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HYPERBOLIFICATION OF DYNAMICAL SYSTEMS: THE CASE OF CONTINUOUS-TIME SYSTEMS

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We present a new method to generate chaotic hyperbolic systems. The method is based on the knowledge of a chaotic hyperbolic system and the use of a synchronization technique. This procedure is called hyperbolification of dynamical systems. The aim of this process is to create or enhance the hyperbolicity of a dynamical system. In other words, hyperbolification of dynamical systems produces chaotic hyperbolic (structurally stable) behaviors in a system that would not otherwise be hyperbolic. The method of hyperbolification can be outlined as follows. We consider a known n-dimensional hyperbolic chaotic system as a drive system and another n-dimensional system as the response system plus a feedback control function to be determined in accordance with a specific synchronization criterion. We then consider the error system and apply a synchronization method, and find sufficient conditions for the errors to converge to zero and hence the synchronization between the two systems to be established. This means that we construct a $2n$-dimensional continuous-time system that displays a robust hyperbolic chaotic attractor. An illustrative example is given to show the effectiveness of the proposed hyperbolification method.

1. INTRODUCTION

Generally, the dynamics of a dynamical system is interesting if it has a closed, bounded, and hyperbolic attractor. In fact, the coexistence of highly complicated long-term behavior, sensitive dependence on the initial conditions, and the overall stability of the orbit structure are the most important features resulting from hyperbolicity. In strange attractors of the hyperbolic type, all orbits in phase space are of the saddle type, and the invariant sets of trajectories approach the original one in forward or backward time directions, i. e., the stable and unstable manifolds intersect transversally.

The hyperbolic theory of dynamical systems is widely used for characterizing chaotic behavior of realistic nonlinear systems, but it has never been applied to any physical process with a continuous-time dynamics. Generally, best-known physical systems do not belong to the class of systems with hyperbolic attractors [1, 2]. Because hyperbolic strange attractors are robust (structurally stable) [3], it is interesting to find physical examples of hyperbolic chaos, i. e., noise generators and transmitters in chaos-based communications. Recently, some continuous-time dynamical systems were constructed and confirmed to be hyperbolic. The proof was given based on the corresponding Poincaré map [4]. The method most used for such a construction involves coupled self-sustained oscillators with alternating excitation and invokes the numerical analysis to visualize diagrams illustrating the phase transfer [4–12], where an additional coupling allows transfering the phases simultaneously from one partner to the other in order to obtain the desired chaotic map on a circle or a torus (robust hyperbolic behavior). We note that some of the constructed hyperbolic systems have six variables [9] or eight variables [7].

Realistic examples of physical systems with hyperbolic chaotic attractors are of considerable significance because they open the possibility for real applications of the hyperbolic theory of dynamical systems. As far as we know, all examples of chaotic hyperbolic continuous-time systems were constructed based

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on well-known discrete hyperbolic chaotic maps. In fact, the many applications of chaos synchronization in secure communications [3, 4, 6–23] make it much more important to synchronize two different chaotic systems [13–15, 17–19, 23–25]. Also, it was shown in [26] that the frictionless motion of a mechanical system called triple linkage can be described in terms of a geodesic flow on a surface with everywhere negative Gaussian curvature. In fact, this system is expected to have a hyperbolic chaotic attractor in the presence of friction and an appropriate feedback control law. These two examples show the importance of the hyperbolic nature of dynamical systems modeling real-world phenomena.

In this paper, we present a new method for such a construction based on a known chaotic hyperbolic continuous-time system and a synchronization method, namely, the active control method presented in [15, 17, 23, 25]. An illustrative example is given to show the effectiveness of the proposed hyperbolification method.

2. HYPERBOLIFICATION OF DYNAMICAL SYSTEMS

In this section, we present our method for hyperbolification of continuous-time dynamical systems. This is a partial answer to a question posed in [27]. Indeed, let

$$x_1' = f\left(x_1\right)$$

be a known hyperbolic chaotic system regarded as a drive system, where

$$x_1 = \left(x_1^1, x_1^2, \ldots, x_1^n\right) \in \mathbb{R}^n.$$

Let

$$x_2' = g\left(x_2\right) + U\left(t\right)$$

be the response system, where $U\left(t\right)$ is a feedback control function (in fact, the function $U\left(t\right)$ depends on the time t and the dynamical variables x_1 and x_2) to be determined in accordance with a specific synchronization criterion. Let the error states be

$$e_i = x_2^i - x_1^i, \quad i = 1, 2, \ldots, n.$$

We then consider the error system and apply the synchronization method, and then find sufficient conditions for which the errors $\left(e_i\right)_{1 \le i \le n}$ converge to zero as $t \to \infty$, and hence synchronization between the two systems is achieved. This means that we construct a $2n$-dimensional continuous-time system with a robust hyperbolic chaotic attractor.

The synchronization criterion used in this paper is the active control method presented in [15, 17, 23, 25]. We also use a 4-dimensional continuous-time dynamical system as a drive system. This system corresponds to the 3-dimensional Smale–Williams attractor, the composed equations studied in [4] and given by

$$\begin{aligned} x_1' &= -2\pi u_1 + \left(h_1 + A_1 \cos\frac{2\pi t}{N}\right) x_1 - \frac{1}{3} x_1^3, \\ u_1' &= 2\pi\left(x_1 + \varepsilon_2 y_1 \cos 2\pi t\right), \\ y_1' &= -4\pi v_1 + \left(h_2 - A_2 \cos\frac{2\pi t}{N}\right) y_1 - \frac{1}{3} y_1^3, \\ v_1' &= 4\pi\left(y_1 + \varepsilon_1 x_1^2\right) \end{aligned} \tag{1}$$

which were first introduced in [8]. System (1) is a nonautonomous nonlinear system consisting of two coupled van der Pol oscillators whose frequencies are ω_0 and $2\omega_0$, where $h_{1,2}$, $A_{1,2}$, $\varepsilon_{1,2}$, and N are real constants. System (1) exhibits a Smale–Williams-type strange attractor when it is represented by a 4-dimensional stroboscopic Poincaré map. In this case, the hyperbolicity is verified numerically by analyzing the distribution of the angle φ between the stable and unstable subspaces of manifolds of the resulting chaotic invariant set. System (1) has been constructed as a laboratory device [4], and experimental and numerical solutions were found.

The response system is given by the general equation

$$\begin{aligned} x_2' &= f_1\left(x_2, u_2, y_2, v_2\right) + z_1\left(t\right), \\ u_2' &= f_2\left(x_2, u_2, y_2, v_2\right) + z_2\left(t\right), \\ y_2' &= f_3\left(x_2, u_2, y_2, v_2\right) + z_3\left(t\right), \\ v_2' &= f_4\left(x_2, u_2, y_2, v_2\right) + z_4\left(t\right), \end{aligned} \tag{2}$$

where $f_i\left(x_2, u_2, y_2, v_2\right)$, $1 \le i \le 4$, are smooth functions. We assume that system (2) without the active control functions $z_1(t)$, $z_2(t)$, $z_3(t)$, and $z_4(t)$ displays bounded solutions. The required smoothness of system (2) means that there is a derivative at every point. The advantages of smoothness can be seen in the fact that the local picture can be given by a derivative. Also in the hyperbolic case, the concept of a tangent space, which splits into expanding and contracting directions, requires smoothness of the system under consideration. The functions $z_1(t)$, $z_2(t)$, $z_3(t)$, and $z_4(t)$ are the active control functions to be determined. Let the error states be

$$e_1 = x_2 - x_1, \quad e_2 = u_2 - u_1,$$

$$e_3 = y_2 - y_1, \quad e_4 = v_2 - v_1.$$

Using the active control method, for the active control function

$$U = [z_1(t), z_2(t), z_3(t), z_4(t)]^T$$

we obtain

$$z_1(t) = \left(\alpha - A_1 \cos\frac{2\pi t}{N}\right) e_1 + 2\pi e_2 - \\ - \left(f_1 - \left(h_1 + A_1 \cos\frac{2\pi t}{N}\right) x_2 + 2\pi u_2 + \frac{1}{3}x_1^3\right),$$
$$z_2(t) = -2\pi e_1 + \beta e_2 - 2\pi\varepsilon_2 e_3 \cos 2\pi t - \\ - (f_2 - 2\pi x_2 - 2\pi\varepsilon_2 y_2 \cos 2\pi t),$$
$$z_3(t) = \left(\gamma + A_2 \cos\frac{2\pi t}{N}\right) e_3 + 4\pi e_4 - \\ - \left(f_3 + 4\pi v_2 - \left(h_2 - A_2 \cos\frac{2\pi t}{N}\right) y_2 + \frac{1}{3}y_1^3\right),$$
$$z_4(t) = -4\pi e_3 + \delta e_4 - \left(f_4 - 4\pi y_2 - 4\pi\varepsilon_1 x_1^2\right),$$

where α, β, γ, and δ are real parameters to be chosen such that the error states e_i, $1 \le i \le 4$, converge to zero, and the response system (2) becomes

$$x_2' = \left(\alpha - A_1 \cos\frac{2\pi t}{N}\right) e_1 + 2\pi e_2 + \\ + \left(h_1 + A_1 \cos\frac{2\pi t}{N}\right) x_2 - 2\pi u_2 + \frac{1}{3}x_1^3,$$
$$u_2' = -2\pi e_1 + \beta e_2 - (2\pi\varepsilon_2 \cos 2\pi t)\, e_3 + \\ + 2\pi x_2 + (2\pi\varepsilon_2 \cos 2\pi t)\, y_2, \quad (3)$$
$$y_2' = \left(\gamma + A_2 \cos\frac{2\pi t}{N}\right) e_3 + 4\pi e_4 - 4\pi v_2 + \\ + \left(h_2 - A_2 \cos\frac{2\pi t}{N}\right) y_2 - \frac{1}{3}y_1^3,$$
$$v_2' = -4\pi e_3 + \delta e_4 + 4\pi y_2 + 4\pi\varepsilon_1 x_1^2.$$

We note that equation (3) is an 9-dimensional dynamical system (where t is a variable) relating solutions of the drive system (1) and the response system (2). With the particular choice of the functions $z_1(t)$, $z_2(t)$, $z_3(t)$, and $z_4(t)$, the closed loop system is given by

$$e_1' = (h_1 + \alpha)\, e_1, \quad e_2' = \beta e_2,$$
$$e_3' = (h_2 + \gamma)\, e_3, \quad e_4' = \delta e_4,$$

whose eigenvalues are $h_1 + \alpha$, β, $h_2 + \gamma$, and δ. Then, for any set of parameters α, β, γ, and δ such that

$$\alpha < -h_1, \quad \beta < 0, \quad \gamma < -h_2, \quad \delta < 0,$$

the linear system for e_i, $1 \le i \le 4$, is asymptotically stable. This choice leads to the error states e_1, e_2,

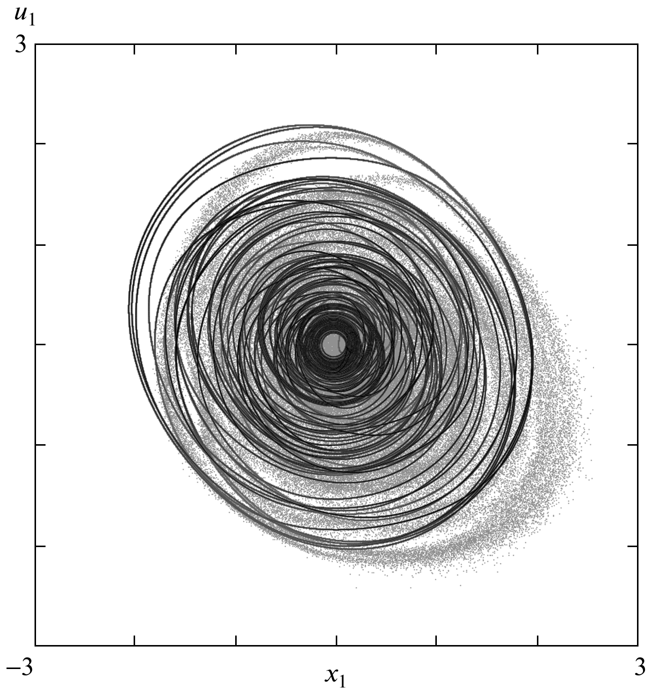

Fig. 1. The chaotic attractor of the drive system (1) for $N = 8$, $A_1 = 1.5$, $A_2 = 6$, $\varepsilon_{1,2} = 0.1$, and $h_{1,2} = 0$

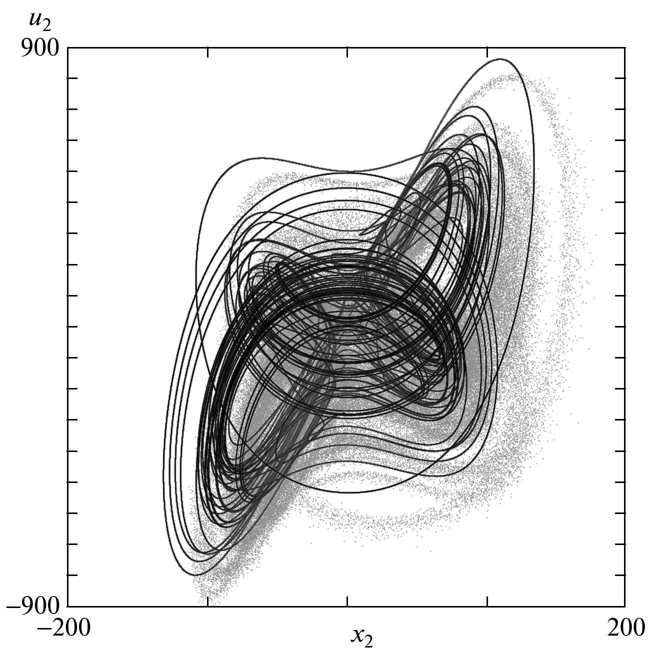

Fig. 2. The chaotic attractor of the response system (4) for $\sigma = 10$, $b = 8/3$, $s = 1619$, and $r = 2289$

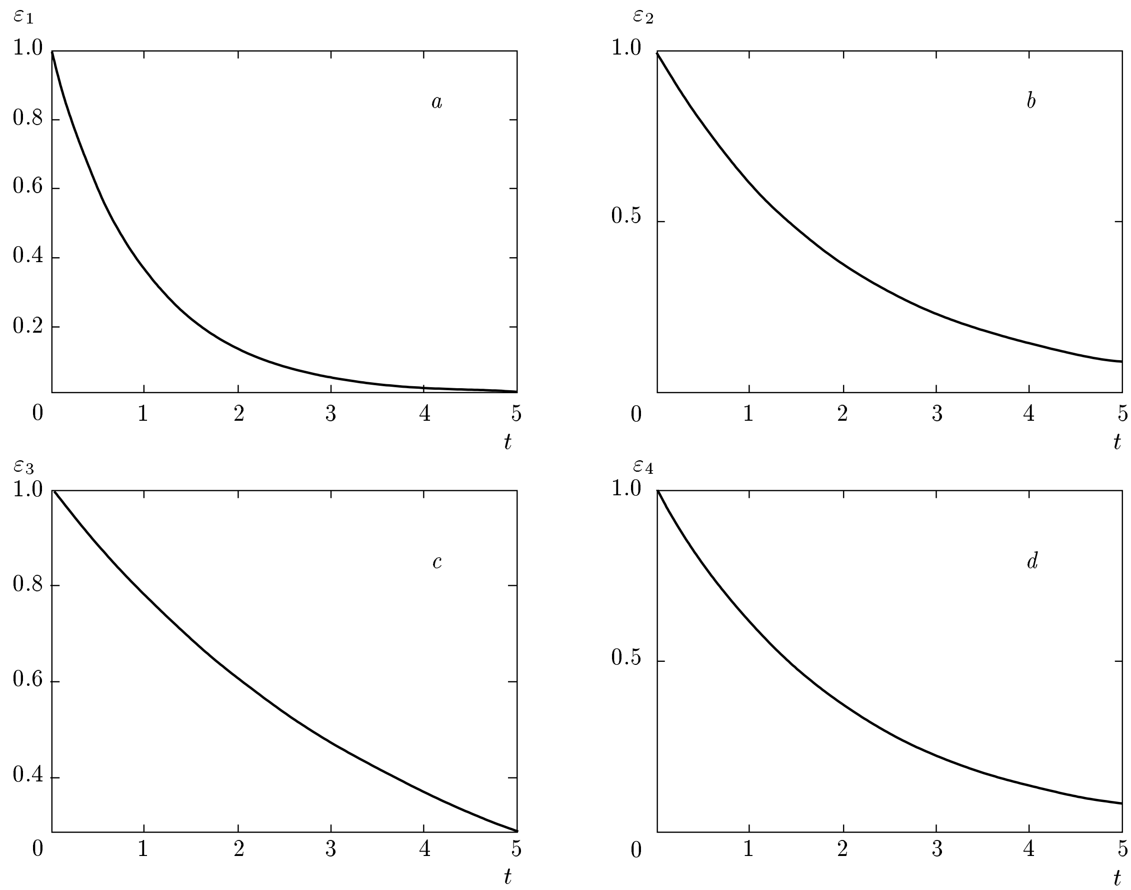

Fig. 3. The dynamics of synchronization errors states, $e_i(t)$, $1 \le i \le 4$, for systems (1) and (4)

e_3, and e_4 converging to zero as $t \to \infty$, and hence the synchronization between the general system (3) and the chaotic hyperbolic system (1) is achieved. For the parameters $h_{1,2}$, $A_{1,2}$, $\varepsilon_{1,2}$, and N with which system (1) displays robust (hyperbolic) chaos (for example $N = 8$, $A_1 = 1.5$, $A_2 = 6$, $\varepsilon_{1,2} = 0.1$, and $h_{1,2} = 0$ as shown in [4]), it drives another chaotic attractor resulting from the general system (3), which is also robust hyperbolic because the system error between (1) and (3) converges to zero for large time t.

3. NUMERICAL SIMULATION

In this section, we take the Lorenz–Stenflo system given by

$$\begin{aligned} x_2' &= -\sigma(x_2 - u_2) + s v_2, \\ u_2' &= -x_2 y_2 + r x_2 - u_2, \\ y_2' &= x_2 u_2 - b y_2, \\ v_2' &= -x_2 - \sigma v_2, \end{aligned} \tag{4}$$

as the response system and system (1) as the drive system. Here σ, r, b, $s \in \mathbb{R}$ are the bifurcation parameters of system (4). Lorenz–Stenflo system (4) describes finite-amplitude, low-frequency, short-wave-length, acoustic gravity waves in a rotational system [28]. The drive system (1) displays robust (hyperbolic) chaos for $N = 8$, $A_1 = 1.5$, $A_2 = 6$, $\varepsilon_{1,2} = 0.1$, and $h_{1,2} = 0$ [4]. Its attractor is shown in Fig. 1, and the response system (4) displays chaos for $\sigma = 10$, $b = 8/3$, $s = 1619$, $r = 2289$, with an attractor as shown in Fig. 2.

For the active control function

$$U = [z_1(t), z_2(t), z_3(t), z_4(t)]^T$$

defined above, we choose α, β, γ, and δ as

$$\alpha = -1 < -h_1 = 0, \quad \beta = -0.5 < 0,$$

$$\gamma = -0.25 < -h_2 = 0, \quad \delta = -0.5 < 0.$$

The dynamics of synchronization errors states $e_i(t)$, $1 \le i \le 4$, for systems (1) and (4) are shown in Fig. 3.

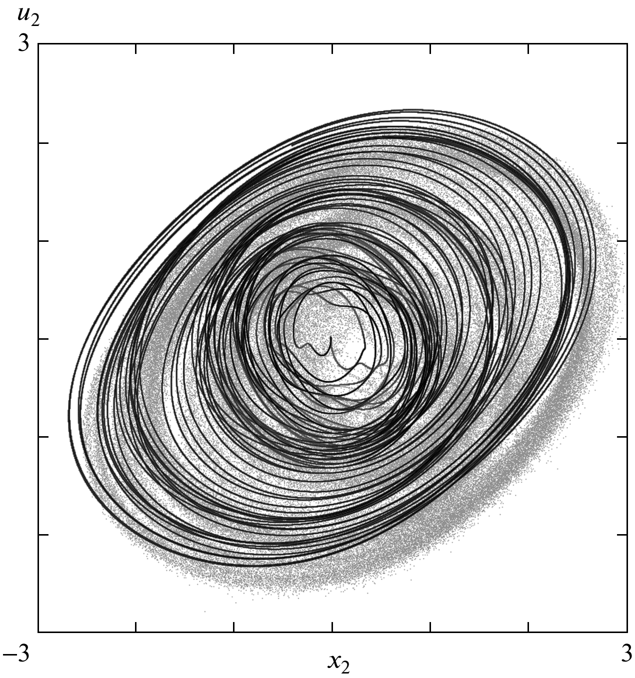

Fig. 4. The chaotic attractor of system (3) for $N = 8$, $A_1 = 1.5$, $A_2 = 6$, $\varepsilon_{1,2} = 0.1$, $h_{1,2} = 0$, $\sigma = 10$, $b = 8/3$, $s = 1619$, and $r = 2289$

Finally, it is clear that the synchronization error converges to zero, and therefore synchronization between the two systems (1) and (4) is achieved. The solution of the response system (3) is shown in Fig. 4 (the largest Lyapunouv exponent of this system is about 0.085). It seems that the dynamics of system (3) is inspired by the one of system (1). This fact is exactly the main meaning of the claim that system (1) drives system (4).

We note that it is possible to use other synchronization methods such as those in [13–15, 17–19, 23–25] or other known hyperbolic systems such as those in [4–12] to generate chaotic attractors with a hyperbolic structure just like system (1).

Finally, our proposed method to hyperbolification of continuous-time dynamical systems opens new directions in studying the nature of chaos in these systems and improves possibilities for *robust* real-world applications of hyperbolic systems, which are structurally stable. Structural stability means the robustness of solutions of the governing dynamical equations if the changes are sufficiently small.

4. CONCLUSION

We have presented a new method to generate chaotic hyperbolic systems based on the knowledge of a chaotic hyperbolic system and the use of a synchronization technique. This process creates hyperbolicity in a dynamical system and generates structurally stable chaotic attractors. An illustrative example is given to show the effectiveness of the proposed hyperbolification method.

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