ELECTROMAGNETIC PLANE WAVES WITH NEGATIVE PHASE VELOCITY IN CHARGED BLACK STRINGS

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We investigate the propagation regions of electromagnetic plane waves with negative phase velocity in the ergosphere of static charged black strings. For such a propagation, some conditions for negative phase velocity are established that depend on the metric components and the choice of the octant. We conclude that these conditions remain unaffected by the negative values of the cosmological constant.

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1. INTRODUCTION

The phenomenon of negative phase velocity (NPV) propagation is important due to one of its consequences, negative refraction [1, 2]. This is the property of light propagation in a medium that occurs when the phase velocity of a plane wave has a negative projection onto the time-average Poynting vector. Alternatively, it is the plane wave propagation mode in which the wave vector and the time-average Poynting vector are oppositely aligned [3–5]. Negative refraction is an electromagnetic phenomenon in which light rays are refracted at the interface in a sense reverse to that normally expected. This property of negative refraction has generated considerable attention in the electromagnetic, optics, and material research communities [2, 3, 6].

Metamaterials are synthetic materials with unusual refractive index properties used to obtain the negative refraction effect. Negative values of the permittivity ϵ and permeability μ are responsible for these unusual refractive index properties. This was originally proposed by Veselago [7]. The direct consequence of this property is the development of a wave propagation medium called the Veselago medium. The presence of the Veselago medium alters the propagation of plane waves such that the electric field, the magnetic field, and the wave vector follow a left-hand rule instead of the right-hand rule. This leads to the construction of left-handed metamaterials [8]. Lenses with extremely low distortion are one of the most useful applications of NPV supporting artificial metamaterial. These are widely used in the modern optics, for communication, entertainment, and data storage as well as for retrieval purposes [9–12].

The characteristics of NPV materials lead to the concept of anisotropic and bianisotropic materials that provide industrial benefits in modern technology [2, 4, 13]. The application of NPV propagation in astrophysical scenarios has been explored in the last few years. It was shown in [14, 15] that the vacuum can support NPV propagation for particular spacetimes. The same authors [13] proved that the de Sitter spacetime supports NPV propagation, whereas the anti-de Sitter metric does not admit such a propagation. The propagation of electromagnetic plane waves with NPV in the Schwarzschild–de Sitter spacetime was investigated in [16]. Some regions supporting NPV propagation within the ergosphere of an uncharged rotating and charged rotating black holes were explored in [17, 18]. Plasma wave properties of the Schwarzschild and Schwarzschild–de Sitter horizons in a Veselago medium were discussed in [19].

In this paper, we investigate propagation of electromagnetic plane waves of static charged black strings described by a cylindrical symmetric spacetime with a negative cosmological constant. The regions of NPV propagation are explored. The format of the paper is as follows. In the next section, we review the mathematical formalism. Section 3 describes the static charged

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black strings and plane wave propagation in **R**. In Sec. 4, we investigate the conditions of NPV. Finally, we discuss and summarize the results in the last section.

2. REVIEW OF THE MATHEMATICAL FORMALISM

In this section, we review the mathematical formulation needed to discuss the propagation of electromagnetic waves in the vacuum in a curved spacetime. This is based on the formal analogy between electromagnetic waves in a flat spacetime in a fictitious instantaneously responding medium and in the curved spacetime in free space. Tamm [20] originally proposed this approach which was used by many authors [21–26].

The source-free covariant Maxwell equations for a curved spacetime are

$$F_{\alpha\beta;\nu} + F_{\beta\nu;\alpha} + F_{\nu\alpha;\beta} = 0, \quad F^{\beta\alpha}_{;\beta} = 0,$$
$$\alpha, \beta = 0, 1, 2, 3.$$

For a flat spacetime, these equations reduce to

$$F_{\alpha\beta,\nu} + F_{\beta\nu,\alpha} + F_{\nu\alpha,\beta} = 0, \quad (-g)^{1/2} F^{\beta\alpha}_{,\beta} = 0. \tag{1}$$

Here, $F^{\alpha\beta}$ and $F_{\alpha\beta}$ are the contravariant and covariant electromagnetic field tensors and

$$g = \det[g_{\alpha\beta}].$$

The semicolon $(;)$ and comma $(,)$ respectively indicate covariant and ordinary derivatives. These equations can be rewritten as

$$\begin{gathered} B_{i,i} = 0, \quad B_{i,0} + \varepsilon_{ijk} E_{j,k} = 0, \quad D_{i,i} = 0, \\ -D_{i,0} + \varepsilon_{ijk} H_{j,k} = 0, \quad i, j, k = 1, 2, 3, \end{gathered} \tag{2}$$

where B_i, E_j, D_i, and H_j are the components of the magnetic field vector $\mathbf{B}$, electric field vector $\mathbf{E}$, displacement field vector $\mathbf{D}$, and magnetizing field vector $\mathbf{H}$ and ε_{ijk} is the three-dimensional Levi-Civita symbol.

The electromagnetic field vectors $\mathbf{E}$, $\mathbf{B}$, $\mathbf{D}$, and $\mathbf{H}$ are

$$\begin{gathered} E_i = F_{i0}, \quad B_i = (1/2)\varepsilon_{ijk} F_{jk}, \\ D_i = (-g)^{1/2} F^{i0}, \quad H_i = (1/2)\varepsilon_{ijk} (-g)^{1/2} F^{jk}. \end{gathered} \tag{3}$$

These vectors satisfy the constitutive relations of an equivalent instantaneously responding medium that can describe the electromagnetic response of the vacuum in a curved spacetime. These constitutive relations are

$$\mathbf{D} = \epsilon_0 \underline{\underline{\gamma}} \mathbf{E}, \quad \mathbf{B} = \mu_0 \underline{\underline{\gamma}} \mathbf{H}, \tag{4}$$

where $\varepsilon_0 = 8.854 \cdot 10^{-12}$ Fm^{-1} and $\mu_0 = 4\pi \times \times 10^{-12}$ Hm^{-1} in SI units. The dyadic $\underline{\underline{\gamma}}$ can be expressed in the metric form

$$\gamma_{ab} = -(-g)^{1/2} \frac{g^{ab}}{g_{00}}. \tag{5}$$

In the 3×3 dyadic form, Eqs. (2) and (4) can be written as [15, 17]

$$\begin{gathered} \nabla \times \mathbf{E}(ct, \mathbf{r}) + \frac{\partial \mathbf{B}(ct, \mathbf{r})}{\partial t} = 0, \\ \nabla \times \mathbf{H}(ct, \mathbf{r}) - \frac{\partial \mathbf{D}(ct, \mathbf{r})}{\partial t} = 0, \end{gathered} \tag{6}$$

$$\begin{gathered} \mathbf{D}(ct, \mathbf{r}) = \epsilon_0 \underline{\underline{\gamma}}(ct, \mathbf{r}) \cdot \mathbf{E}(ct, \mathbf{r}), \\ \mathbf{B}(ct, \mathbf{r}) = \mu_0 \underline{\underline{\gamma}}(ct, \mathbf{r}) \cdot \mathbf{H}(ct, \mathbf{r}). \end{gathered} \tag{7}$$

3. CHARGED BLACK STRINGS AND WAVE PROPAGATION

Static charged black strings with a negative cosmological constant have the line element of the form [27, 28]

$$\begin{aligned} ds^2 = &-\left(\alpha^2 r^2 - \frac{b}{\alpha r} + \frac{c^2}{\alpha^2 r^2}\right) dt^2 + \\ &+ \left(\alpha^2 r^2 - \frac{b}{\alpha r} + \frac{c^2}{\alpha^2 r^2}\right)^{-1} dr^2 + \\ &+ r^2 d\theta^2 + \alpha^2 r^2 dz^2, \end{aligned} \tag{8}$$

where

$$\alpha^2 = -\frac{\Lambda}{3}, \quad b = 4GM, \quad c^2 = 4GQ^2,$$
$$h(r) = \frac{2Q}{\alpha r} + \text{const},$$
$$-\infty < t < \infty, \quad 0 \le r < \infty,$$
$$-\infty < z < \infty, \quad 0 \le \theta \le 2\pi.$$

Here, M is the mass and Q is the linear charge density per unit length of the z line of the black strings, G is the gravitational constant, and $\Lambda < 0$ is the cosmological constant. The black hole horizons are found by setting $g_{00} = 0$,

$$r_{\pm} = \frac{b^{\frac{1}{3}} \sqrt{s + \sqrt{2(s^2 - 4p^2 - s)}}}{2\alpha}, \tag{9}$$

where

$$s = \left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - 4\left(\frac{4p^2}{3}\right)^3}\right)^{1/3} + \left(\frac{1}{2} - \frac{1}{2}\sqrt{1 - 4\left(\frac{4p^2}{3}\right)^3}\right)^{1/3}, \quad p^2 = \frac{c^2}{b^{\frac{4}{3}}}.$$

Here, r_- and r_+ represent the inner and outer event horizons. In order to have a physical region, we take r_+ only and neglect the inner event horizon.

For $Q = 0$, Eq. (8) yields the line element of static black strings

$$ds^2 = -\left(\alpha^2 r^2 - \frac{b}{\alpha r}\right) dt^2 + \left(\alpha^2 r^2 - \frac{b}{\alpha r}\right)^{-1} dr^2 + r^2 d\theta^2 + \alpha^2 r^2 dz^2, \quad (10)$$

where mass is the only parameter and the respective event (outer) horizon is

$$r = r_+ = \frac{b^{\frac{1}{3}}}{\alpha}. \quad (11)$$

Because $\underline{\underline{\gamma}}$ is a second-rank Cartesian tensor [13, 22, 23], we convert metric (8) into Cartesian coordinates as

$$g_{ab} = \begin{pmatrix} -f & 0 & 0 & 0 \\ 0 & \dfrac{x^2 + y^2 f}{r^2 f} & \dfrac{xy(1-f)}{r^2 f} & 0 \\ 0 & \dfrac{xy(1-f)}{r^2 f} & \dfrac{y^2 + x^2 f}{r^2 f} & 0 \\ 0 & 0 & 0 & \alpha^2 r^2 \end{pmatrix}, \quad (12)$$

where

$$g = -\alpha^2 r^4,$$

$$f = \alpha^2 r^2 - \frac{b}{\alpha r} + \frac{c^2}{\alpha^2 r^2}.$$

The constitutive relations provide global description of the cylindrically symmetric spacetime. To approximate a nonuniform metric γ_{ab} by a uniform metric $\tilde{\gamma}_{ab}$, we consider the partition of the global spacetime into sufficiently small and adjoining neighborhoods $\mathbf{R}$ at arbitrary locations $(\tilde{x}, \tilde{y}, \tilde{z})$. We usually solve differential equations with nonhomogeneous coefficients by this method. The uniform metric is defined as [16, 17]

$$[\tilde{\gamma}_{ab}] = \begin{pmatrix} \dfrac{\alpha \tilde{f}\tilde{x}^2 + \tilde{y}^2}{\tilde{f}} & \dfrac{\alpha(-1+\tilde{f})\tilde{x}\tilde{y}}{\tilde{f}} & 0 \\ \dfrac{\alpha(-1+\tilde{f})\tilde{x}\tilde{y}}{\tilde{f}} & \dfrac{\alpha \tilde{f}\tilde{y}^2 + \tilde{x}^2}{\tilde{f}} & 0 \\ 0 & 0 & \dfrac{1}{\alpha \tilde{f}} \end{pmatrix}, \quad (13)$$

where

$$\det\left[\underline{\underline{\tilde{\gamma}}}\right] = \frac{\alpha(\tilde{f}\tilde{x}^2 + \tilde{y}^2)(\tilde{x}^2 + \tilde{f}\tilde{y}^2)}{\tilde{f}^3},$$

$$\tilde{f} = \alpha^2 \tilde{r}^2 - \frac{b}{\alpha \tilde{r}} + \frac{c^2}{\alpha^2 \tilde{r}^2}.$$

To discuss the propagation of plane waves in the medium defined by constitutive relations (7), we consider the plane-wave solutions

$$\begin{aligned} \mathbf{E} &= \operatorname{Re}\mathbf{E}_0 \exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)], \\ \mathbf{H} &= \operatorname{Re}\mathbf{H}_0 \exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)], \end{aligned} \quad (14)$$

where $\mathbf{k}$ is the wave vector, $\mathbf{r}$ is the position vector within the neighborhood $\mathbf{R}$ containing an arbitrary location $(\tilde{x}, \tilde{y}, \tilde{z})$, t denotes the time and ω is the angular frequency. Also, $\mathbf{E}_0$ and $\mathbf{H}_0$ represent complex-valued amplitudes. When Eq. (14) is used in Eq. (6), an eigenvector equation is obtained after some algebraic manipulations. The resulting equation is given by

$$\left[\left(k_0^2 \det\left[\underline{\underline{\tilde{\gamma}}}\right] - \mathbf{k}\cdot\underline{\underline{\tilde{\gamma}}}\cdot\mathbf{k}\right)\underline{\underline{I}} + \mathbf{k}\mathbf{k}\cdot\underline{\underline{\tilde{\gamma}}}\right]\cdot\mathbf{E}_0 = 0, \quad (15)$$

where

$$k_0 = \omega\sqrt{\epsilon_0 \mu_0}.$$

The corresponding dispersion relation can be written as

$$k_0^2 \det\left[\underline{\underline{\tilde{\gamma}}}\right]\left(k_0^2 \det\left[\underline{\underline{\tilde{\gamma}}}\right] - \mathbf{k}\cdot\underline{\underline{\tilde{\gamma}}}\cdot\mathbf{k}\right)^2 = 0. \quad (16)$$

Since $\det\left[\underline{\underline{\tilde{\gamma}}}\right]$ is nonzero, this equation leads to

$$\mathbf{k}\cdot\underline{\underline{\tilde{\gamma}}}\cdot\mathbf{k} = k_0^2 \det\left[\underline{\underline{\tilde{\gamma}}}\right]. \quad (17)$$

With this value used in Eq. (15), it follows that $\mathbf{k}\mathbf{k}\cdot\underline{\underline{\tilde{\gamma}}}\cdot\mathbf{E}_0 = 0$, which shows that $\mathbf{k}\cdot\underline{\underline{\tilde{\gamma}}}$ and $\mathbf{E}_0$ are orthogonal.

Let the wave vector $\mathbf{k}$ be described as

$$\mathbf{k} = k\hat{\mathbf{k}} = \frac{k(\hat{\mathbf{u}}_x, \hat{\mathbf{u}}_y, \hat{\mathbf{u}}_z)}{3}, \quad (18)$$

where $\hat{\mathbf{u}}_x$, $\hat{\mathbf{u}}_y$, and $\hat{\mathbf{u}}_z$ represent unit vectors along $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$ axes and k is the magnitude of the wave vector. Hence,

$$\mathbf{k}\cdot\underline{\underline{\tilde{\gamma}}} = \frac{k\left[(\tilde{\gamma}_{11}+\tilde{\gamma}_{21})\hat{\mathbf{u}}_x + (\tilde{\gamma}_{12}+\tilde{\gamma}_{22})\hat{\mathbf{u}}_y + \tilde{\gamma}_{33}\hat{\mathbf{u}}_z\right]}{3}.$$

Furthermore,

$$\mathbf{k}\cdot\underline{\underline{\tilde{\gamma}}}\cdot\mathbf{k} = \frac{k^2\left[(\tilde{\gamma}_{11}+\tilde{\gamma}_{21}) + (\tilde{\gamma}_{12}+\tilde{\gamma}_{22}) + \tilde{\gamma}_{33}\right]}{9}, \quad (19)$$

where $\tilde{\gamma}_{11}$, $\tilde{\gamma}_{12}$, $\tilde{\gamma}_{21}$, $\tilde{\gamma}_{22}$, $\tilde{\gamma}_{33}$ are the components of the metric $[\gamma_{ab}]$. Substituting Eq. (19) in (17), we obtain

$$k^2 = \frac{9k_0^2 \det\left[\underline{\underline{\tilde{\gamma}}}\right]}{(\tilde{\gamma}_{11}+\tilde{\gamma}_{12}) + (\tilde{\gamma}_{12}+\tilde{\gamma}_{22}) + \tilde{\gamma}_{33}}.$$

Inserting the values of $\tilde{\gamma}_{11}$, $\tilde{\gamma}_{12}$, $\tilde{\gamma}_{21}$, $\tilde{\gamma}_{22}$, $\tilde{\gamma}_{33}$ in the above equation, we obtain

$$k^2 = \frac{9k_0^2\alpha^2(\tilde{f}\tilde{x}^2+\tilde{y}^2)(\tilde{x}^2+\tilde{y}^2\tilde{f})}{\tilde{f}^2\left[\alpha^2(\tilde{f}(\tilde{x}+\tilde{y})^2+(\tilde{x}-\tilde{y})^2)+1\right]},$$

which yields the wave numbers $k = k^{\pm}$

$$k^{\pm} = 3k_0\sqrt{\frac{\alpha^2(\tilde{f}\tilde{x}^2+\tilde{y}^2)(\tilde{x}^2+\tilde{y}^2\tilde{f})}{\tilde{f}^2\left[\alpha^2(\tilde{f}(\tilde{x}+\tilde{y})^2+(\tilde{x}-\tilde{y})^2)+1\right]}}. \quad (20)$$

For propagation of waves, the values of wave numbers $k^{\pm}$ must be real, which lead to

$$\tilde{f}^2\left[\alpha^2(\tilde{f}(\tilde{x}+\tilde{y})^2+(\tilde{x}-\tilde{y})^2)+1\right] \neq 0, \quad \tilde{f} > 0. \quad (21)$$

We impose the condition $f > 0$ because $f < 0$ provides the nonphysical region $r < r_+$. The general solution of Eq. (15) can be written as

$$\mathbf{E}_0 = C_1\mathbf{e}_1 + C_2\mathbf{e}_2, \quad (22)$$

where C_1 and C_2 are complex constants and $\mathbf{e}_1$ and $\mathbf{e}_2$ are two linearly independent eigenvectors given by [15]:

$$\mathbf{e}_1 = (\tilde{\gamma}_{11}+\tilde{\gamma}_{22})\hat{\mathbf{u}}_x - (\tilde{\gamma}_{11}+\tilde{\gamma}_{12})\hat{\mathbf{u}}_y, \quad (23)$$

$$\mathbf{e}_2 = \tilde{\gamma}_{33}(\tilde{\gamma}_{11}+\tilde{\gamma}_{12})\hat{\mathbf{u}}_x + \tilde{\gamma}_{33}(\tilde{\gamma}_{12}+\tilde{\gamma}_{22})\hat{\mathbf{u}}_y - \left[(\tilde{\gamma}_{11}+\tilde{\gamma}_{12})^2 + (\tilde{\gamma}_{12}+\tilde{\gamma}_{22})^2\right]\hat{\mathbf{u}}_z. \quad (24)$$

The application of Fourier transformation (14) to Eqs. (6) and (7) leads to

$$\mathbf{k}\times\mathbf{H}_0 = -\omega\epsilon_0\underline{\underline{\tilde{\gamma}}}\cdot\mathbf{E}_0, \quad \mathbf{k}\times\mathbf{E}_0 = \omega\mu_0\underline{\underline{\tilde{\gamma}}}\cdot\mathbf{H}_0. \quad (25)$$

Combining Eqs. (22) and (25) yields the magnetic induction [15, 29]

$$\mathbf{H}_0 = \frac{k\left[\underline{\underline{\tilde{\gamma}}}^{-1}\cdot\hat{\mathbf{k}}\times(C_1\mathbf{e}_1 + C_2\mathbf{e}_2)\right]}{\omega\mu_0}. \quad (26)$$

4. CONDITIONS FOR NEGATIVE PHASE VELOCITY

The negative phase velocity is defined by [30]

$$\mathbf{k}\cdot\langle\mathbf{P}\rangle_t < 0, \quad (27)$$

where

$$\langle\mathbf{P}\rangle_t = \frac{1}{2}\,\mathrm{Re}\{\mathbf{E}_0\times\mathbf{H}_0^*\}$$

is the time-average Poynting vector, which gives the associated rate of energy flow over time and $\mathbf{H}_0^*$ is the complex conjugate of $\mathbf{H}_0$. Substituting Eqs. (22) and (26) in the above equation leads to

$$\langle\mathbf{P}\rangle_t = \frac{k}{2\omega\mu_0}[|C_1|^2(\mathbf{e}_1\times\mathbf{p}\times\mathbf{e}_1) + |C_2|^2(\mathbf{e}_2\times\mathbf{p}\times\mathbf{e}_2) + 2|C_1||C_2|(\mathbf{e}_2\times\mathbf{p}\times\mathbf{e}_1)],$$

where

$$\mathbf{p} = \underline{\underline{\gamma}}^{-1}\cdot\hat{\mathbf{k}}.$$

Inserting this value in Eq. (27), we obtain

$$\frac{k^2}{2\omega\mu_0}[|C_1|^2(\mathbf{e}_1\times\mathbf{p}\times\mathbf{e}_1) + |C_2|^2(\mathbf{e}_2\times\mathbf{p}\times\mathbf{e}_2) + 2|C_1||C_2|(\mathbf{e}_2\times\mathbf{p}\times\mathbf{e}_1)]\cdot\hat{\mathbf{k}} < 0. \quad (28)$$

This is satisfied if the following three inequalities hold simultaneously:

$$(\mathbf{e}_1\times\mathbf{p}\times\mathbf{e}_1)\cdot\hat{\mathbf{k}} < 0, \quad (\mathbf{e}_2\times\mathbf{p}\times\mathbf{e}_2)\cdot\hat{\mathbf{k}} < 0, \quad (\mathbf{e}_2\times\mathbf{p}\times\mathbf{e}_1)\cdot\hat{\mathbf{k}} < 0. \quad (29)$$

Now,

$$(\mathbf{e}_1\times\mathbf{p}\times\mathbf{e}_1)\cdot\hat{\mathbf{k}} = \frac{1}{9}\Bigg[(\tilde{\gamma}_{11}+\tilde{\gamma}_{12})^2\frac{(\tilde{x}^2+\tilde{y}^2\tilde{f})}{\alpha\tilde{r}^4} + (\tilde{\gamma}_{12}+\tilde{\gamma}_{22})^2\frac{(\tilde{f}\tilde{x}^2+\tilde{y}^2)}{\alpha\tilde{r}^4} + \alpha\tilde{f}\left[(\tilde{\gamma}_{11}+\tilde{\gamma}_{12})^2+(\tilde{\gamma}_{12}+\tilde{\gamma}_{22})^2\right] + \frac{\alpha^2}{\tilde{f}^2}\Big[\tilde{f}(\tilde{x}^2-\tilde{y}^2)^2 + 2\tilde{x}^2\tilde{y}^2(1+\tilde{f}^2)\left[(\tilde{x}+\tilde{y})^2+\tilde{f}(\tilde{x}-\tilde{y})^2\right]\Big]\frac{1}{\alpha\tilde{r}^4} + \left[(\tilde{\gamma}_{11}+\tilde{\gamma}_{12})^2+(\tilde{\gamma}_{12}+\tilde{\gamma}_{22})^2\right]\frac{(1-\tilde{f})\tilde{x}\tilde{y}}{\alpha\tilde{r}^4} + \frac{\alpha^2}{\tilde{f}^2}(\tilde{x}^3\tilde{y}+\tilde{y}\tilde{x}^3)(-1+\tilde{f}^2)\frac{(\tilde{x}+\tilde{y})^2+\tilde{f}(\tilde{x}-\tilde{y})^2}{\alpha\tilde{r}^4}\Bigg], \quad (30)$$

$$(\mathbf{e}_2 \times \mathbf{p} \times \mathbf{e}_1)\cdot\hat{\mathbf{k}} = \frac{2}{9\alpha\tilde{f}^3\tilde{r}^4} \times \times \Bigg[(1-\tilde{f})^2(\tilde{x}^2-\tilde{y}^2) - \tilde{f}(\tilde{x}^2+\tilde{y}^2)^2 + \frac{\tilde{r}^4\alpha^2}{2\tilde{f}} \times \times \Big[(\tilde{f}\tilde{x}^2+\tilde{y}^2+(-1+\tilde{f})\tilde{x}\tilde{y})^2 + (\tilde{f}\tilde{y}^2+\tilde{x}^2 + + (-1+\tilde{f})\tilde{x}\tilde{y})^2\Big]\Bigg](1-\tilde{f})(\tilde{x}^2-\tilde{y}^2), \quad (31)$$

$$(\mathbf{e}_2\times\mathbf{p}\times\mathbf{e}_2)\cdot\hat{\mathbf{k}} = \frac{1}{9}\Bigg[\left[(\tilde{\gamma}_{11}+\tilde{\gamma}_{12})^2+(\tilde{\gamma}_{12}+\tilde{\gamma}_{22})^2\right] \times \times \left[\frac{-\alpha^2[\tilde{f}(\tilde{x}+\tilde{y})^2+(\tilde{x}-\tilde{y})^2]+1}{\tilde{f}} + \frac{2(\tilde{x}^2+\tilde{y}^2)^2}{\alpha\tilde{f}\tilde{r}^4}\right] + + \left[(\tilde{\gamma}_{11}+\tilde{\gamma}_{12})^2+(\tilde{\gamma}_{12}+\tilde{\gamma}_{22})^2\right]^2 \frac{\tilde{f}(\tilde{x}-\tilde{y})^2+(\tilde{x}+\tilde{y})^2}{\alpha\tilde{r}^4} + + \tilde{\gamma}_{33}^2(\tilde{\gamma}_{12}+\tilde{\gamma}_{22})^2\frac{\tilde{f}\tilde{y}^2+\tilde{x}^2+(1-\tilde{f})\tilde{x}\tilde{y}}{\alpha\tilde{r}^4} + + \tilde{\gamma}_{33}^2(\tilde{\gamma}_{11}+\tilde{\gamma}_{12})^2\frac{\tilde{f}\tilde{x}^2+\tilde{y}^2+(1-\tilde{f})\tilde{x}\tilde{y}}{\alpha\tilde{r}^4} + + \tilde{\gamma}_{33}^2\frac{\alpha^2}{\tilde{f}^2}\left[\tilde{f}(\tilde{x}^2-\tilde{y}^2)^2+2\tilde{x}^2\tilde{y}^2(1+\tilde{f}^2) + (-1+\tilde{f}^2) \times \times (\tilde{x}^3\tilde{y}+\tilde{x}\tilde{y}^3)\right]\left[-\frac{\tilde{f}(\tilde{x}-\tilde{y})^2+(\tilde{x}+\tilde{y})^2}{\alpha\tilde{r}^4}\right]\Bigg]. \quad (32)$$

To be consistent with Eqs. (21) and (29), we construct some sufficient conditions. After analyzing the values of Eqs. (30)–(32), it follows that sufficient conditions for NPV depend on the choice of the octants

$$\{(\tilde{x}<0,\ \tilde{y}<0),\ (\tilde{x}>0,\ \tilde{y}>0),$$
$$(\tilde{x}<0,\ \tilde{y}>0),\ (\tilde{x}>0,\ \tilde{y}<0)\}$$

and the values

$$\{(\tilde{x}^2<\tilde{y}^2),\ (\tilde{x}^2>\tilde{y}^2)\}.$$

Using Eq. (21) and the above conditions on $(\tilde{x},\ \tilde{y})$, we obtain the following four cases:

I. $\alpha>0$, $\tilde{f}>0$, $(\tilde{x}<0,\ \tilde{y}<0)$ or $(\tilde{x}>0,\ \tilde{y}>0)$, $\tilde{x}^2<\tilde{y}^2$ or $\tilde{x}^2>\tilde{y}^2$,

II. $\alpha>0$, $\tilde{f}>0$, $(\tilde{x}<0,\ \tilde{y}>0)$ or $(\tilde{x}>0,\ \tilde{y}<0)$, $\tilde{x}^2<\tilde{y}^2$ or $\tilde{x}^2>\tilde{y}^2$,

III. $\alpha<0$, $\tilde{f}>0$, $(\tilde{x}<0,\ \tilde{y}<0)$ or $(\tilde{x}>0,\ \tilde{y}>0)$, $\tilde{x}^2<\tilde{y}^2$ or $\tilde{x}^2>\tilde{y}^2$,

IV. $\alpha<0$, $\tilde{f}>0$, $(\tilde{x}<0,\ \tilde{y}>0)$ or $(\tilde{x}>0,\ \tilde{y}<0)$, $\tilde{x}^2<\tilde{y}^2$ or $\tilde{x}^2>\tilde{y}^2$.

Further, $\tilde{f}$ can be either
1) $\tilde{f}>1$,
2) $\tilde{f}<1$,
3) $\tilde{f}=1$.

From all the above cases, we conclude that if they are taken together with option 3), there is a possibility when all the three inequalities in Eq. (29) are not satisfied simultaneously. This leads to following possibilities:

(i) If one of the terms in the square brackets in Eq. (28) can have a negative value, then the sum of the other two positive terms has to be less than that value, such that Eq. (28) become negative and provide NPV propagation.

(ii) Similarly, if two of the three terms in square brackets in Eq. (28) have negative value, then their sum must be greater than the remaining positive term in order to make expression (28) negative.

Since the location of **R** within the spacetime is arbitrary, these conditions are applied generally for NPV propagation.

5. SUMMARY

In this paper, we have investigated some possible conditions for NPV propagation of static charged black string. We can summarize the result as follows.

1) $f>1$ and $f<1$ provide regions r_1 and r_2 for the possible NPV propagation. We note that $f=1$ yields a region r_3 where all the conditions of cases (1) and (2) do not hold and Eq. (29) cannot be satisfied. Some other conditions have to be formulated to make NPV propagation possible in this region. The values of regions r_1, r_2, and r_3 depend on the parameters of temporal component of the given metric (8).

2) If $Q=0$, then f becomes

$$f=\alpha^2r^2-\frac{b}{\alpha r}.$$

In this case, the conditions $f<1$, $f>1$, and $f=1$ respectively provide the possible NPV regions

$$r<\frac{(2/3)^{1/3}\alpha}{\left(9\alpha^6b+\sqrt{3}\sqrt{-4\alpha^{12}+27\alpha^{12}b^2}\right)\frac{1}{3}} + \frac{\left(9\alpha^6b+\sqrt{3}\sqrt{-4\alpha^{12}+27\alpha^{12}b^2}\right)\frac{1}{3}}{(2/3)^{1/3}\alpha},$$

$$r>\frac{(2/3)^{1/3}\alpha}{\left(9\alpha^6b+\sqrt{3}\sqrt{-4\alpha^{12}+27\alpha^{12}b^2}\right)\frac{1}{3}} + \frac{\left(9\alpha^6b+\sqrt{3}\sqrt{-4\alpha^{12}+27\alpha^{12}b^2}\right)\frac{1}{3}}{(2/3)^{1/3}\alpha},$$

$$r = \frac{(2/3)^{1/3}\alpha}{\left(9\alpha^6 b + \sqrt{3}\sqrt{-4\alpha^{12} + 27\alpha^{12}b^2}\,\right)\dfrac{1}{3}} + \\ + \frac{\left(9\alpha^6 b + \sqrt{3}\sqrt{-4\alpha^{12} + 27\alpha^{12}b^2}\,\right)\dfrac{1}{3}}{(2/3)^{1/3}\alpha}.$$

This shows that NPV conditions derived for static charged black strings can be reduced to static black strings without charge by using the corresponding value of f.

It was shown in [13, 16] that negative values of Λ do not support NPV propagation in some spherically symmetric spacetimes. We mention here that the conditions for NPV propagation in a cylindrically symmetric spacetime remains unaffected for $\Lambda < 0$. It would be interesting to explore the NPV regions for regular black holes [31].

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