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THERMODYNAMIC BEHAVIOR OF PARTICULAR $f(R,T)$-GRAVITY MODELS

M. Sharif[*], *M. Zubair*[**]

Department of Mathematics, University of the Punjab
Quaid-e-Azam Campus, Lahore-54590, Pakistan

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We investigate the thermodynamics at the apparent horizon of the FRW universe in $f(R,T)$ theory in the nonequilibrium description. The laws of thermodynamics are discussed for two particular models of the $f(R,T)$ theory. The first law of thermodynamics is expressed in the form of the Clausius relation $T_h d\hat{S}_h = \delta Q$, where δQ is the energy flux across the horizon and $d\hat{S}$ is the entropy production term. Furthermore, the conditions for the generalized second law of thermodynamics to be preserved are established with the constraints of positive temperature and attractive gravity. We illustrate our results for some concrete models in this theory.

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1. INTRODUCTION

Recent astrophysical observations indicate that expansion of the universe is presently in an accelerated epoch. The most compelling evidence for this is found in measurements of type-Ia supernovae (SNeIa) [1], which is supported by renowned observations [2–5]. The mysterious component of energy named dark energy (DE) is often introduced to explain this behavior of the universe. However, the mechanism responsible for the accelerated expansion is still under debate.

Two approaches have been used to illustrate the issue of current cosmic acceleration. Introducing an "exotic cosmic fluid" in the framework of the Einstein gravity [6–8] is one direction to deal such issue, but this approach did not fully explain the current empirical data. The other way is to discuss the modified theories of gravity such as $f(R)$ [9, 10], $f(\mathcal{T})$ [11], where $\mathcal{T}$ is the torsion scalar in teleparallel, and $f(R,T)$, where R and T are the Ricci sclar and the trace of the energy–momentum tensor [12, 13]. The $f(R,T)$ theory modifies the Einstein Lagrangian by coupling matter and geometry. In fact, this modified gravity generalizes the $f(R)$ theory and necessitates an arbitrary function of R and T. A comprehensive review of the problem of DE and modified theories was recently presented in [14].

[*] E-mail: msharif.math@pu.edu.pk
[**] E-mail: mzubairkk@gmail.com

Black hole thermodynamics suggests that there is a fundamental connection between gravitation and thermodynamics [15]. Hawking radiation [16] together with a proportionality relation between temperature and surface gravity, as well as the connection between the horizon entropy and the area of a black hole [17] further support this idea. Jacobson [18] was the first to deduce the Einstein field equations from the Clausius relation

$$T_h d\hat{S}_h = \delta\hat{Q}$$

together with the condition that the entropy is proportional to the horizon area. In case of a general spherically symmetric spacetime, it was shown that the field equations can be stated as the first law of thermodynamics (FLT) [19].

The relation between the Friedmann–Robertson–Walker (FRW) equations and the FLT was shown in [20] for

$$T_h = \frac{1}{2\pi}\tilde{r}_A, \quad S_h = \frac{\pi\tilde{r}_A^2}{G}.$$

The field equations for an FRW background were also formulated in the Gauss–Bonnet and Lovelock theories by using the corresponding entropy relation for static spherically symmetric black holes. It was shown in [21] that the correct field equations cannot be found by simply using the Clausius relation in nonlinear theories of gravity. The authors of [21] remarked that a nonequilibrium description of thermodynamics is needed, whereby the Clausius relation is modified to

$$T_h dS_h = \delta Q + d_\jmath S,$$

where $d_\jmath S$ is the entropy production term. In Refs. [22–26], it was shown that the FRW field equations in general relativity (GR) and modified theories can be rewritten as

$$dE = T_h\, dS_h + W\, dV$$

(a unified FLT on the trapping horizon suggested in [23]) with the work term

$$W = \frac{1}{2}(\rho - p).$$

A generalized procedure to construct the FLT and the generalized second law of thermodynamics (GSLT) at the apparent horizon of a Friedmann universe was developed in [27]. The validity conditions of the GSLT were studied in modified theories of gravity. In [28], it was shown that equilibrium thermodynamics is achievable for extended theories of gravity and entropy correction terms can be confined to mass-like functions. Other alternative approaches [29–33] have also been developed to reinterpret the nonequilibrium correction. In [34], we have explored the GSLT in the $f(R,T)$ theory and found necessary conditions for its validity. It was shown that the equilibrium description is not feasible by redefining the dark energy components in the $f(R,T)$ theory.

In this paper, the thermodynamics laws are examined for two particular models of the $f(R,T)$ theory. We show that the FRW equations can be rewritten in the form of FLT

$$T_h d\hat{S}_h + T_h d_\jmath \hat{S}_h = -d\hat{E} + W_{tot} dV.$$

We formulate the GSLT and explore the conditions to validate this law. The paper is arranged as follows. In Sec. 2, we present a brief introduction to the $f(R,T)$ theory. Section 3 is devoted to a discussion of the FLT and GSLT corresponding to the Friedmann equations of particular $f(R,T)$ models. Finally, concluding remarks are given in Sec. 4.

2. $f(R,T)$ GRAVITY: AN OVERVIEW

The $f(R,T)$ modified gravity is described by the action [12]

$$\mathcal{I} = \int dx^4 \sqrt{-g}\left[\frac{f(R,T)}{2\kappa} + \mathcal{L}_m\right], \tag{1}$$

where $\kappa = 8\pi G$ and $\mathcal{L}_m$ defines the matter substances of the universe. The matter energy–momentum tensor $T^{(m)}_{\alpha\beta}$ is defined as [35]

$$T^{(m)}_{\alpha\beta} = -\frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}_m\right)}{\delta g^{\alpha\beta}}. \tag{2}$$

The field equations can be found by varying the action of the $f(R,T)$ gravity with respect to the metric tensor,

$$R_{\alpha\beta} f_R(R,T) - \frac{1}{2} g_{\alpha\beta} f(R,T) + \\ + (g_{\alpha\beta}\Box - \nabla_\alpha\nabla_\beta) f_R(R,T) = \\ = 8\pi G T^{(m)}_{\alpha\beta} - f_T(R,T) T^{(m)}_{\alpha\beta} - f_T(R,T)\Theta_{\alpha\beta}, \tag{3}$$

where f_R and f_T are derivatives of $f(R,T)$ with respect to R and T. The field equations depend on the source term $\Theta_{\mu\nu}$, and hence every selection of $\mathcal{L}_m$ generates a particular set of field equations.

We consider the perfect fluid as a matter source with the matter Lagrangian $\mathcal{L}_m = p_m$, whence $\Theta_{\alpha\beta}$ is given by

$$\Theta_{\alpha\beta} = -2T^{(m)}_{\alpha\beta} + p_m g_{\alpha\beta}. \tag{4}$$

Substituting this value in Eq. (3) yields

$$R_{\alpha\beta} f_R - \frac{1}{2} g_{\alpha\beta} f + (g_{\alpha\beta}\Box - \nabla_\alpha\nabla_\beta) f_R = \\ = 8\pi G T^{(m)}_{\alpha\beta} + T^{(m)}_{\alpha\beta} f_T - p_m g_{\alpha\beta} f_T. \tag{5}$$

The spatially homogeneous and isotropic, $(n+1)$-dimensional FRW universe is defined as

$$ds^2 = h_{\alpha\beta} dx^\alpha dx^\beta + \tilde{r}^2 d\hat{\Omega}^2_{n-1}, \tag{6}$$

where

$$h_{\alpha\beta} = \mathrm{diag}(-1, a^2/(1-kr^2))$$

is the 2-dimensional metric, $a(t)$ is the scale factor, and k is the cosmic curvature;

$$\tilde{r} = a(t) r, \quad x^0 = t, \quad x^1 = r,$$

and $d\hat{\Omega}^2_{n-1}$ is the metric of a $(n-1)$-dimensional sphere. For $n = 3$, we have the $(3+1)$-dimensional FRW metric in the Einstein gravity, while one can have $n \geq 4$ in other gravity theories.

3. THERMODYNAMICS IN THE $f(R,T)$ GRAVITY

We now discuss the laws of thermodynamics for two particular choices of an $f(R,T)$ gravity [12].

3.1. $f(R,T) = f_1(R) + f_2(T)$

We consider the $f(R,T)$ model with

$$f(R,T) = f_1(R) + f_2(T), \tag{7}$$

where f_1 and f_2 are arbitrary functions of R and T. The corresponding field equations are

$$R_{\alpha\beta}f_{1R}(R)-\frac{1}{2}g_{\alpha\beta}f_1(R)+(g_{\alpha\beta}\Box-\nabla_\alpha\nabla_\beta)f_{1R}(R) = \\ = 8\pi G T^{(m)}_{\alpha\beta} + T^{(m)}_{\alpha\beta} f_{2T}(T) + \frac{1}{2}g_{\alpha\beta}f_2(T), \quad (8)$$

where

$$f_{1R}(R) = \frac{df_1}{dR}, \quad f_{2T} = \frac{df_2}{dT}.$$

The choice $f_2(T) = 0$ implies the field equation of the $f(R)$ gravity. In the FRW background, the field equations become

$$\left(H^2 + \frac{k}{a^2}\right) = \frac{16\pi G_{Eff}}{n(n-1)}(\rho_m + \rho_{dc}), \quad (9)$$

$$\left(\dot{H} - \frac{k}{a^2}\right) = -\frac{8\pi G_{Eff}}{(n-1)}(\rho_m + \rho_{dc} + p_{dc}), \quad (10)$$

where

$$G_{Eff} = \frac{1}{f_{1R}}\left(G + \frac{f_{2T}}{8\pi}\right),$$

and

$$\rho_{dc} = \frac{1}{8\pi G\mathcal{D}}\left[\frac{1}{2}(Rf_{1R}-f_1-f_2)-nH\dot{R}f_{1RR}\right], \quad (11)$$

$$p_{dc} = \frac{1}{8\pi G\mathcal{D}}\left[-\frac{1}{2}(Rf_{1R} - f_1 - f_2) + \\ + (n-1)H\dot{R}f_{1RR} + \ddot{R}f_{1RR} + \dot{R}^2 f_{1RRR}\right], \quad (12)$$

and

$$\mathcal{D} = \left(1 + \frac{f_{2T}(R,T)}{8\pi G}\right).$$

Substituting Eqs. (11) and (12) in the conservation equation [34], we obtain

$$q_t = \frac{n(n-1)}{16\pi G}\left(H^2 + \frac{k}{a^2}\right)\partial_t\left(\frac{f_{1R}}{\mathcal{D}}\right). \quad (13)$$

Clearly, this reduces to the energy transfer relation in the $f(R)$ theory if

$$f(R,T) = f_1(R)$$

(see [32, 33]). If the effective gravitational coupling is constant, we obtain

$$q_t = 0.$$

3.1.1. First law of thermodynamics

We now construct the FLT for the above $f(R,T)$ model. The condition

$$h^{v\lambda}\partial_v\tilde{r}\partial_\lambda\tilde{r} = 0$$

gives the radius $\tilde{r}_A$ of the apparent horizon as

$$\tilde{r}_A = \left(H^2 + \frac{k}{a^2}\right)^{-1/2}.$$

The associated temperature is

$$T_h = \frac{|\kappa_{sg}|}{2\pi},$$

where

$$\kappa_{sg} = \frac{1}{2\sqrt{-h}}\partial_\mu\left(\sqrt{-h}h^{\mu\nu}\partial_\nu\tilde{r}_A\right) = \\ = -\frac{1}{\tilde{r}_A}\left(1 - \frac{1}{2H}\frac{d[\ln\tilde{r}_A]}{dt}\right)$$

is the surface gravity [20]. The temperature

$$T_h = \frac{1}{2\pi\tilde{r}_A}(1-\eta)$$

is positive for

$$\eta = \frac{1}{2H}\frac{d[\ln\tilde{r}_A]}{dt} < 1.$$

Applying the definition of $\tilde{r}_A$, we express the positivity condition for T_h as

$$\dot{H} - \frac{k}{a^2} > -2\left(H^2 + \frac{k}{a^2}\right). \quad (14)$$

In GR, the horizon entropy is defined as

$$S_h = \frac{A}{4G}$$

(see [15–17]), where

$$A = n\hat{\Omega}_n\tilde{r}_A^{n-1} = n\pi^{n/2}[\Gamma(n/2+1)]^{-1}\tilde{r}_A^{n-1}$$

is the area of the apparent horizon. It was proposed in [36] that in modified gravitational theories, the horizon entropy is associated with a Noether charge entropy. In [37], the Wald entropy was shown to be equivalent to

$$S_h = \frac{A}{4G_{Eff}},$$

where G_{Eff} is the effective gravitational coupling. We can define the Wald entropy in the $f(R,T)$ theory as [34]

$$\hat{S}_h = \frac{A}{4G_{Eff}}, \quad (15)$$

where

$$G_{Eff} = \frac{G\mathcal{D}(R,T)}{f_{1R}}$$

for the first $f(R,T)$ model. Following [34], we can obtain the FLT in the form

$$T_h d\hat{S}_h = \delta Q,$$

where the energy flux δQ is

$$\delta Q = -d\hat{E} + \frac{n}{2}\tilde{r}_A^{n-1}(\rho_t - p_t)\,d\tilde{r}_A + + \frac{n\hat{\Omega}_n(n-1)\tilde{r}_A^{n-2}}{16\pi G}\, d\left(\frac{f_{1R}}{\mathcal{F}}\right) = = -d\hat{E} + W_t d\mathbb{V} + \mathbb{V} q_t dt + T_h S_h d\left(\frac{f_{1R}}{\mathcal{D}}\right) \quad (16)$$

and

$$W_t = -\frac{1}{2}T^{(t)\mu\nu}h_{\mu\nu} = \frac{1}{2}(\rho_t - p_t)$$

is the total work density [23]. Thus, the FLT can be expressed as

$$T_h d\hat{S}_h + T_h d_\jmath \hat{S}_h = -d\hat{E} + W_{tot} d\mathbb{V}, \quad (17)$$

where

$$d_\jmath \hat{S}_h = -\frac{n\hat{\Omega}_n\left(H^2 + \dfrac{k}{a^2}\right)^{(1-n)/2}\left((n+1)H^2 + \dot{H} + n\dfrac{k}{a^2}\right) d(f_{1R}/\mathcal{D})}{4G\left(2H^2 + \dot{H} + \dfrac{k}{a^2}\right)}$$

is the entropy production term developed for this model. This characterizes a nonequilibrium treatment of thermodynamics. The FLT for a flat FRW universe in the $f(R)$ theory [32, 33] can be retrieved from this result. For $f(R,T) = R$, the term $d_\jmath \hat{S}_h$ vanishes, which leads to the FLT in the Einstein gravity.

3.1.2. Generalized second law of thermodynamics

We now investigate the validity of the GSLT in $f(R,T)$ theory for this model. The FLT determines the horizon entropy given by Eq. (17). The composition of the entire matter and energy fluids within the horizon is given by Gibb's equation [39]

$$T_t d\hat{S}_t = dE_t + p_t d\mathbb{V}, \quad (18)$$

where T_t and $\hat{S}_t$ are the temperature and entropy of all contents within the horizon. The temperature within the horizon is related to T_h [27] as

$$T_t = bT_h,$$

where

$$0 < b < 1$$

to ensure that

$$0 < T_t < T_h.$$

We consider $\hat{S}$ to be the sum of matter entropy within the horizon, the horizon entropy, and the nonequilibrium entropy production term.

The GSLT states that the time derivative of the total entropy is not decreasing with time, i. e.,

$$T_h \dot{\hat{S}} = T_h(\dot{\hat{S}}_h + d_\jmath \dot{\hat{S}}_h + \dot{\hat{S}}_t) \geqslant 0, \quad (19)$$

where

$$d_\jmath \dot{\hat{S}}_h = \partial_t (d_\jmath \hat{S}_h).$$

Inserting Eqs. (17) and (18) in the above inequality, we obtain

$$\frac{n(n-1)\hat{\Omega}_n}{16\pi HG}\left[2H\dot{\tilde{r}}_A\left[(b-1) + \dot{\tilde{r}}_A \tilde{r}_A^{n-4}(2-b)\right] \times \times \left(\frac{f_{1R}}{\mathcal{D}}\right) + (1-b)H\tilde{r}_A\partial_t\left(\frac{f_{1R}}{\mathcal{D}}\right)\right] \geqslant 0, \quad (20)$$

where

$$\dot{\tilde{r}}_A = -\tilde{r}_A^{-3} H\left(\dot{H} - \frac{\kappa}{a^2}\right).$$

We can impose the constraint

$$\mathcal{D}/f_{1R} > 0$$

for G_{Eff} to be positive. Using the positive temperature condition

$$\dot{H} - \frac{k}{a^2} > -2\left(H^2 + \frac{k}{a^2}\right)$$

with the temperature parameter $b < 1$ then relation (20) becomes

$$\frac{n(n-1)\hat{\Omega}_n\left(H^2 + \dfrac{k}{a^2}\right)^{-n/2+1}}{16\pi G\mathcal{D}} \times \times \left[4Hf_{1R} + (1-b)\mathcal{D}\partial_t\left(\frac{f_{1R}}{\mathcal{D}}\right)\right] \geqslant 0. \quad (21)$$

Hence, the GSLT can be satisfied if

$$\partial_t(f_{1R}/\mathcal{D}) > 0.$$

If

$$\partial_t(f_{1R}/\mathcal{D}) < 0,$$

then the GSLT is protected only if

$$\left|\frac{\partial_t(f_{1R}/\mathcal{D})}{f_{1R}/\mathcal{D}}\right| \leqslant \frac{4H}{1-b}.$$

If the gravitational coupling constant is indeed a constant, i. e.,

$$\partial_t(f_{1R}/\mathcal{D}) = 0,$$

then the GSLT always holds. The condition to preserve the GSLT in the $f(R)$ theory can be reproduced if

$$f(R,T) = f_1(R).$$

For

$$k = 0, \quad f_2(T) = 0,$$

we obtain the inequality already constructed in [27] in nonlinear gravity. In the thermal-equilibrium limit $b \sim 1$, the constraint to protect the GSLT is

$$\frac{n(n-1)\hat{\Omega}_n \left(H^2 + \dfrac{k}{a^2}\right)^{-(n/2+1)}}{16\pi G\mathcal{D}} \times \\ \times \left[H\left(\dot{H} - \frac{k}{a^2}\right)^2 f_{1R}\right] \geqslant 0. \quad (22)$$

Relation (22) depends on the choice of $f(R,T)$; for instance,

$$f_1(R) = R, \quad f_2(T) = 0$$

results in

$$\frac{n(n-1)\hat{\Omega}_n \left(H^2 + \dfrac{k}{a^2}\right)^{-(n/2+1)}}{16\pi G} \times \\ \times \left[H\left(\dot{H} - \frac{k}{a^2}\right)^2\right] \geqslant 0,$$

which is the GSLT validity condition in the Einstein gravity.

Here, we discuss the validity of the GSLT for some particular forms of $f(R,T)$ gravity:

(i) $f_1(R) = f(R), \quad f_2(T) = \lambda T$,

(ii) $f_1(R) = R, \quad f_2(T) = 2f(T)$.

In the first case, we consider the $f(R,T)$ model corresponding to the power-law solution $a(t) = a_0 t^m$ [13]

$$f(R,T) = \alpha_{k\omega}(-R)^k + \lambda T, \quad (23)$$

where

$$\alpha_{k\omega} = \\ = \frac{2^{3-2k}3^{k-1}kA(k(4k-3(1+\omega))^{1-k}(1+\omega)^{2k-2}}{k^2(6\omega+8)-k(9\omega+13)+3(\omega+1)}.$$

For this model, the Hubble and deceleration parameters are

$$H = \frac{2k}{3(1+\omega)}$$

and

$$q = -1 + \frac{3(1+\omega)}{2k}.$$

The validity of the GSLT in the $(3+1)$-dimensional flat FRW universe for model (23) requires the condition

$$T_h\dot{\hat{S}} = \frac{9(1+\omega)^2\alpha_{k\omega}}{8k^2\tilde{G}} \times \\ \times \left(\frac{4k[4k-3(1+\omega)]}{3(1+\omega)^2t^2}\right)^{k-1} \geqslant 0, \quad (24)$$

where

$$\tilde{G} = G + \frac{\lambda}{8\pi}.$$

We present some constraints for the particular values

$$k = -2, -1, 1, 2.$$

For $k = 1$, this solution represents the ΛCDM model and the constraint on the GSLT is given by

$$T_h\dot{\hat{S}} = \frac{9(1+\omega)^2 A}{8\tilde{G}} \geqslant 0,$$

which is true if $A > 0$ with $\omega \leqslant 3$.

For $k = 2$, the GSLT is valid if

$$T_h\dot{\hat{S}} = \frac{A(5-3\omega)^2}{2\tilde{G}t^2} \geqslant 0,$$

which requires $A < 0$.

For $k = -1, -2$, we find $T_h\dot{\hat{S}} \geqslant 0$ if $A > 0$ with $\omega \geqslant 0$. This choice would favor the expanding universe because $q < -1$.

The higher powers of curvature can be made available for larger values of k, and we can examine the validity of the GSLT. If we consider the dust case $\omega = 0$, then the possible role of λ and k can be seen from the graphical description shown in Fig. 1.

In the second case, the GSLT for the

$$f(R,T) = R + 2f(T)$$

model requires the following inequality to be satisfied:

$$T_h\dot{\hat{S}} = \frac{\dot{H}^2}{2H^4\hat{G}} \geqslant 0,$$

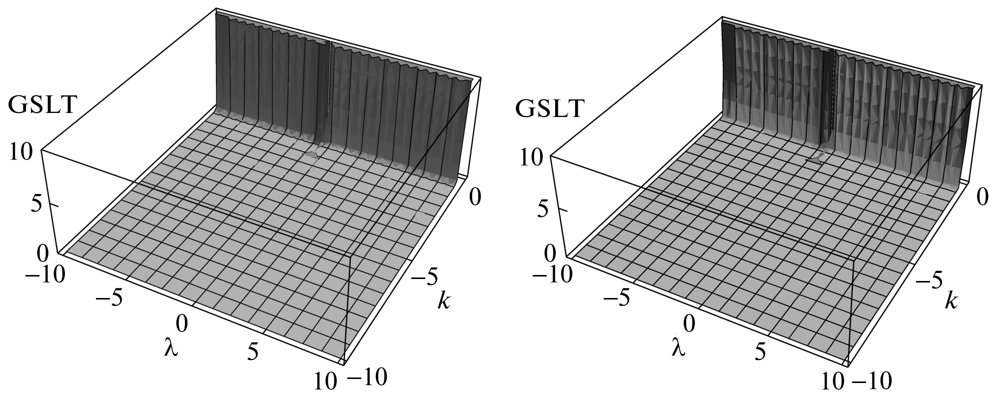

Fig. 1. Evolution of the GSLT for different values of the parameters k and λ (a) for the present epoch $z = 0$ and (b) for $z = -0.9$

where

$$\hat{G} = G + \frac{2f(T)}{8\pi}.$$

Here, we consider the power-law solution of the form

$$f(T) = a_1 T + a_2 T^k,$$

where a_1 and a_2 are parameters. Following [13] for the dust case, we set

$$a_1 = 1, \quad a_2 = \frac{2^{3-2k}3^{k-1}k^{3-2k}}{4+2k}.$$

Then the above inequality takes the form

$$T_h \dot{\hat{S}} = \\ = \frac{9\pi(4+2k)}{k^2[(4+2k)(8\pi G+1)+2^{3-2k}3^{k-1}k^{4-2k}T^{k-1}]} \geqslant 0.$$

This shows that the GSLT holds for the $f(T)$ power-law model, and its validity is shown in Fig. 2.

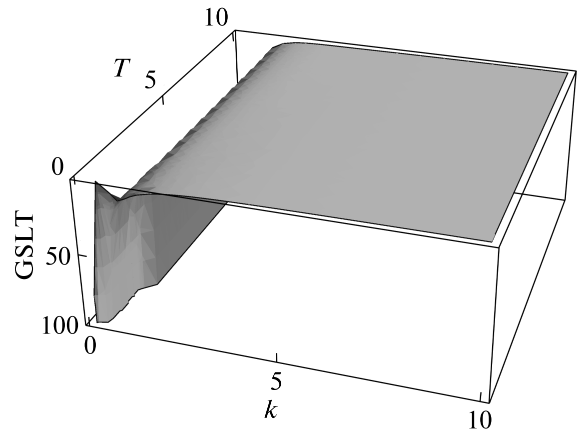

Fig. 2. Evolution of GSLT for case (ii) versus T and k

3.2. $f(R,T) = f_1(R) + f_2(R)f_3(T)$

A more general $f(R,T)$ gravity model is of the form [12]

$$f(R,T) = f_1(R) + f_2(R)f_3(T), \tag{25}$$

where f_i ($i = 1, 2$) are functions of R and f_3 is function of T. For a dust matter source, the field equation is obtained as

$$R_{\alpha\beta}[f_{1R} + f_{2R}f_3] - \frac{1}{2}g_{\alpha\beta}f_1 + \\ + (g_{\alpha\beta}\Box - \nabla_\alpha\nabla_\beta)[f_{1R} + f_{2R}f_3] = \\ = 8\pi G T^{(m)}_{\alpha\beta} + T^{(m)}_{\alpha\beta} f_2 f_{3T} + \frac{1}{2}g_{\alpha\beta}f_2 f_3. \tag{26}$$

An equivalent Einstein field equation can be obtained with

$$G_{Eff} = \frac{1}{f_{1R} + f_{2R}f_3}\left(G + \frac{f_2 f_{3T}}{8\pi}\right),$$

whereas

$$\hat{T}^{(dc)}_{\alpha\beta} = \frac{1}{f_{1R} + f_{2R}f_3} \times \\ \times \left[\frac{1}{2}g_{\alpha\beta}(f_1 + f_2 f_3) - R(f_{1R} + f_{2R}f_3)) + \\ + (\nabla_\alpha\nabla_\beta - g_{\alpha\beta}\Box)(f_{1R} + f_{2R}f_3)\right]. \tag{27}$$

In the discussion in what follows, we set

$$\mathcal{J}(R,T) = f_{1R}(R) + f_{2R}(R)f_3(T).$$

For this $f(R,T)$ model, the field equations are identical to Eqs. (9), (10), whereas

$$\hat{\rho}_{dc} = \frac{1}{8\pi G\mathcal{B}} \times \left[\frac{1}{2}(R\mathcal{J} - f_1 - f_2 f_3) - nH(\dot{R}\mathcal{J}_R + \dot{T}\mathcal{J}_T)\right], \quad (28)$$

$$\hat{p}_{dc} = \frac{1}{8\pi G\mathcal{B}}\left[-\frac{1}{2}(R\mathcal{J} - f_1 - f_2 f_3) + (n-1)H(\dot{R}\mathcal{J}_R + \dot{T}\mathcal{J}_T) + \ddot{R}\mathcal{J}_R + \dot{R}^2\mathcal{J}_{RR} + 2\dot{R}\dot{T}\mathcal{J}_{RT} + \ddot{T}\mathcal{J}_T + \dot{T}^2\mathcal{J}_{TT}\right] \quad (29)$$

and

$$\mathcal{B}(R,T) = \left(1 + \frac{f_2(R) f_{3T}(T)}{8\pi G}\right),$$

which includes contributions from both matter and geometry. The total energy exchange term for this model is given by

$$q_t = \frac{n(n-1)}{16\pi G}\left(H^2 + \frac{k}{a^2}\right)\partial_t\left(\frac{\mathcal{J}}{\mathcal{B}}\right). \quad (30)$$

We now analyze the validity of the FLT and GSLT for the above model.

3.2.1. First law of thermodynamics

The Wald entropy $\hat{S}_h = A/4G_{Eff}$ for function (25) becomes

$$\hat{S}_h = \frac{n\hat{\Omega}_n \tilde{r}_A^{n-1}\mathcal{J}}{4G\mathcal{B}}. \quad (31)$$

In this case, the FLT involves the energy flux δQ and entropy production terms of the form [34]

$$\delta Q = -d\hat{E} + \frac{n}{2}\Omega_n \tilde{r}_A^{n-1}(\rho_{tot} - p_{tot})\, d\tilde{r}_A + \frac{n(n-1)}{16\pi G}\Omega_n \tilde{r}_A^{n-2} d\left(\frac{\mathcal{J}}{\mathcal{B}}\right) = -d\hat{E} + W_{tot}d\mathbb{V} + \mathbb{V}q_{tot}dt + T_h\hat{S}_h d\left(\frac{\mathcal{J}}{\mathcal{B}}\right), \quad (32)$$

$$d_\jmath\hat{S}_h = -\frac{1}{T_h}\mathbb{V}q_{tot}dt - S_h d\left(\frac{\mathcal{J}}{\mathcal{B}}\right) = -\frac{n\Omega_n\left(H^2 + \frac{k}{a^2}\right)^{(n-1)/2}\left((n+1)H^2 + \dot{H} + n\frac{k}{a^2}\right) d(\mathcal{J}/\mathcal{B})}{4G\left(2H^2 + \dot{H} + \frac{k}{a^2}\right)}. \quad (33)$$

The $f(R,T)$ gravity model with

$$f(R,T) = f_1(R) + f_2(R) f_3(T)$$

involves the explicit nonminimal gravitational coupling between matter and curvature. Results obtained using this theory would be different from other models such as the $f(R)$ theory. The coupling of matter and geometry reveals that the matter energy–momentum tensor is no longer conserved and there is an energy transfer between the two components. Due to this interaction, the energy exchange term q_t is nonzero, and hence the entropy production term would be an additional term in this modified gravity. Hence, the FLT is established in a more general $f(R,T)$ gravity and entropy production is induced in a nonequilibrium treatment of thermodynamics [21, 34]. It was shown in recent papers [33] that the entropy production term can be incorporated by a redefinition of the field equations. However, in this theory, such a treatment is not useful, as shown in [34].

3.2.2. Generalized second law of thermodynamics

To develop the GSLT for the second model, we consider Gibbs equation (18). The horizon entropy is determined from the FLT. The necessary constraint for the validity of the GSLT is shown in Eq. (19). For the $f(R,T)$ model in (25), we obtain

$$\frac{n(n-1)\hat{\Omega}_n}{16\pi HG}\left[2H\dot{\tilde{r}}_A\left[(b-1) + \dot{\tilde{r}}_A\tilde{r}_A^{n-4}(2-b)\right] \times \left(\frac{\mathcal{J}}{\mathcal{B}}\right) + (1-b)H\tilde{r}_A\partial_t\left(\frac{\mathcal{J}}{\mathcal{B}}\right)\right] \geqslant 0, \quad (34)$$

where

$$\dot{\tilde{r}}_A = -\tilde{r}_A^{-3}H\left(\dot{H} - \frac{\kappa}{a^2}\right).$$

The effective gravitational coupling constant for this model is

$$G_{Eff} = \frac{G\mathcal{B}}{\mathcal{J}}.$$

We can impose the condition $\mathcal{B}/\mathcal{J} > 0$ to keep $G_{Eff} > 0$. For the positive-temperature condition

$$\dot{H} - \frac{k}{a^2} > -2\left(H^2 + \frac{k}{a^2}\right)$$

with $b < 1$, Eq. (34) reduces to

$$\frac{n(n-1)\hat{\Omega}_n\left(H^2+\frac{k}{a^2}\right)^{-n/2+1}}{16\pi G\mathcal{B}}\times \times\left[4H\mathcal{J}+(1-b)\mathcal{B}\partial_t\left(\frac{\mathcal{J}}{\mathcal{B}}\right)\right]\geqslant 0. \quad (35)$$

This shows that the GSLT is valid only if

$$\partial_t(\mathcal{J}/\mathcal{B}) > 0.$$

If the gravitational coupling constant is indeed a constant, the GSLT is always protected. If

$$\partial_t(\mathcal{J}/\mathcal{B}) < 0,$$

then the GSLT can hold only if

$$\left|\frac{\partial_t(\mathcal{J}/\mathcal{B})}{\mathcal{J}/\mathcal{B}}\right| \leqslant \frac{4H}{1-b}.$$

The GSLT in the $f(R)$ theory can be retrieved for $f_3(T) = 0$. If $T_t \sim T_h$, then condition (35) becomes

$$\frac{n(n-1)\hat{\Omega}_n\left(H^2+\frac{k}{a^2}\right)^{-(n/2+1)}}{16\pi G\mathcal{B}}\times \times\left[H\left(\dot{H}-\frac{k}{a^2}\right)^2\mathcal{J}\right]\geqslant 0.$$

We consider the $f(R,T)$ model in (25) with

$$f_1(R) = R, \quad f_2(R) = R^p,$$

and

$$f_3(R) = T^q \quad (p, q > 0);$$

then in 4-dimensional flat FRW metric, the GSLT becomes

$$T_h\dot{\hat{S}} = \frac{\dot{H}^2(1+pR^{p-1}T^q)}{2H^4G\left(1+\frac{R^pT^q}{8\pi G}\right)} \geqslant 0. \quad (36)$$

For the power law dependence

$$a(t) = a_0t^m$$

with

$$\rho = \rho_0 a^{-3},$$

this can be written as

$$T_h\dot{\hat{S}} = = \frac{8\pi[1+p(6m(2m-1)t^{-2}]^{p-1}(\rho_0t^{-3m})^q}{2m^2[8\pi G+(6m(2m-1)t^{-2})^{p-1}(\rho_0t^{-3m})^q]} \geq \geq 0. \quad (37)$$

We have examined the validity of relation (37) and developed constraints on the parameters m, p, and q. The results are shown in Figs. 3 and 4.

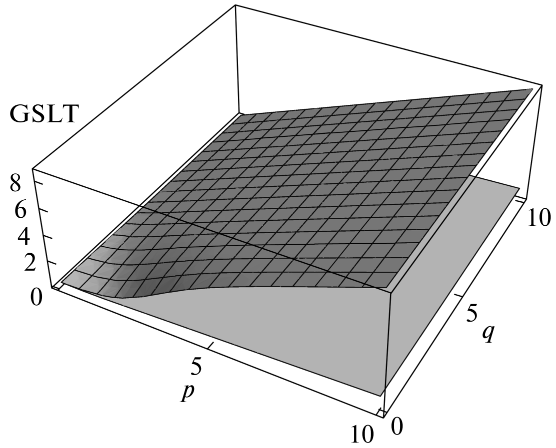

Fig. 3. Evolution of the GSLT for model (36) with $m = 2$. The curve with a larger slope corresponds to $z = -0.9$, while the other curve represents the present value ($z = 0$)

4. CONCLUSIONS

In this paper, the thermodynamic properties have been discussed in a more general $f(R,T)$ theory. The nonequilibrium treatment of thermodynamics is addressed for two particular models of the $f(R,T)$ gravity. In this modified theory, accelerated expansion can result not just from the scalar-curvature part of the entire energy density of the universe, but can include a matter component as well. The consequences of the $f(R,T)$ theory may contribute to significant results when compared to other modified gravitational theories, applicable to various problems of contemporary interest such as accelerated cosmic expansion, gravitational collapse, dark matter, and the detection of gravitational waves [40]. The detection of gravitational waves could be an excellent way to test general relativity and modified theories of gravity. Corda [40] has investigated the detection of gravitational waves in the $f(R)$ theory, and it would be appealing to explore this issue in the $f(R,T)$ theory.

It is shown that the representation of equilibrium thermodynamics is not executable in this theory [34]. Hence, the nonequilibrium treatment of thermodynamics is used to discuss the laws of thermodynamics. Here, we studied two particular models of the $f(R,T)$ theory to show the consequences of explicit coupling of matter and geometry. The gravitational coupling between matter and higher-derivative terms of curvature describes a transfer of energy and momentum beyond that normally existing in curved spaces. This interaction leads to the entropy production term in this modi-

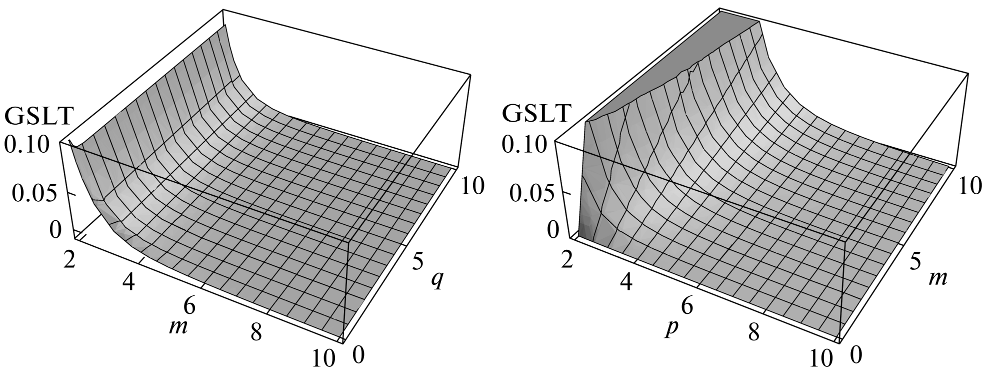

Fig. 4. We can choose the specific value of one parameter and vary the others. In the left panel, we choose $p = 1$ and show the constraints on m and q. The right panel represents the parametric values of m and p for the fixed $q = 1$

fied gravity. The FLT is formulated by using the Wald entropy relation. We remark that an entropy production term is produced in this work but no such term is present in GR, Gauss–Bonnet [22], Lovelock [22], and braneworld [24] theories of gravity.

The validity of the GSLT has also been investigated in this work. We have found that the GSLT holds under the conditions of the attractive nature of gravity and temperature being positive. In fact, it is natural to assume the relation $T_t = bT_h$, and the proportionality constant b can be considered equal to unity, implying that the system is in thermal equilibrium. Generally, the horizon temperature cannot match the temperature of all energy sources within the horizon, and the two mechanisms must experience interaction for some interval of time ahead of achieving the thermal equilibrium. Moreover, the gravitational curvature–matter coupling in the $f(R,T)$ theory can produce the unscripted flow of energy between the horizon and the fluid. Also, the energy fluid of dark components does not permit the effective gravitational constant to be an approximate constant. In the limited choice of thermal equilibrium, we assume that T_t is very close to T_h. We find that the GSLT is satisfied in both phantom and quintessence regimes of the universe, which seems to be consistent with the results in [41]. Furthermore, we have also developed constraints on some concrete $f(R,T)$ models corresponding to power-law solutions. It is significant to remark that the equilibrium treatment of thermodynamics in the $f(R,T)$ theory would benefit from further study.

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