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CENTER-OF-MASS ENERGY FOR THE PLEBANSKI–DEMIANSKI BLACK HOLE

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We study the center-of-mass energy of the particles colliding in the vicinity of acceleration and event horizons of the Plebanski and Demianski class of black holes. We calculate the collision energy of uncharged particles in the center-of-mass frame that are freely falling along the equatorial plane of a charged accelerating and rotating black hole with an NUT parameter. This energy turns out to be infinite in the nonextremal case, while in the extremal case, it becomes infinitely large near the event horizon only if the particle has the critical angular momentum. We conclude that the center-of-mass energy depends on the rotation and the NUT parameter.

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Black holes (BHs) are the most important prediction of general relativity and are studied by detecting their effects on the nearby matter. One of the interesting features of BHs is that they can behave like a particle accelerator (accelerating charged particles to high speed). Recently, the physics of ultra-high energies in the context of particle accelerators is receiving much attention. Therefore, the study of naturally occurring processes in the vicinity of astrophysical objects is of great significance. Collision energies up to 10 TeV can be observed by the largest terrestrial accelerators like the Tevatron and Large Hadron Collider. In this regard, the center-of-mass energy (CME) provides the collision energy required for the production of new particles. Black holes can accelerate and collide particles with an unlimited CME. This infinite increase in energy in the center-of-mass (CM) frame is due to the blue-shifting of particles near the horizon.

The possibility of obtaining infinite growth of energy in the CM frame due to particles colliding near the horizon of a BH was discussed by Banados, Silk, and West (the BSW effect) [1]. They showed that the rotating BH can behave as a particle accelerator and observed high CMEs for particles propelling along the equatorial plane in the locality of the Kerr (extremal) BH. In [2], it was pointed out that the CME is finite for the Kerr (nonextremal) BH. An infinite CME due to particle scattering in the Kerr (nonextremal) BH was found in [3]; an infinite CME for particles colliding in the Kerr–de Sitter (extremal) BH was observed in [4]. The infinite CME at the cosmological horizon of the Reissner–Nordstrom (RN)–de Sitter BH was studied in [5]. The infinite CME for the critical particles (with fine-tuned angular momentum) colliding along the equatorial plane of the Sen (extremal and nonextremal) BH and the Kerr–Newman BH were discussed in [6].

The BSW effect near the event horizon of the Kerr–Taub–NUT BH was investigated in [7]. In [8], the authors discussed the collision of particles around the four-dimensional Kaluza–Klein (extremal) BH and found the infinitely large CME near the horizon in both rotating and nonrotating cases. Joshi and Patil [9] found that the CME turns out to be high in the naked singularity of the RN and Kerr BHs. The same authors [10] proved that a high CME can also be seen in regular BHs for particular values of the parameters m and q. The BSW effect for the Ayon–Beato–Garcia–Bronnikov BH, the Einstein–Maxwell–dilaton–axion BH, and the Banados–Teitelboim–Zanelli BH was discussed in [11]. In [12], the CME was generalized for charged particles moving in an electromagnetic field and braneworld BHs were discussed. It was proved in [13] that a nonrotating but charged RN (extremal or nonextremal) BH can also serve as an accelerator with an arbitrarily high CME of charged particles colliding near the horizon.

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The effect of acceleration on the CME for particles involved in nonequatorial motion and colliding in the Kerr–Newman BH was discussed in [14]. Arbitrarily high CME for colliding particles with nonequatorial motion near the horizon of an (extremal) Kerr–Newman BH were obtained in [15]. In [16], nonequatorial motion of particles colliding in dirty BHs was discussed and the CME was found to grow without bound. This generalizes the results of the equatorial motion. Collisions of the innermost orbit particle in a nonequatorial plane of an (extremal) Kerr BH were studied in [17], and the CME was found to be unboundedly high. The CME of a Plebanski–Demianski (PD) (extremal) BH with a zero NUT parameter near the acceleration and event horizons was studied in [18].

In a recent paper, we have studied the CME of a PD (nonextremal) BH with a zero NUT parameter near the event horizon [19]. In this paper, we explore the CME for charged accelerating and rotating (extremal and nonextremal) BHs with an NUT parameter near the event and acceleration horizons. The particles are assumed to be colliding in the equatorial plane. In general, the NUT parameter is associated with the twisting property of the BH. Plebanski and Demianski presented a class of type-D BHs known as the family of PD BHs [20]. These are described by the metric

$$ds^2 = -f(r,\theta)\,dt^2 + \frac{1}{g(r,\theta)}dr^2 - 2H(r,\theta)\,dt\,d\phi + \Sigma(r,\theta)\,d\theta^2 + K(r,\theta)\,d\phi^2,$$

where f, g, H, Σ, and K are functions that describe different BHs in this class.

We consider a PD BH with a NUT parameter, described by the metric [21]

$$ds^2 = -\frac{1}{\Omega^2}\left\{\frac{Q}{\rho^2}\left[dt - \left(a\sin^2\theta + 4l\sin^2\frac{\theta}{2}\right)d\phi\right]^2 - \frac{\rho^2}{Q}dr^2 - \frac{\tilde{P}}{\rho^2}\left[a\,dt - (r^2+(a+l)^2)\,d\phi\right]^2 - \frac{\rho^2}{\tilde{P}}\sin^2\theta\,d\theta^2\right\}, \quad (1)$$

with

$$\Omega = 1 - \frac{\alpha}{\omega}(l + a\cos\theta)\,r, \quad \rho^2 = r^2 + (l + a\cos\theta)^2,$$

$$Q = \left[(\omega^2 k + e^2 + g^2)\left(1 + \frac{2\alpha l}{\omega}r\right) - 2Mr + \frac{\omega^2 k}{a^2 - l^2}r^2\right]\left[1 - \frac{\alpha(a+l)}{\omega}r\right] \times \left[1 + \frac{\alpha(a-l)}{\omega}r\right],$$

$$\tilde{P} = \sin^2\theta(1 - a_3\cos\theta - a_4\cos^2\theta) = P\sin^2\theta,$$

$$a_3 = 2\frac{\alpha a}{\omega}M - 4\frac{\alpha^2 al}{\omega^2}(\omega^2 k + e^2 + g^2),$$

$$a_4 = \frac{-\alpha^2 a^2}{\omega^2}(\omega^2 k + e^2 + g^2).$$

Here, M and a respectively represent the mass and rotation of BH, and the parameters e and g are the electric and magnetic charges. Moreover, α is the acceleration of a BH and l is the NUT parameter. The rotation parameter ω in terms of a and k is given by

$$\left(\frac{\omega^2}{a^2 - l^2} + 3\alpha^2 l^2\right)k = 1 + \frac{2\alpha l}{\omega}M - \frac{3\alpha^2 l^2}{\omega^2}(e^2 + g^2).$$

It is interesting to mention here that all the parameters α, M, e, g, and a vary independently, but ω depends on the rotation and NUT parameters. For $\alpha = 0$, the metric reduces to the Kerr–Newman BH with an NUT parameter. In the absence of an NUT parameter, it reduces to a charged accelerating and rotating BH. Further, the limit $\alpha = 0$ leads to the Kerr–Newman BH, and $a = 0$ yields the RN BH. In addition, if $e = 0 = g$, then we have a Schwarzschild BH, while the limit $l = 0 = a$ leads to the C-metric.

The horizons of BH (1) can be found by setting $g(r,\theta) = 0$, which yields

$$(\omega^2 k + e^2 + g^2)\left(1 + \frac{2\alpha l}{\omega}r\right) - 2Mr + \frac{\omega^2 k}{a^2 - l^2}r^2 = 0,$$

which is quadratic in r with the roots

$$r_\pm = \frac{a^2 - l^2}{\omega^2 k}\left\{-\left[(\omega^2 k + e^2 + g^2)\frac{\alpha l}{\omega} - M\right] \pm \left\{\left[(\omega^2 k + e^2 + g^2)\frac{\alpha l}{\omega} - M\right]^2 - \frac{\omega^2 k}{a^2 - l^2}(\omega^2 k + e^2 + g^2)\right\}^{1/2}\right\},$$

where $r_\pm$ respectively represent the outer (event) and inner horizons. For the existence of horizons, the condition is

$$[(\omega^2 k + e^2 + g^2)\frac{\alpha l}{\omega} - M]^2 \geq \frac{\omega^2 k}{a^2 - l^2}(\omega^2 k + e^2 + g^2). \quad (2)$$

Also,

$$r_{\alpha_1} = \frac{\omega}{\alpha(a+l)}, \quad r_{\alpha_2} = -\frac{\omega}{\alpha(a-l)}$$

are acceleration horizons. The angular velocity at the outer horizon is

$$\Omega_H = -\frac{g_{t\phi}}{g_{\phi\phi}},$$

which, in our case, takes the form

$$\Omega_H = \frac{a}{r_+^2 + (a+l)^2}. \tag{3}$$

We consider a particle exhibiting the geodesic motion in the PD BH. Let

$$U^a = (U^t, U^r, U^\theta, U^\phi)$$

be the four-velocity of the particle, which is restricted to equatorial motion ($\theta = \pi/2$), leading to

$$U^\theta = 0.$$

We can define the energy and angular momentum of the particle as

$$E = -g_{ab}(\partial_t)^a U^b = -g_{tt}U^t - g_{t\phi}U^\phi,$$
$$L = g_{ab}(\partial_\phi)^a U^b = g_{t\phi}U^t + g_{\phi\phi}U^\phi.$$

These are conserved throughout the motion, termed as constants of motion. With Eq. (1), these quantities become

$$E = \frac{1}{\Omega^2\rho^2}[Q - Pa^2]U^t + \frac{1}{\Omega^2\rho^2}\left[Pa(r^2 + (a+l)^2) - Q(a+2l)\right]U^\phi, \tag{4}$$

$$L = -\frac{1}{\Omega^2\rho^2}\left[Pa(r^2 + (a+l)^2) - Q(a+2l)\right]U^t + \frac{1}{\Omega^2\rho^2}\left[(P(r^2 + (a+l)^2)^2 - Q(a+2l)^2)\right]U^\phi. \tag{5}$$

These lead to the four-velocity components

$$U^t = \frac{1}{PQ(r^2 + l^2)}\{E(P(r^2+(a+l)^2)^2 - Q(a+2l)^2) - L(Pa(r^2 + (a+l)^2) - Q(a+2l))\}, \tag{6}$$

$$U^\phi = \frac{1}{PQ(r^2 + l^2)}\{E(Pa(r^2 + (a+l)^2) - Q(a+2l)) + L(Q - Pa^2)\}. \tag{7}$$

Using the normalization condition,

$$g_{ab}U^aU^b = -1,$$

we find the radial component of the velocity as

$$U^r = \pm\left[-\frac{Q}{(r^2+l^2)\Omega^2} + \frac{1}{P(r^2+l^2)^2\Omega^2} \times \left[E^2[P(r^2 + (a+l)^2)^2 - Q(a+2l)^2] - L^2(Q-Pa^2) - 2EL(Pa(r^2+(a+l)^2) - Q(a+2l))\right]\right]^{1/2},$$

where $\pm$ correspond to the radially ingoing and outgoing particles.

We introduce the effective potential as

$$U^{r^2} + V_{eff}(r) = 0, \tag{8}$$

where

$$V_{eff}(r) = \frac{Q}{(r^2+l^2)\Omega^2} - \frac{1}{P(r^2+l^2)^2\Omega^2} \times \{E^2[P(r^2 + (a+l)^2)^2 - Q(a+2l)^2] - L^2(Q-Pa^2) - 2EL\left[Pa(r^2+(a+l)^2) - Q(a+2l)\right]\}.$$

The conditions for a circular orbit are

$$V_{eff}(r) = 0, \quad \frac{dV_{eff}(r)}{dr} = 0.$$

Because the timelike component of the four-velocity is greater than zero (causally connected), Eq. (6) leads to

$$E(P(r^2 + (a+l)^2)^2 - Q(a+2l)^2) \geq L\left[Pa(r^2 + (a+l)^2) - Q(a+2l)\right], \tag{9}$$

which reduces (at the horizon) to

$$E \geq \frac{aL}{r_+^2 + (a+l)^2}.$$

Furthermore, the angular velocity of the BH (at $r = r_+$) is

$$\Omega_H = \frac{a}{r_+^2 + (a+l)^2},$$

which yields

$$E \geq \Omega_H L. \tag{10}$$

We now discuss the CME for two colliding particles with rest masses m_1 and m_2 moving in the equatorial plane. In terms of the four-momentum

$$p_i^a = m_i U_i^a, \quad i = 1, 2, \quad a = t, r, \theta, \phi,$$

the CME of two particles is [7]

$$E_{cm}^2 = -p_i^a p_{ai},$$

which yields

$$E_{cm}^2 = 2m_1m_2\left[\frac{(m_1-m_2)^2}{2m_1m_2} + (1-g_{ab}U_1^aU_2^b)\right]. \quad (11)$$

The CME of these particles turns out to be

$$\frac{E_{cm}}{\sqrt{2m_1m_2}} = \sqrt{\frac{(m_1-m_2)^2}{2m_1m_2} + \frac{M(r)-N(r)}{T(r)}}, \quad (12)$$

where

$$M(r) = PQ\rho^2 + \{(E_1L_2+E_2L_1)\times \times[Q(a+2l) - Pa(r^2+(a+l)^2)] + +E_1E_2(P(r^2+(a+l)^2)^2 - Q(a+2l)^2) - L_1L_2(Q-Pa^2)\},$$

$$N(r) = \sqrt{n_1(r)n_2(r)},$$

$$n_i(r) = E_i^2[P(r^2+(a+l)^2)^2 - Q(a+2l)^2] - - PQ\rho^2 - L_i^2(Q-Pa^2) - - 2E_iL_i[Pa(r^2+(a+l)^2) - Q(a+2l)],$$

$$T(r) = P\left[(\omega^2k+e^2+g^2)\left(1+\frac{2\alpha l}{\omega}r\right) - - 2Mr + \frac{\omega^2k}{a^2-l^2}r^2\right]\left[1-\frac{\alpha(a+l)}{\omega}r\right]\times \times\left[1+\frac{\alpha(a-l)}{\omega}r\right]\rho^2.$$

Here E_i and L_i are respectively the conserved energy and angular momentum for the ith particle. This result indicates that the CME is affected by the rotation and NUT parameters. In the nonextremal limits, the CME given by Eq. (12) diverges at the acceleration horizons if we take the mass of both particles to be equal:

$$\lim_{r\to r_+}\frac{E_{cm}^2}{2m^2} = \infty. \quad (13)$$

For the extremal PD BH, the rotation and NUT parameters satisfy the relation

$$(a^2-l^2)\left(1+\frac{2\alpha l}{\omega}M - \frac{3\alpha^2l^2}{\omega^2}\right) + e^2 + g^2 = 1.$$

For $\alpha = 0$, this condition reduces to

$$a^2 - l^2 + e^2 + g^2 = 1,$$

and further, for $l = 0$, it becomes

$$a^2 + e^2 + g^2 = 1$$

(see [18]). For the CME near the event horizon, the term in the right side of Eq. (12) becomes undetermined. Using l'Hospital rule, we then find

$$\frac{E_{cm}}{\sqrt{2m_1m_2}} = \sqrt{\frac{(m_1-m_2)^2}{2m_1m_2} + \frac{M'(r)-N'(r)}{T'(r)}}, \quad (14)$$

with

$$M'(r)|_{r=r_+} = E_1E_2[4Pr_+(r_+^2+(a+l)^2) - Q'(a+2l)] - - L_1L_2Q' - (E_1L_2+E_2L_1)(2Par_+ - Q'(a+2l)),$$

$$N'(r)|_{r=r_+} = \frac{1}{2\sqrt{n_1(r_+)n_2(r_+)}}\times \times\{n_1'(r_+)n_2(r_+) + n_2'(r_+)n_1(r_+)\},$$

$$n_i'(r)|_{r=r_+} = -PQ'(r_+^2+l^2) + + E_i^2[4Pr_+(r_+^2+(a+l)^2) - Q'(a+2l)^2] - - L_i^2Q' - 2E_iL_i(2Par_+ - Q'(a+2l)),$$

$$T'(r)|_{r=r_+} = PQ'(r_+^2+l^2).$$

When

$$a = 0, \quad l = 0, \quad m_1 = m_2 = m,$$

we have

$$E_{cm} = \sqrt{2}m\left\{1 + \frac{E_1E_2r^2}{Q} - \frac{L_1L_2}{Pr^2} - - \frac{1}{Q}\left(-Q + E_1^2r^2 - \frac{L_1^2Q}{Pr^2}\right)^{1/2}\times \times\left(-Q + E_2^2r^2 - \frac{L_2^2Q}{Pr^2}\right)^{1/2}\right\}^{1/2}. \quad (15)$$

Expanding this at $Q = 0$ yields

$$E_{cm} = 2m\sqrt{1+\frac{L_1-L_2}{4P}}, \quad (16)$$

which is the E_{cm} of the extremal Kerr–Newman BH with $a = 0$. This shows that the CME can be unlimited if one of these particles has a very large angular momentum. We consider

$$L_{H_i} = \frac{E_i}{\Omega_H} = \frac{E_i(r^2+(a+l)^2)}{a};$$

for

$$L_{H_1} = L_{H_2} = L_H, \quad E_1 = E_2 = 1,$$

the CME of the extremal PD BH in Eq. (14) is

$$E_{cm} = \\ = 2m\sqrt{1 + \frac{(L_1 - L_2)^2 L_H}{4(L_1 - L_H)(L_2 - L_H)P(1 + l^2)a}}. \quad (17)$$

This implies that the CME is finite when both L_1 and L_2 are finite. However, if one particle has the critical angular momentum

$$L_H = \frac{(a+l)^2}{a} + \frac{1}{a},$$

the CME is arbitrarily large. When $l = 0$, we obtain the result for an extremal charged accelerating and rotating BH with

$$L_H = a + \frac{1}{a}$$

(see [18]). Further, for $\alpha = 0$, it reduces to the results for the Kerr BH [22].

We have found that the CME depends not only on the rotation but also on the NUT parameter. We conclude that the CME decreases with the increase in rotation and NUT parameters. For the extremal BH, the CME becomes infinite for the particles having the critical angular momentum. We note that a BH is extremal when the CME is infinite at the event horizon, but is nonextremal when the CME is infinite at both the event and acceleration horizons. Our results generalize the results already available in the literature. If $a = 0 = l$, our result reduces to the one for the Kerr–Newman (extremal) BH. For $\alpha = 0$, we obtain the CME for the Kerr–Taub–NUT BH [7]. It has been shown that for the zero NUT parameter, the collision energy of the particles in the background of a PD (extremal) BH depends on the rotation parameter [18]. Here, we have considered a nonzero NUT parameter and found that the CME also depends on the NUT parameter. It is interesting to mention here that there is a rapid increase in the CME with the decrease in the NUT parameter.

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