorname{rot}_z \mathbf{u})/\partial z^2$ in Eq. (3.3) should be of some order in powers of ε, we have

$$\varepsilon^k = \frac{1}{R} \quad \text{for some integer } k.$$

Hence, we consider the asymptotic behavior of $\mathbf{u}$ with R as a large parameter. In these terms, the incompressibility condition $\operatorname{div} \mathbf{u} = 0$ becomes

$$\operatorname{div}_z \mathbf{u} + \varepsilon \operatorname{div}_{\zeta\eta} \mathbf{u} = 0,$$

i. e.,

$$\operatorname{div}_z \mathbf{u}_{k+1} = -\operatorname{div}_{\zeta\eta} \mathbf{u}_k.$$

Hence,

$$\frac{\partial}{\partial z}(\mathbf{u}_{k+1})_z = -\left(\frac{\partial(\mathbf{u}_k)_\xi}{\partial\xi} + \frac{\partial(\mathbf{u}_k)_\eta}{\partial\eta}\right).$$

For $k = 0$, it follows from (3.2) that

$$\frac{\partial(\mathbf{u}_1)_z}{\partial z} = -\left(\frac{\partial C_0}{\partial\xi} - C_0\frac{\partial\phi}{\partial\eta}\right)\sin(z+\phi) - \\ -\left(\frac{\partial C_0}{\partial\eta} + C_0\frac{\partial\phi}{\partial\xi}\right)\cos(z+\phi),$$

i. e.,

$$(\mathbf{u}_1)_z = \left(\frac{\partial C_0}{\partial\xi} - C_0\frac{\partial\phi}{\partial\eta}\right)\cos(z+\phi) - \\ -\left(\frac{\partial C_0}{\partial\eta} + C_0\frac{\partial\phi}{\partial\xi}\right)\sin(z+\phi) + \delta_1(\xi,\eta,\tau).$$

We note that due to the last equation

$$(\mathbf{u}_1)_z = (\operatorname{rot}_{\xi\eta} \mathbf{u}_0(\xi,\eta,z,\tau))_z + \delta_1. \tag{3.4}$$

However, it also follows from (3.2) that

$$\delta_1 = -\frac{\partial\phi}{\partial\tau}\,. \tag{3.5}$$

We seek other terms of the asymptotic expansion in the same form as $\mathbf{u}_1$ in (3.4):

$$\mathbf{u}_k(\xi,\eta,z,\tau) = \mathbf{w}_k(\xi,\eta,z,\tau) + \\ + \operatorname{rot}_{\xi\eta} \mathbf{w}_{k-1} + \boldsymbol{\delta}_k(\xi,\eta,\tau), \tag{3.6}$$

where

$$\mathbf{w}_k = \gamma_0^{(k)}(\xi,\eta,\tau)\mathbf{e}_1(z) + \gamma_1^{(k)}(\xi,\eta,\tau)\mathbf{h}_1(z),$$

i. e.,

$$\operatorname{rot}_z \mathbf{w}_k = \mathbf{w}_k,$$

$$\boldsymbol{\delta}_k = \begin{pmatrix} 0 \\ 0 \\ \delta_k(\xi,\eta,\tau) \end{pmatrix}.$$

As is proved in Ref. [27], if we seek the terms of the asymptotic expansion in form (3.6), then $k = 2$ and

$$\varepsilon = R^{1/2}\,.$$

4. SCALE-INVARIANT STREAMLINE EQUATIONS

Now, due to (3.2) and (3.4), we can write the equations for large-scale streamlines:

$$\begin{cases} \dot\xi = C_0(\xi,\eta)\sin(z+\phi(\xi,\eta,\tau)), \\ \dot\eta = C_0(\xi,\eta)\cos(z+\phi(\xi,\eta,\tau)), \end{cases} \tag{4.1}$$

and

$$\frac{dz}{d\tau} = \left(\frac{\partial C_0}{\partial\xi} - C_0\frac{\partial\phi}{\partial\eta}\right)\cos(z+\phi) - \\ -\left(\frac{\partial C_0}{\partial\eta} + C_0\frac{\partial\phi}{\partial\xi}\right)\sin(z+\phi) - \frac{\partial\phi(\xi,\eta,\tau)}{\partial t}.$$

However, due to (4.1),

$$\frac{d\phi}{d\tau} = \frac{\partial\phi}{\partial\xi}\dot\xi + \frac{\partial\phi}{\partial\eta}\dot\eta + \frac{\partial\phi}{\partial\tau} = \\ = C_0\frac{\partial\phi}{\partial\xi}\sin(z+\phi) + C_0\frac{\partial\phi}{\partial\eta}\cos(z+\phi) + \frac{\partial\phi}{\partial\tau},$$

whence

$$\frac{d(z+\phi)}{d\tau} = \frac{\partial C_0}{\partial\xi}\cos(z+\phi) - \frac{\partial C_0}{\partial\eta}\sin(z+\phi)\,. \tag{4.2}$$

Streamline equations (4.1) and (4.2) are actually the scaled equations (2.10). Equations (4.1) are identical to the first and second equations in system (2.10). We impose the requirement of scaling invariance on the streamline equations. Hence, we consider the streamlines $(\bar\xi(\tau), \bar\eta(\tau), \bar z(\tau))$ satisfying Eqs. (4.1) and the equation

$$\frac{d(\bar z(\tau) + \phi(\bar\xi(\tau), \bar\eta(\tau), \tau))}{d\tau} \equiv 0. \tag{4.3}$$

We call these streamlines "quasi-stationary trajectories". Due to (4.2), we then have

$$\operatorname{tg}(\bar z + \phi(\bar\xi,\bar\eta,\tau)) = \frac{\partial C_0/\partial\xi}{\partial C_0/\partial\eta}. \tag{4.4}$$

Then

$$\sin(\bar z + \phi(\bar\xi,\bar\eta,\tau)) = \frac{\partial C_0/\partial\xi}{\sqrt{\left(\dfrac{\partial C_0}{\partial\xi}\right)^2 + \left(\dfrac{\partial C_0}{\partial\eta}\right)^2}},$$

$$\cos(\bar z + \phi(\bar\xi,\bar\eta,\tau)) = \frac{\partial C_0/\partial\eta}{\sqrt{\left(\dfrac{\partial C_0}{\partial\xi}\right)^2 + \left(\dfrac{\partial C_0}{\partial\eta}\right)^2}},$$

and we can rewrite (4.1) for the quasi-stationary trajectories $(\bar{\xi}(\tau), \bar{\eta}(\tau), \bar{z}(\tau))$,

$$\begin{aligned} \dot{\bar{\xi}} &= \frac{C_0 \partial C_0/\partial \xi}{\sqrt{\left(\frac{\partial C_0}{\partial \xi}\right)^2 + \left(\frac{\partial C_0}{\partial \eta}\right)^2}}, \\ \dot{\bar{\eta}} &= \frac{C_0 \partial C_0/\partial \eta}{\sqrt{\left(\frac{\partial C_0}{\partial \xi}\right)^2 + \left(\frac{\partial C_0}{\partial \eta}\right)^2}}, \end{aligned} \tag{4.5}$$

or, in the vector form,

$$\dot{\bar{\Sigma}} = C_0\left(\bar{\Sigma}\right) \frac{\mathrm{grad} C_0\left(\bar{\Sigma}\right)}{\left|\mathrm{grad} C_0\left(\bar{\Sigma}\right)\right|}, \tag{4.6}$$

where

$$\bar{\Sigma} = \begin{pmatrix} \bar{\xi}(\tau) \\ \bar{\eta}(\tau) \end{pmatrix}.$$

Thus, the change of variables ("scaling") and the requirement of the scaling invariance enabled the separation of "slow"(ξ, η) and "fast" z variables for the quasi-stationary equations. Hence, (4.6) states that the tangent vector to the quasistationary trajectory $(\bar{\xi}(\tau), \bar{\eta}(\tau))$ is collinear to the gradient vector of the function $C_0(\xi, \eta)$:

$$\mathrm{grad}\, C_0(\bar{\xi}, \bar{\eta}) = \begin{pmatrix} \dfrac{\partial C_0}{\partial \xi} \\ \dfrac{\partial C_0}{\partial \eta} \end{pmatrix},$$

and the integral lines of the gradient vector field for the function

$$C_0(\xi, \eta) = \left[(Ab_0 + \gamma_0(\xi, \eta))^2 + \gamma_1^2(\xi, \eta)\right]^{1/2}$$

might be regarded as "large-scale structures", where $\gamma_0(\xi, \eta)$ and $\gamma_1(\xi, \eta)$ are the initial small longwave amplitude modulations ("noise") of the dual Beltrami flows

$$\mathbf{e}_1(z) = \begin{pmatrix} \sin z \\ \cos z \\ 0 \end{pmatrix}, \quad \mathbf{h}_1(z) = \begin{pmatrix} \cos z \\ -\sin z \\ 0 \end{pmatrix},$$

and the projection of the "quasistationary" streamline onto the (ξ, η) plane is an integral line of the grad $C_0(\xi, \eta)$ vector field.

It is well known (and can be easily demonstrated [27]) that the integral lines of a gradient field connect the "stationary points" where the gradient of $C_0(\xi, \eta)$ vanishes:

$$|\,\mathrm{grad}\, C_0(\bar{\xi}, \bar{\eta})\,| = 0.$$

As is proved in Ref. [27], the streamline projection of any perturbation of a quasistationary solution onto the $\xi\eta$ plane is a curve asymptotically approaching a "limit curve" defined by the equations for the quasistationary streamlines. Thus, "large scale structures" are formed from these stable ($t \sim 1/\varepsilon$) curves in the xy plane. Hence, the question of the streamline behavior under long-wavelength perturbations of the Trkal solution for the force-free Navier–Stokes equation is reduced to the determination of gradient lines for the function

$$C_0(\xi, \eta) = \left[(Ab_0 + \gamma_0(\xi, \eta))^2 + \gamma_1^2(\xi, \eta)\right]^{1/2}.$$

The stationary points of $C_0(\xi, \eta)$ are either points of maximum ("sources") or minimum values ("sinks") or saddle points.

Each trajectory starts at some maximal point and ends at some minimal point. The saddle point has one incoming and one outgoing trajectory, the separatrices. The plane domain is thus partitioned by the separatrices into invariant subdomains containing the trajectories (plane streamlines) that connect one maximum critical point with one minimum critical point, while there are no other critical points inside these subdomains, i. e., the trajectories inside the subdomains are homotopic. The assumption of the "long-wavelength perturbation" means that $\gamma_0(\xi, \eta)$ and $\gamma_1(\xi, \eta)$ are two-periodic functions in the $\xi\eta$ plane. Therefore, $\gamma_0(\xi, \eta)$ and $\gamma_1(\xi, \eta)$ are finite trigonometric polynomials in two variables, because all spatial frequencies otherwise participate in any Fourier representation of these functions and the concept of the "long-wavelength perturbation" has no meaning.

We restrict ourselves to the case where only the first harmonics are present in these trigonometric polynomials. The case of periodic boundary conditions was investigated by Arnold [30], who wrote these functions in the form

$$\begin{aligned} \bar{\gamma}_0(\xi, \eta) = a\cos\xi + b\sin\xi + c\cos\eta + d\sin\eta + \\ + p\cos(\xi + \eta) + q\sin(\xi + \eta) \end{aligned}$$

and proved that they have six stationary points and allow two different topological pictures (with respect to the diffeomorphism group of the torus) for the level lines. These two pictures (and consequently the gradient line pictures) are determined by the structure of the six stationary points:

one maximum point, three saddle points, and two minimum points;

two maximum points, three saddle points and one minimum point.

It can be easily seen that the plane is divided into "curved polygons", the stationary points are the polygon vertices, and the polygon sides are the separatrices, the gradient lines that separate subsets of homotopic gradient lines. Each polygon is an invariant set under the gradient flow. This unveils the $3D$ picture of a streamline-vortex tube, which is in fact a "curved upright prism" with vertical edges growing from the stationary points, based on the plane "curved polygon" made by the separatrices.

5. EXPLICIT TIME DEPENDENCE AND THE EMERGENCE OF THE UPWARD FLOW

To elucidate the $3D$ behavior of quasistationary trajectories, we have to find other terms of the asymptotic expansion

$$\bar{u}(\xi,\eta,z,\tau) = \mathbf{u}_0(\xi,\eta,z,\tau) + \varepsilon\mathbf{u}_1 + \varepsilon^2\mathbf{u}_2 + \ldots, \quad \operatorname{div}\mathbf{u} = 0. \tag{5.1}$$

We seek other terms in the same form as $\mathbf{u}_1$ in (3.4). For quasistationary trajectory (4.3), we have

$$\bar{z}(\tau) = c_0(\xi_0,\eta_0,\tau_0) - \phi(\bar{\xi}(\tau),\bar{\eta}(\tau),\tau), \quad \xi_0 = \bar{\xi}(\tau_0),\, \eta_0 = \bar{\eta}(\tau_0), \tag{5.2}$$

i.e., we must find $\phi(\xi,\eta,\tau)$. As is proved in Ref. [27], $\phi(\xi,\eta,\tau)$ satisfies the equation

$$\frac{\partial^2\phi}{\partial\tau^2} = -\frac{C_0^2(\xi,\eta)}{2}\Delta_{\xi\eta}\phi - C_0\left(\frac{\partial C_0}{\partial\xi}\frac{\partial\phi}{\partial\xi} + \frac{\partial C_0}{\partial\eta}\frac{\partial\phi}{\partial\eta}\right) \tag{5.3}$$

with the initial conditions

$$\phi(\xi,\eta,0) = \operatorname{arctg}\frac{\gamma_1(\xi,\eta,0)}{\gamma_0(\xi,\eta,0)+A},$$

$$\frac{\partial}{\partial\tau}\phi(x,y,0) = \delta_0(\xi,\eta).$$

Thus, we have a Cauchy problem (with periodic boundary conditions) for an elliptic partial differential equation, which is a classic example of an "ill-posed problem", i.e., the case of instability in time of the phase $\phi(\xi,\eta,\tau)$ as well as of the upward velocity. Because (5.3) was derived without involvement of the viscous terms, we are dealing with the Eulerian instability of the phase. The negative sign before the Laplacian in (5.3) is usually considered as a manifestation of the so-called "negative viscosity" [20, 24, 25] at times $\tau \sim 1$, ($t \sim 1/\varepsilon$). In the case where the initial upward flow is absent, $\phi(\xi,\eta,\tau)$ is nontrivial only if the initial long-wavelength amplitude modulation $\gamma_1(\xi,\eta,0) = \gamma_1(\xi,\eta)$ of the dual flow is nonvanishing, i.e., the explicit time dependence of the flow and the emergence of the upward flow, which makes the flow essentially three dimensional, are due to the initial spatial gradient of the second coefficient in (3.1). As a result, we have found $z(t)$ in (5.2).

In fact, we can speak about the "pseudo chaotic" behavior of $z(t)$, due to (5.3), as opposed to stable structures defined by (4.6). The same function $C_0(\xi,\eta)$ determines (4.6) as well as (5.2), i.e., order and "pseudo chaos" emerge from the same cause: small long-wavelength amplitude modulation in the $\xi\eta$ plane of a pair of dual anisotropic Beltrami flows [11].

This is because the initial conditions in (5.3) involve the function $C_0(\xi,\eta)$ that occurs as the coefficient at the highest-order derivative, and hence a small variation in the initial conditions can cause a large variation of the solution (in fact, this is the so-called "Hadamard example" of instability with respect to initial conditions of the Cauchy problem for elliptic equations) as well as a variation of $C_0(\xi,\eta)$. Thus, variation of the initial conditions in (5.3) can cause instability of the solution and of the vertical velocity, while the solutions of (4.5) remain stable.

It can be easily seen from (3.4) and (5.1) that

$$\mathbf{u} - \operatorname{rot}\mathbf{u} = \varepsilon\boldsymbol{\delta}_1 + O(\varepsilon^2),$$

where

$$\operatorname{rot} \to \varepsilon\operatorname{rot}_{\xi\eta} + \operatorname{rot}_z.$$

The last term in the right-hand side contains all the higher-order terms in the asymptotic expansion, while the first term in the right-hand side is a vector, which is parallel to the z axis. Therefore, up to terms of the order of ε^2, both velocity and vorticity vectors belong to the tangent plane of a vertical surface, which contains a curve in the xy plane, determined by Eqs. (4.5). This is true even when the first term in the right-hand side of the last equation is not small, i.e., when the velocity and vorticity cease to be almost collinear. This surface can therefore be considered a "streamline sheet" as well as a "vortex sheet". Streamline sheets, which contain homotopic quasistationary trajectories of the same subdomain in the xy plane, connecting two fixed stationary points, form invariant $3D$ domains called "streamline tubes". Clearly, up to terms of the order of ε^2, "streamline tubes" are at the same time "vortex tubes". These streamline–vortex tubes are stable at times $\tau \sim 1$, $(t \sim 1/\varepsilon)$.

Hence, $R^{1/2}$ is the characteristic size of the invariant domains in the xy plane or the characteristic "diameter" of the $3D$ invariant streamline–vortex tubes. For times $t \sim R^{1/2}$, the phase $\phi(\xi,\eta,\tau)$ between two dual Beltrami flows satisfies Eq. (5.3) with a negative Laplacian in the right-hand side, i. e., the large scale viscosity becomes negative, while the upward velocity becomes unstable. In fact, the upward velocity would decrease at times $t \propto R^2$ [26, 27].

6. SINGULARITY OF VORTICITY AT STATIONARY POINTS AND STRINGS OF SINGULAR VORTICITY

Due to (4.1) and (3.6), the quasistationary velocity field can be considered as

$$\mathbf{u} = C_0(\bar\Sigma)\frac{\text{grad } C_0(\bar\Sigma)}{|\text{grad } C_0(\bar\Sigma)|} + \frac{1}{\sqrt{R}}\bar{\boldsymbol\delta}(\bar\Sigma,\tau) + \frac{1}{\sqrt{R}}\mathbf{w}_1, \quad (6.1)$$

where, as was demonstrated in Ref. [27],

$$\mathbf{w}_1 = \left|\tilde{\tilde C}(\bar\xi,\bar\eta,\tau)\right| \begin{pmatrix} \sin(z+\phi(\bar\xi,\bar\eta,\tau)+\tilde\phi_0(\bar\xi,\bar\eta)) \\ \cos(z+\phi(\bar\xi,\bar\eta,\tau)+\tilde\phi_0(\bar\xi,\bar\eta)) \\ 0 \end{pmatrix} =$$

$$= \frac{\left|\tilde{\tilde C}(\bar\xi,\bar\eta,\tau)\right|}{|\text{grad}\, C_0(\bar\xi,\bar\eta)|} \begin{pmatrix} \dfrac{\partial C_0}{\partial \xi}\cos\tilde\phi_0 + \dfrac{\partial C_0}{\partial\eta}\sin\tilde\phi_0 \\ \dfrac{\partial C_0}{\partial \eta}\cos\tilde\phi_0 - \dfrac{\partial C_0}{\partial\xi}\sin\tilde\phi_0 \\ 0 \end{pmatrix} =$$

$$= \frac{\left|\tilde{\tilde C}(\bar\xi,\bar\eta,\tau)\right|}{|\text{grad}\, C_0(\bar\xi,\bar\eta)|}(\cos\tilde\phi_0\,\text{grad}\, C_0 + \sin\tilde\phi_0\,\text{ngrad}\, C_0),$$

and

$$\text{ngrad}\, C_0 = \begin{pmatrix} \dfrac{\partial C_0}{\partial\eta}, \\ -\dfrac{\partial C_0}{\partial\xi} \end{pmatrix}$$

is a vector normal to grad C_0. In Eq. (6.1),

$$\overline\Sigma = \begin{pmatrix} \bar\xi \\ \bar\eta \end{pmatrix},$$

$$C_0^2(\bar\Sigma) = (A+\gamma_0(\bar\Sigma))^2 + \gamma_1^2(\bar\Sigma),$$

$$\bar{\boldsymbol\delta}(\bar\Sigma,\tau) = \begin{pmatrix} 0 \\ 0 \\ \bar\delta_1(\bar\Sigma,\tau) \end{pmatrix},$$

$$\bar\delta_1 = -\frac{d\phi}{d\tau} = -\frac{C_0}{|\text{grad}\, C_0|}\left(\frac{\partial C_0}{\partial\xi}\frac{\partial\phi}{\partial\xi} + \frac{\partial C_0}{\partial\eta}\frac{\partial\phi}{\partial\eta}\right) - \\ -\frac{\partial\phi}{\partial\tau} = -\frac{C_0\cdot(\text{grad}\, C_0, \text{grad}\,\phi)}{|\text{grad}\, C_0|} - \frac{\partial\phi}{\partial\tau}.$$

We want to find the behavior of

$$\text{rot}\,\mathbf{u} = \frac{1}{\sqrt{R}}\text{rot}_{\xi\eta}\left(C_0\frac{\text{grad}\, C_0}{|\text{grad}\, C_0|}\right) - \\ - \frac{1}{R}\text{rot}_{\xi\eta}\,\bar{\boldsymbol\delta}_1(\xi,\eta,\tau) + \frac{1}{R}\text{rot}_{\xi\eta}\,\mathbf{w}_1$$

in the neighborhood of a stationary point (ξ_0,η_0) of the function $C_0(\xi,\eta)$:

$$\frac{\partial C_0}{\partial\xi}(\xi=\xi_0,\eta=\eta_0) = \frac{\partial C_0}{\partial\eta}(\xi=\xi_0,\eta=\eta_0) = 0.$$

Then

$$C_0(\xi,\eta) = C_0(\xi_0,\eta_0) + \frac{1}{2}(B\tilde\Sigma,\tilde\Sigma) + \ldots, \quad (6.2)$$

where

$$\tilde\Sigma = \begin{pmatrix} \xi-\xi_0 \\ \eta-\eta_0 \end{pmatrix} = \begin{pmatrix} \tilde\xi \\ \tilde\eta \end{pmatrix}$$

and B is the Hessian of the function $C_0(\xi,\eta)$ at (ξ_0,η_0):

$$B = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad a = \frac{\partial^2 C_0}{\partial\xi^2}(\xi_0,\eta_0),$$

$$b = \frac{\partial^2 C_0}{\partial\xi\partial\eta}(\xi_0,\eta_0), \; c = \frac{\partial^2 C_0}{\partial\eta^2}(\xi_0,\eta_0),$$

while the truncated terms are of the order of $O(\rho^3)$,

$$\rho^2 = \tilde\xi^2 + \tilde\eta^2.$$

Therefore,

$$\text{grad}\, C_0(\tilde\xi,\tilde\eta) = B\tilde\Sigma + \ldots \quad (6.3)$$

and

$$\left|\text{grad}\, C_0(\tilde\Sigma)\right| = |\,B\tilde\Sigma\,| + \ldots \quad (6.4)$$

Because

$$\text{rot}(f\cdot\mathbf{G}) = f\cdot\text{rot}\,\mathbf{G} + [\text{grad}\, f\times\mathbf{G}]$$

and

$$\text{grad}\frac{1}{\left|B\tilde\Sigma\right|} = -\frac{B^2\tilde\Sigma}{\left|B\tilde\Sigma\right|^3},$$

after some transformations, we obtain

$$\frac{1}{\sqrt{R}}\,\mathrm{rot}_{\xi\eta}\,C_0\frac{\mathrm{grad}\,C_0}{|\mathrm{grad}\,C_0|} = = -\frac{2C_0}{\sqrt{R}}\frac{\left[B^2\tilde{\Sigma}\times B\tilde{\Sigma}\right]}{\left|B\tilde{\Sigma}\right|^3}+\ldots \tag{6.5}$$

Because B is a symmetric matrix, we can consider the right-hand side of (6.5) in the eigenbasis with the coordinates $(\tilde{\tilde{\xi}},\tilde{\tilde{\eta}})$:

$$\frac{[B^2\tilde{\Sigma}\times B\tilde{\Sigma}]}{\left|B\tilde{\Sigma}\right|^3} = \frac{\lambda_1\lambda_2(\lambda_1-\lambda_2)\tilde{\tilde{\xi}}\,\tilde{\tilde{\eta}}}{(\lambda_1^2\tilde{\tilde{\xi}}^2+\lambda_2^2\tilde{\tilde{\eta}}^2)^{3/2}}\begin{pmatrix}0\\0\\1\end{pmatrix} = = \frac{\lambda_1\lambda_2(\lambda_1-\lambda_2)\sin^2\varphi}{\tilde{\tilde{\rho}}(\lambda_1^2\cos^2\varphi+\lambda_2^2\sin^2\varphi)^{3/2}}\begin{pmatrix}0\\0\\1\end{pmatrix},$$

where

$$\tilde{\tilde{\rho}}^2 = \tilde{\tilde{\xi}}^2+\tilde{\tilde{\eta}}^2 = \tilde{\xi}^2+\tilde{\eta}^2 = \tilde{\rho}^2 = = (\xi-\xi_0)^2+(\eta-\eta_0)^2 = \frac{1}{R}((x-x_0)^2+(y-y_0)^2) = = \frac{r^2}{R},\quad \tilde{\tilde{\xi}} = \tilde{\rho}\cos\varphi,\quad \tilde{\tilde{\eta}} = \tilde{\rho}\sin\varphi.$$

Hence,

$$|(\mathrm{rot}\,\mathbf{u})_z| \sim \frac{4C_0(x_0,y_0)}{\sqrt{R}}\,\frac{\lambda_1\lambda_2\left|(\lambda_1-\lambda_2)\sin^2\varphi\right|}{(\lambda_1^2+\lambda_2^2+(\lambda_1^2-\lambda_2^2)\sin^2\varphi)}\times \times\frac{\sqrt{R}}{r} = K(\varphi)C_0\frac{|\det B|^{3/2}}{r}.$$

In the same way, we can prove that the upward vector

$$\frac{1}{2}\,\mathrm{rot}\,\mathbf{w}_1 = \frac{1}{2}\,\mathrm{rot}_{\xi\eta}\frac{\left|\tilde{\tilde{C}}(\bar{\xi},\bar{\eta},\tau)\right|}{|\mathrm{grad}\,C_0|}\times \times(\cos\tilde{\phi}_0\mathrm{grad}\,C_0+\sin\tilde{\phi}_0\,\mathrm{ngrad}\,C_0)$$

has singularities of the same type at the stationary points, i. e., the upward component of the vorticity has a singularity of the K/r type at each stationary point (ξ_0,η_0) of the function

$$C_0(\xi,\eta) = C_0\left(\frac{x}{\sqrt{R}},\frac{y}{\sqrt{R}}\right),$$

where $r = \sqrt{(x-x_0)^2+(y-y_0)^2}$ and K is independent of the Reynolds number R because we derived the zeroth term in the asymptotic expansion of the large scale vorticity in powers of $\varepsilon = R^{-1/2}$.

To assess the $\xi\eta$-plane component of the vorticity $(1/R)\,\mathrm{rot}_{\xi\eta}\,\bar{\boldsymbol{\delta}}_1(\tilde{\xi},\tilde{\eta},\tau)$, we have to investigate the behavior of $\bar{\delta}_1(\xi,\eta,\tau)$ in the vicinity of the stationary point. According to (5.3), $\phi(\xi,\eta,\tau)$ is a solution of the Cauchy problem

$$\begin{gathered}\frac{\partial^2\phi}{\partial\tau^2} = -\frac{C_0^2(\xi,\eta)}{2}\Delta_{\xi\eta}\phi - -C_0\left(\frac{\partial C_0}{\partial\xi}\frac{\partial\phi}{\partial\xi}+\frac{\partial C_0}{\partial\eta}\frac{\partial\phi}{\partial\eta}\right),\\ \phi(\xi,\eta,0) = \mathrm{arctg}\frac{\gamma_1(\xi,\eta,0)}{\gamma_0(\xi,\eta,0)+A},\\ \frac{\partial\phi}{\partial\tau}(\xi,\eta,0) = 0,\end{gathered} \tag{6.6}$$

where $\gamma_0(\xi,\eta,0) = \gamma_0(\xi,\eta)$ and $\gamma_1(\xi,\eta,0) = \gamma_1(\xi,\eta)$ are small initial perturbations with a finite number of terms in the Fourier expansion. We can seek a solution of (5.3) in the vicinity of the stationary point as a series in powers of $\tilde{\xi} = \xi-\xi_0$ and $\tilde{\eta} = \eta-\eta_0$. Then the zeroth approximation yields

$$\begin{gathered}\frac{\partial^2\phi_0}{\partial\tau^2} = -\frac{1}{2}C_0(\xi_0,\eta_0)\,\Delta_{\tilde{\xi}\tilde{\eta}}\phi_0(\tilde{\xi},\tilde{\eta},\tau),\\ \phi_0(\tilde{\xi},\tilde{\eta},0) = \mathrm{arctg}\frac{\gamma_1(\tilde{\xi},\tilde{\eta})}{\gamma_0(\tilde{\xi},\tilde{\eta})+A} = \\ = \mathrm{arctg}\frac{\bar{\varepsilon}\,\tilde{\gamma}_1(\tilde{\xi},\tilde{\eta})}{\bar{\varepsilon}\,\tilde{\gamma}_0(\tilde{\xi},\tilde{\eta})+A} = \bar{\varepsilon}\,\frac{\tilde{\gamma}_1}{A}+\ldots,\end{gathered} \tag{6.7}$$

where $\bar{\varepsilon}$ is a small parameter ("initial noise"). Hence, the time dependence is determined by $\gamma_1(\xi,\eta)$ (and therefore $\tilde{\gamma}_1(\tilde{\xi},\tilde{\eta})$). As in Sec. 6, we regard $\gamma_1(\xi,\eta)$ (and therefore $\tilde{\gamma}_1(\tilde{\xi},\tilde{\eta})$) as a finite double trigonometric polynomial,

$$\tilde{\gamma}_1(\tilde{\xi},\tilde{\eta}) = \sum_{m,n}\alpha_{mn}\exp\left[i(m\tilde{\xi}+n\tilde{\eta})\right],$$

where m and n are bounded. Then

$$\phi_0(\tilde{\xi},\tilde{\eta},\tau) = \frac{\bar{\varepsilon}}{A}\sum_{m,n}\times \times\exp\left[i(m\tilde{\xi}+n\tilde{\eta})\right]\exp\left[C_0\sqrt{\frac{\bar{m}^2+n^2}{2}}\tau\right].$$

Thus, the growth of ϕ_0 in time is determined by

$$\max_{m,n}\sqrt{m^2+n^2} = \sqrt{\bar{m}^2+\bar{n}^2}\,.$$

We consider only the fastest growing term:

$$\phi_0(\tilde\xi,\tilde\eta,\tau) = \alpha_{\bar m\bar n}\frac{\bar\varepsilon}{A}\times$$
$$\times\sum_{m,n}\exp\left(C_0\sqrt{\frac{\bar m^2+\bar n^2}{2}}\,\tau\right)\exp\left[i(\bar m\tilde\xi+\bar n\tilde\eta)\right]+\ldots$$

Hence,

$$\frac{\partial\phi_0}{\partial\tau} = \sqrt{\bar m^2+\bar n^2}\,\alpha_{\bar m\bar n}\frac{\bar\varepsilon}{A}\times$$
$$\times\exp\left[i(\bar m\tilde\xi+\bar n\tilde\eta)\right]\exp\left[C_0\sqrt{\frac{\bar m^2+\bar n^2}{2}}\,\tau\right],$$

$$\operatorname{grad}\phi_0 = \sqrt{\bar m^2+\bar n^2}\,\alpha_{\bar m\bar n}\frac{\bar\varepsilon}{A}\times$$
$$\times\exp\left[i(\bar m\tilde\xi+\bar n\tilde\eta)\right]\exp\left[C_0\sqrt{\frac{\bar m^2+\bar n^2}{2}}\,\tau\right]\begin{pmatrix}\bar m\\ \bar n\end{pmatrix},$$

i. e.,

$$|\operatorname{grad}\phi_0| = K\exp\left[C_0\sqrt{\frac{\bar m^2+\bar n^2}{2}}\,\tau\right],$$

where $C_0 = C_0(\xi_0,\eta_0)$. We now consider the $\xi\eta$-plane component of the vorticity:

$$\frac{1}{R}\operatorname{rot}_{\xi\eta}\bar{\boldsymbol\delta}_1 = \frac{1}{R}\times$$
$$\times\begin{pmatrix}\dfrac{\partial}{\partial\eta}\left(C_0(\tilde\Sigma)\dfrac{(\operatorname{grad}C_0,\operatorname{grad}\phi_0)}{|\operatorname{grad}C_0|}\right)+\dfrac{\partial^2\phi_0}{\partial\tau\partial\eta}\\ -\dfrac{\partial}{\partial\xi}\left(C_0(\tilde\Sigma)\dfrac{(\operatorname{grad}C_0,\operatorname{grad}\phi_0)}{|\operatorname{grad}C_0|}\right)-\dfrac{\partial^2\phi_0}{\partial\tau\partial\xi}\end{pmatrix} =$$
$$= \frac{1}{R}\begin{pmatrix}(B\tilde\Sigma,\operatorname{grad}\phi_0)\dfrac{\partial}{\partial\eta}\dfrac{C_0(\tilde\Sigma)}{\left|B\tilde\Sigma\right|}\\ -(B\tilde\Sigma,\operatorname{grad}\phi_0)\dfrac{\partial}{\partial\xi}\dfrac{C_0(\tilde\Sigma)}{\left|B\tilde\Sigma\right|}\end{pmatrix}+\ldots,$$

where the truncated terms are bounded functions of $\tilde\xi$ and $\tilde\eta$ for any given τ. It can be easily checked that

$$\begin{pmatrix}\dfrac{\partial C_0(\tilde\Sigma)}{\partial\eta}\\ -\dfrac{\partial C_0(\tilde\Sigma)}{\partial\xi}\end{pmatrix} = \tilde B\tilde\Sigma+\ldots,$$

where $\tilde B = \begin{pmatrix}b & c\\ -a & -b\end{pmatrix}$, and hence

$$B\tilde B = \begin{pmatrix}0 & ac-b^2\\ b^2-ac & 0\end{pmatrix} = \det B\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

Thus, it can be easily seen that

$$\begin{pmatrix}\dfrac{\partial}{\partial\eta} & \dfrac{C_0(\tilde\Sigma)}{\left|B\tilde\Sigma\right|}\\ -\dfrac{\partial}{\partial\xi} & \dfrac{C_0(\tilde\Sigma)}{\left|B\tilde\Sigma\right|}\end{pmatrix} = -\frac{C(\Sigma_0)\det B\begin{pmatrix}\tilde\eta\\ -\tilde\xi\end{pmatrix}}{\left|B\tilde\Sigma\right|^3}+\ldots,$$

where the truncated terms are bounded for a given τ, and $\Sigma_0 = \begin{pmatrix}\xi_0\\ \eta_0\end{pmatrix}$. It can be demonstrated that

$$\left|\frac{1}{R}\operatorname{rot}_{\xi\eta}\bar{\boldsymbol\delta}_1\right| = \frac{1}{R}\left|\left(B\tilde\Sigma,\operatorname{grad}\phi_0\right)\right|\times$$
$$\times\left\|\begin{pmatrix}\dfrac{\partial}{\partial\eta} & \dfrac{C_0(\tilde\Sigma)}{\left|B\tilde\Sigma\right|}\\ -\dfrac{\partial}{\partial\xi} & \dfrac{C_0(\tilde\Sigma)}{\left|B\tilde\Sigma\right|}\end{pmatrix}\right\| =$$
$$= \frac{\exp\left[C_0\sqrt{\dfrac{\bar m^2+\bar n^2}{2}}\dfrac{t}{\sqrt R}\right]}{\sqrt R r}K_1(\psi)\det B,$$

where ψ is the angle between $\mathbf e_1 = B\tilde\Sigma/\left|B\tilde\Sigma\right|$ and $\mathbf e_2 = \operatorname{grad}\phi_0/|\operatorname{grad}\phi_0|$.

Thus, in the original ("fast") variables, the amplitude of the vorticity-plane component in the vicinity of the stationary point (x_0,y_0) is

$$\frac{K_1(\psi)C_0\left(\dfrac{x_0}{\sqrt R},\dfrac{y_0}{\sqrt R}\right)\det B\exp\left[C_0\sqrt{\dfrac{\bar m^2+\bar n^2}{2R}}\,t\right]}{r\sqrt R}$$

where

$$r = \sqrt{(x-x_0)^2+(y-y_0)^2}.$$

Therefore, the vertical line $x = x_0$, $y = y_0$ is a "string", while the vorticity line rotates around it. Hence, vertical edges of the large-scale streamline–vorticity prism are the strings of singular vorticity.

7. CONCLUSIONS: EXPLICIT TIME DEPENDENCE AND THREE-DIMENSIONALITY JOINTLY EMERGING FROM THE ENERGY DENSITY GRADIENT

We consider the Beltrami triplet with variable coefficients (2.9) as the source of the emergence of large-scale streamline tubes for large values of the Reynolds number R. Although (2.9) clearly indicates the possibility of an upward flow, induced by the presence of the dual Trkal flows with time-dependent amplitudes, there is no inherent clue to an equation for the phase. In (2.9), the absolute value of the velocity in the xy plane is constant. The breakthrough comes by variation of C_0 in the xy plane: it is now supposed to be a bounded smooth function of x and y with a small gradient. Such a function has maximum and minimum points. These are the points of the maximum and minimum values of the plane velocity. The liquid flows from the points of the maximal velocity to the points of the minimal velocity, i. e., the liquid flows along the gradient lines of the function. The distance between these points is determined by the ratio of the velocity change between maximum and minimum values, and the average value of the gradient of the function. Since the first number is finite, while the gradient is small, the distance between maximum and minimum points is large. This is how the plane large-scale streamlines emerge.

The emergence of an unstable upward flow (which is tied to the phase $\phi(\xi, \eta, \tau)$ between the coupled Trkal flows), i. e., the appearance of the "twins" — the explicit time dependence and three-dimensionality of the flow, is induced by a small gradient variation (i. e., by "long-wavelength perturbation") of C_0 (in fact, of $\gamma_1(\xi, \eta, 0) = \gamma_1(\xi, \eta)$) in the xy plane. It might be seen as a manifestation of the "hydrodynamic instability", which in this case is actually the "Eulerian phase instability". The equation for the phase between two dual Trkal flows, which becomes a function of x and y, might be deduced through a rigorous procedure for the asymptotic expansion of the perturbed solution of the force-free Navier–Stokes equation. It turns out that the only possible value of the expansion parameter of the asymptotic expansion consistent with terms of type (5.2) equals $R^{1/2}$.

Hence, the inverse of the "average" gradient of the function C_0, and the distance between maximum and minimum points in the xy plane are close to $R^{1/2}$, as is the characteristic size of the area in the xy plane. Thus, in the case of an anisotropic helical solution for the force-free Navier–Stokes equation at a large Reynolds number R, the initial coupling of large-scale amplitude-modulated dual pair of Trkal (Beltrami) flows together with the orthogonal constant velocity vector form a triplet, which is transformed by a long-wavelength perturbation into a large-scale streamline tube, the plane streamlines being stable at times of the order of $R^{1/2}$. These streamlines, which are gradient lines of the energy density in the orthogonal plane to the anisotropy direction, can be regarded as large-scale structures with the typical size $R^{1/2}$. The gradient lines connect the "stationary points", where the energy density gradient vanishes. The streamlines inside the domains that do not contain "stationary points" are homotopic. The domains of homotopic plane streamlines are bounded by the "separatrices" determining both invariant subsets of the plane flow (invariant under the flow of the liquid) and the invariant $3D$ polygon prisms ("tubes"); the latter are also invariant under velocity and vorticity field flows and are typically characterized by the asymptotic collinearity of the velocity and vorticity vectors. The component of the $3D$ large-scale velocity that is parallel to the anisotropy direction is tied to the phase $\phi(\xi, \eta, \tau)$ between coupled Trkal flows and can be obtained directly as a solution of the Cauchy problem for an elliptic-type equation (the typical case of an ill-posed problem) whose coefficients are determined by the initial conditions. This velocity component outlives the initial Trkal flow and vanishes at times of the order of $t \propto R^2$.

If we call the initial Trkal flow with a finite amplitude A the dominant mode, then the amplitude long-wavelength modulation of the dominant mode $\gamma_0(\xi, \eta)$ is responsible for the emergence of the gradient line picture, while the long-wavelength amplitude modulation $\gamma_1(\xi, \eta)$ of the dual mode is responsible for the unstable upward flow, i. e., for the emergence of the "twins" — the explicit time dependence and three-dimensionality of the secondary flow. Thus, the large-scale streamline–vortex tubes are metastable coherent structures. Although the stationary points inside streamline–vortex tubes are singular points of vorticity, the vorticity lines remain inside the tube, while rotating around the "strings" — vertical lines of singularity that are growing from the stationary points in the xy plane.

REFERENCES

1. E. Levich and A. Tsinober, Phys. Lett. A **93**, 293 (1983).

2. E. Levich and E. Tzvetkov, Phys. Lett. A **100**, 53 (1984).

3. E. Levich, Concepts of Phys. **VI**, 239 (2009).

4. A. Tsinober and E. Levich, Phys. Lett. A **99**, 321 (1983).

5. H. K. Moffatt, J. Fluid Mech. **159**, 359 (1985).

6. H. K. Moffatt, J. Fluid Mech. **166**, 359 (1986).

7. H. K. Moffatt, in Proc. IUTAM Symp. on Geometry and Statistics of Turbulence, Series: Fluid Mechanics and Its Applications, ed. by T. Kambe, T. Nakano, and T. Miyauchi, Vol. 59, ISBN: 978-0-7923-6711-6 (2001).

8. V. Trkal, Časopis št Mat. **48**, 302 (1918).

9. Y. Andreopoulos, Elektrokhimiya **44**, 422 (2008).

10. Yeonatek Choi, Byong-Gu, and Chagoon Lee, Phys. Rev. E **80**, 017309 (2009).

11. M. Farge, G. Pellegrino, and K. Schneider, Phys. Rev. Lett. **87**, 054501 (2001).

12. M. S. Lilley, S. Lovejoy, K. Strawbridge et al., Quart. J. Roy. Meteor. Soc. **134**, 301 (2008).

13. S. Lovejoy, S. Hovde, A. Tuck, and D. Schertzer, Geophys. Res. Lett. **34**, L14802 (2007).

14. S. Lovejoy, D. Schertzer, M. Lilley et al., Quart. J. Roy. Meteor. Soc. **134**, 277 (2008).

15. P. D. Mininni, A. Alexakis, and A. Pouquet, arXiv: physics/0602148v2 [physics.flu-dyn] (2008).

16. P. D. Mininni, A. Alexakis, and A. Pouquet, arXiv:physics/07809.1939v1 [physics.flu-dyn] (2008).

17. J. Molinari and D. Vollaro, Month. Weater Rev. **136**, 4355 (2008).

18. A. Pouquet, A Few Issues in Turbulence and How to Cope with Them, Using Computers, CU Boulder, pouquet@ucar.edu (2011).

19. A. Radkevich, S. Lovejoy, K. Strawbridge et al., Quart. J. Roy. Meteor. Soc. **134**, 315 (2008).

20. A. M. Selvam, *Chaotic Climatic Dynamics*, Lunivers Press, Frome, United Kingdom (2007).

21. G. Sivashinsky, Physica D **17**, 243 (1985).

22. A. Libin and G. I. Sivashinsky, Quart. Appl. Math., **XLVIII**, 611 (1990).

23. A. Libin and G. I. Sivashinsky, Phys. Lett. **144**, 172 (1990).

24. L. Shtilman and G. Sivashinsky, J. de Phys. **47**, 1137 (1986).

25. G. Sivashinsky and V. Yakhot, Phys. Rev. A **35**, 810 (1987).

26. A. Libin, G. I. Sivashinsky, and E. Levich, Phys. Fluids **3**, 2984 (1987).

27. A. Libin, Theor. Math. Phys. **165**, 1534 (2010).

28. H. K. Moffatt, in *Advances in Turbulence*, Springer-Verlag, Berlin–Heidelberg (1987).

29. P. R. Baldwin and G. M. Townsend, Phys. Rev. E **51**, 2059 (1995).

30. V. Arnold, Trudy Mat. Inst. Steklova **258**, 7 (2007).