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EQUILIBRATION OF A ONE-DIMENSIONAL QUANTUM LIQUID

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Dedicated to the memory of Professor Anatoly Larkin

We review some of the recent results on equilibration of one-dimensional quantum liquids. The low-energy properties of these systems are described by the Luttinger liquid theory, in which the excitations are bosonic quasiparticles. At low temperatures, the relaxation of the gas of excitations toward full equilibrium is exponentially slow. In electronic Luttinger liquids, these relaxation processes involve backscattering of electrons and give rise to interesting corrections to the transport properties of one-dimensional conductors. We focus on the phenomenological theory of the equilibration of a quantum liquid and obtain an expression for the relaxation rate in terms of the excitation spectrum.

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1. INTRODUCTION

The low-energy properties of one-dimensional quantum systems are commonly described in the framework of the so-called Tomonaga–Luttinger liquid [1–9]. This description applies to systems of either bosons or fermions, but regardless of the statistics of the constituent particles, the excitations of the Luttinger liquid are bosons with a linear spectrum. The Hamiltonian of a Luttinger liquid is given by

$$H = \sum_q v|q| b_q^\dagger b_q + \frac{\pi\hbar}{2L}\left[v_N (N - N_0)^2 + v_J J^2\right], \quad (1)$$

where b_q is the annihilation operator of a bosonic excitation with momentum q propagating with velocity v [8]. Hamiltonian (1) assumes that the system has a finite size L and periodic boundary conditions are imposed. Apart from the occupation numbers of bosonic states, the energy of the system depends on two integer numbers, N and J. For fermionic Luttinger liquids, these numbers can be interpreted in terms of the numbers of right- and left-moving particles as $N = N^R + N^L$ and $J = N^R - N^L$. The parameters v_N and v_J have the dimension of velocity and depend on the interactions between the particles; N_0 is some reference number of particles in the system. The Luttinger liquid theory described by Hamiltonian (1) has been successful in predicting a number of interesting phenomena, such as the renormalization of impurity scattering in interacting one-dimensional electron systems [10, 11], subsequently observed in experiments [12–15].

An interesting feature of the model in (1) is the complete absence of coupling between the bosons. As a result, the lifetimes of bosonic excitations are infinite and the system does not relax toward thermal equilibrium. It is important to keep in mind, however, that Eq. (1) is the exact Hamiltonian of the system only for the so-called Luttinger model [2], where the spectrum of the fermions consists of two linear branches $\epsilon_p = \pm v_F p$. In a generic situation, this is an approximation applicable only in the vicinity of the two Fermi points, and hence the Luttinger theory (1) applies only at low energies. In other words, Eq. (1) represents a fixed-point Hamiltonian in the renormalization group sense, which should, in principle, be amended by additional contributions describing various irrelevant perturbations. The latter are the operators of third and higher degrees in b_q and $b_q^\dagger$, which result in scattering of bosonic excitations. They adequately account for the curvature of the spectrum near the Fermi points, which gives rise to a multitude of interesting phenomena studied in the last few years (see [16] for a recent review).

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Another important aspect of the Luttinger model is that the original fermions are classified as belonging to one of two species, the right- and left-moving particles. In realistic systems, there is no fundamental difference between the particles moving in opposite directions, and a right-moving fermion can become a left-moving one upon scattering. These backscattering processes give rise to several interesting phenomena not captured by the Luttinger liquid theory.

One example is the effect of backscattering on the transport properties of the system. Experimentally, transport can be studied in quantum wire devices [17, 18], where a one-dimensional system is smoothly connected to two-dimensional leads. In the absence of interactions, the conductance of a quantum wire is quantized in units of e^2/h, where e is the elementary charge and h is the Planck constant. Interactions between electrons included into the Luttinger liquid theory do not affect conductance quantization [19–21]. On the other hand, the backscattering processes excluded from model (1) reduce the conductance [22]. More detailed theories of conductance of long uniform quantum wires relate the correction to conductance due to electron–electron interactions to the rate of equilibration of the electron liquid [23, 24].

The physics of equilibration of a liquid of one-dimensional fermions is the main subject of this paper. It is another example of a problem where backscattering processes are crucial. For particles with a realistic spectrum, such as $\epsilon_p = p^2/2m$, the relaxation of the system to equilibrium involves backscattering processes changing the numbers N^R and N^L. On the other hand, in a Luttinger liquid, the difference $J = N^R - N^L$ is conserved, even if the irrelevant perturbations are taken into account. As a result, equilibrium states of the system described by Hamiltonian (1) are characterized by different chemical potentials of the two species of particles, μ^R and μ^L, and their relaxation to a single value μ is neglected.

Below, we discuss the mechanism of equilibration of one-dimensional quantum liquids beyond the Luttinger liquid approximation. An expression for the corresponding equilibration rate τ^{-1} was obtained microscopically for the regimes of both weak [23] and strong [25] interactions. An alternative phenomenological approach [26, 27] based on the Luttinger liquid theory is applicable at any interaction strength and results in an expression for the equilibration rate τ^{-1} in terms of the excitation spectrum of the system. The latter can be either measured experimentally or derived microscopically for specific models. In Secs. 2–5, we review the phenomenological approach in [26, 27] and discuss the implications of the results for the equilibration rate to experiments with quantum wires.

2. EQUILIBRIUM STATE OF A UNIFORM LUTTINGER LIQUID

We first discuss the possible equilibrium states of a Luttinger liquid. In general, the equilibrium distribution is determined by the integrals of motion of the system [28]. We assume that the irrelevant perturbations resulting in weak scattering of bosons are added to Hamiltonian (1). Then there are four integrals of motion: energy, momentum, and the numbers of right- and left-moving particles, $N^R = (N + J)/2$ and $N^L = (N - J)/2$. The Gibbs probability of a realization of a given many-particle state i is then given by

$$w_i = \frac{1}{Z} \exp\left(-\frac{E_i + uP_i - \mu^L N^L - \mu^R N^R}{T}\right), \quad (2)$$

where E_i and P_i are the values of the momentum of the system in state i. Obtaining the equilibrium distribution of Bose excitations also requires the expression [8] for the momentum of the Luttinger liquid

$$P = p_F J + \sum_q q\, b_q^\dagger b_q, \quad (3)$$

where the Fermi momentum is defined in terms of the particle density, $p_F = \pi\hbar N/L$.

Using expression (2), we easily obtain the equilibrium form of the occupation numbers of the boson states:

$$N_q = \frac{1}{\exp((v|q| - uq)/T) - 1}. \quad (4)$$

We note that as a result of momentum conservation, the Bose distribution depends not only on temperature but also on the parameter u, which can be regarded as the velocity of the gas of bosonic excitations.

In addition to the bosonic occupation numbers, the state of the liquid depends on the zero modes N and J. In thermal equilibrium, according to Eq. (2) the latter is peaked sharply near an average value J, which satisfies

$$\pi\hbar \frac{v_J J}{L} = up_F + \frac{1}{2}\Delta\mu, \quad (5)$$

where $\Delta\mu = \mu^R - \mu^L$. In a Luttinger liquid, the ratio $j = v_J J/L$ has the meaning of the particle current [8]. Expressing it in terms of the drift velocity v_d as $j = (N/L)v_d$, we find

$$v_d = u + \frac{\Delta\mu}{2p_F}. \quad (6)$$

This expression shows that in an equilibrium of the Luttinger liquid, the gas of excitations moves at a velocity u different from the velocity v_d of the system as a whole. This decoupling is a result of the conservation of J, which allows for the possibility of $\Delta\mu \neq 0$. In a realistic system, the backscattering processes result in the relaxation of $\Delta\mu$ to zero, and the velocities u and v_d equilibrate.

3. EQUILIBRATION RATE

To study the kinetics of equilibration of a Luttinger liquid, we have to consider the corrections to the fixed-point Hamiltonian (1). In the case of a spinless Luttinger liquid, the irrelevant perturbations are terms of third and higher orders in bosonic operators, such as

$$b^\dagger_{q_1+q_2} b_{q_1} b_{q_2}, \quad b^\dagger_{q_1+q_2-q_3} b^\dagger_{q_3} b_{q_1} b_{q_2}, \quad \text{etc.}$$

Such perturbations give rise to scattering of the bosonic excitations and to relaxation of their distribution function toward equilibrium distribution (4). Since the scattering of bosons conserves their total momentum, the resulting distribution is characterized by a velocity u, which can be easily obtained from the initial momentum of the whole gas of excitations. When the distribution approaches the equilibrium form (4), the typical scattering events involve bosons with energies of the order of temperature, and the scattering rate τ_0^{-1} scales as a power of T. For instance, in the case of a strongly interacting system, the equilibration rate of the gas of excitations scales at $\tau_0^{-1} \propto T^5$ [29].

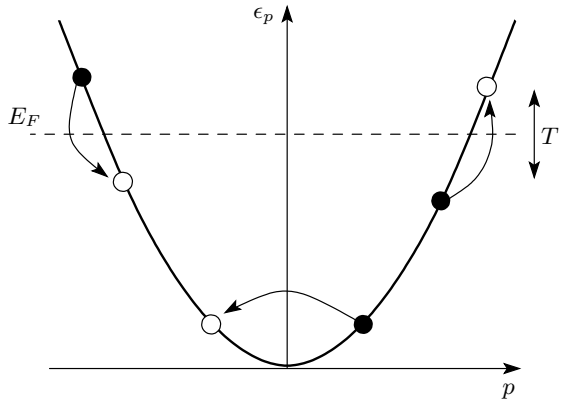

Fig. 1. In the model of weakly interacting fermions, the dominant backscattering process involves three particles: one near the bottom of the band and the other two near the left and right Fermi points [22]

The backscattering processes required for the relaxation of the velocity u toward v_d have been studied microscopically in the regime of weak interaction in Ref. [22]. Because of the constraints imposed by the momentum and energy conservation, the simplest nontrivial process involves three particles, as we show in Fig. 1. The Fermi statistics requires that in the dominant backscattering process, two particles be within the energy range of the order of temperature from the left and right Fermi points, whereas the third one be within T from the bottom of the band. As a result of such a scattering event, the third particle backscatters, i. e., the numbers of right- and left-moving particles change by one. Since the backscattering particle fills a hole deep below the Fermi level, the rate of such processes is exponentially small, $\tau^{-1} \propto e^{-E_F/T}$ [22, 23]. We see below that the backscattering rate is exponentially suppressed at low temperatures for any interaction strength.

The strong suppression of the backscattering rate means that at low temperatures, the equilibration of the quantum liquid proceeds in two steps. First, the bosonic excitations come to thermal equilibrium with each other and their distribution function takes form (4). This thermalization takes a relatively short time of the order of τ_0. Second, over a much longer time τ, the backscattering processes equilibrate the zero mode J with the bosons. During this time, the velocity u of the gas of bosonic excitations approaches the velocity v_d of the liquid as the difference of the chemical potentials of the right and left movers $\Delta\mu$ relaxes to zero (see Eq. (6)). The time dependences of u and $\Delta\mu$ should follow the usual relaxation law

$$\frac{du}{dt} = -\frac{u - v_d}{\tau}, \quad \frac{d}{dt}\Delta\mu = -\frac{\Delta\mu}{\tau}. \tag{7}$$

Expression (7) gives the formal definition of the equilibration time τ.

To study the relaxation rate τ^{-1} at an arbitrary interaction strength, it is tempting to use the Luttinger liquid description of the system. But this approach is incapable of describing the particles near the bottom of the band (see Fig. 1), which are crucial for the equilibration of the system. More precisely, bosonic Hamiltonian (1) provides correct description of the excitation spectrum of a quantum liquid only at low energies, namely, for $|\varepsilon| < D$, where the bandwidth $D \ll vp_F$. Indeed, for such excitations, the spectrum can be linearized and consists of two independent branches, as required in the Luttinger model. On the other hand, any excitation with the energy $|\varepsilon| \sim vp_F$ is not accounted for by Hamiltonian (1).

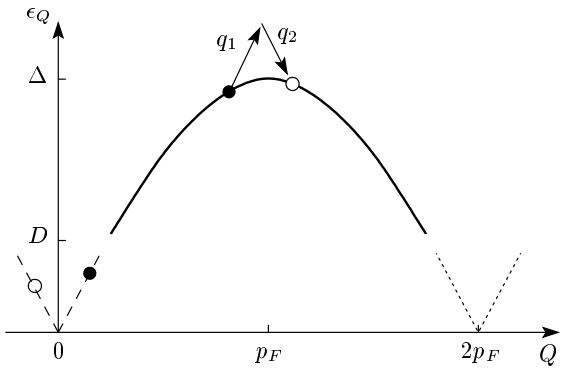

Fig. 2. Spectrum of a hole excitation in a quantum liquid. The states with energies below D are treated as excitations of the Luttinger liquid, whereas the higher-energy states are modeled as a mobile impurity. The hole can change its momentum by $\delta Q = q_1 - q_2$ by absorbing a boson with a momentum q_1 and emitting one with a momentum q_2

This difficulty can be overcome as follows [26]. Because the small probability of an empty state near the bottom of the band plays the crucial role in the physics of equilibration, we first consider the spectrum of the hole excitations. For noninteracting fermions, a hole with a momentum Q can be defined as an excitation of the system obtained by moving a fermion from the state $p_F - Q$ to p_F. For a system with a concave spectrum, such as the one in Fig. 1, the hole represents the ground state of the system with the total momentum Q. We use this observation to generalize the concept of a hole excitation to the case of an arbitrary interaction strength, and define the hole as the ground state of the system with the momentum Q. Because moving a fermion from one Fermi point to the other changes the momentum by $2p_F$ without changing the energy of the system, the energy ε_Q of the hole is a periodic function of momentum and vanishes at $Q = 0, \pm 2p_F, \pm 4p_F, \ldots$

The holes with energies below the bandwidth D have a nearly linear spectrum. They are accounted for in Hamiltonian (1) as superpositions of various bosonic excitations with the same momentum. The holes with energies above D are not included in Hamiltonian (1) and are treated as mobile impurities in the Luttinger liquid [30–38]. The exact value of the crossover energy scale D is not important as long as it is small compared to the maximum energy of the hole $\varepsilon_{p_F} \sim v p_F$ and large compared to the temperature T.

The mechanism of equilibration can be described as follows. For simplicity, we assume from now on that the liquid is at rest, $v_d = 0$. The gas of bosonic excitations equilibrates relatively quickly, and the occupation numbers of bosonic states take form (4), which applies in the region $|q| < D/v$ represented by two straight dashed lines in Fig. 2. In the generic case, the total momentum of the excitations in the initial state is not zero, and hence the Bose distribution in (4) has a boost velocity $u \neq 0$. As a result of interactions between the bosons, a small fraction of the particles are promoted above the energy D, where they are no longer described by Hamiltonian (1). At an arbitrary interaction strength, the properties of these higher-energy excitations are rather complicated, but the lowest-energy excitation at a given momentum Q is a hole. Because $\varepsilon_Q \gg T$, the occupation of the hole states is given by the Boltzmann factor

$$f(Q) = \exp\left(-\frac{\varepsilon_Q - uQ}{T}\right). \tag{8}$$

The presence of the correction $-uQ$ in the exponent is assured by the fact that the hole interacts and exchanges momentum with the thermalized bosons. As a result of many such collisions, the hole, with a small probability, can increase its momentum Q above p_F, after which it is more likely to fall toward $Q = 2p_F$ than return to the vicinity of $Q = 0$ (see Fig. 2). As the hole approaches $Q = 2p_F$, it enters the linear spectrum region at $\varepsilon_Q < D$, shown by dotted lines in Fig. 2. There it can again be viewed as a superposition of bosonic excitations.

As a result of this rare sequence of scattering events, the bosons have transferred the momentum $2p_F$ to the hole. Due to the conservation of the total momentum (3), this decrease in the momentum of the gas of excitations means that the zero mode $J = N^R - N^L$ has increased by 2, i. e., one fermion has been backscattered. Also, the decrease in the total momentum of the bosons means that the velocity u has also decreased, in accordance with relaxation law (7).

The equilibration proceeds very slowly because the hole must pass the point $Q = p_F$ in the momentum space, where the occupation numbers are exponentially small. We therefore expect

$$\tau^{-1} = Ce^{-\Delta/T}, \quad \Delta = \varepsilon_{p_F}. \tag{9}$$

To obtain the prefactor C, the kinetics of the scattering processes should be considered in more detail.

We start by noting that the equilibration rate is controlled by a small region of momentum space near $q = p_F$ where the energy of the hole is close to the maximum, $\Delta - \varepsilon_Q \lesssim T$. The width of this region can

be estimated as $(m^*T)^{1/2}$, where we introduce the effective mass of the hole as

$$\frac{1}{m^*} = -\frac{d^2\varepsilon_Q}{dQ^2}\bigg|_{Q=p_F}. \tag{10}$$

Although the region is narrow compared to p_F, it is wide compared to the typical change of the momentum of the hole in a single collision with bosonic excitations. Indeed, an elementary scattering event consists of the hole absorbing one boson and emitting another (see Fig. 2). Since the bosons are thermalized, the typical change of Q is of the order of T/v, which is much smaller than $(m^*T)^{1/2}$ at $T \ll \Delta$. This estimate allows simplifying the problem considerably.

The motion of the hole in momentum space is random and occurs in steps that are small compared to the size of the critical region near the barrier. Such diffusion in momentum space is described by the Fokker–Planck equation [39] for the time-dependent distribution function $f(Q,t)$:

$$\partial_t f = -\partial_Q F, \tag{11}$$

where the probability current F has the form

$$F = -\frac{B(Q)}{2}\left[\frac{\varepsilon_Q'}{T} + \partial_Q\right] f. \tag{12}$$

Here, the prime denotes the derivative with respect to Q and $B(Q)$ has the meaning of the diffusion constant in momentum space. It is defined as

$$B(Q) = \sum_{\delta Q} [\delta Q]^2 W_{Q,Q+\delta Q} \tag{13}$$

in terms of the rate $W_{Q,Q'}$ of hole scattering from the state Q to Q'.

In the steady-state regime, Fokker–Planck equation (11) is solved by demanding that the probability current F be independent of Q. Finding the value of F requires imposing boundary conditions on the occupation numbers $f(Q)$ on the two sides of the barrier. Assuming that the size of the crossover region in momentum space $(m^*T)^{1/2}$ is small compared with p_F, we can approximate Eq. (8) as

$$f(Q) = \exp\left(-\frac{\varepsilon_Q - up_F}{T}\right), \quad p_F - Q \gg (m^*T)^{1/2}. \tag{14}$$

This expression specifies the boundary condition on $f(Q)$ to the left of the barrier. To find the boundary condition to the right of the barrier, we notice that the hole states with momenta Q and $Q+2p_F$ are identical, and the occupation of states with Q between p_F and $2p_F$ is given by Eq. (8) with $Q \to Q - 2p_F$. This yields

$$f(Q) = \exp\left(-\frac{\varepsilon_Q + up_F}{T}\right), \quad Q - p_F \gg (m^*T)^{1/2}. \tag{15}$$

Solving first-order differential equation (12) with a constant F, we find that boundary conditions (14) and (15) are satisfied for

$$F = u\,\frac{p_F B}{(2\pi m^* T^3)^{1/2}}\, e^{-\Delta/T}, \tag{16}$$

where we take the limit $u \to 0$ and set $B = B(2p_F)$.

A nonvanishing constant F means that $(L/h)F$ holes are passing any given point in momentum space in unit time. Each hole moving from the vicinity of $Q = 0$ to that of $Q = 2p_F$ takes the momentum $2p_F$ out of the bosonic excitations. We therefore conclude that the total momentum of the bosons changes with time at the rate

$$\dot{P}_b = -2p_F \frac{L}{h} F.$$

Given that the momentum of the bosons distributed in accordance with Eq. (4) is $P_b = (\pi L T^2/3\hbar v^3)u$, we find $\dot{u} = -u/\tau$ with

$$\frac{1}{\tau} = \frac{3B}{\pi^{5/2} p_F^2}\left(\frac{vp_F}{T}\right)^3 \left(\frac{p_F^2}{2m^*T}\right)^{1/2} e^{-\Delta/T}. \tag{17}$$

As expected, the equilibration rate has the exponential form (9). To fully evaluate the prefactor, however, we need to study the hole scattering rate $W_{Q,Q'}$ and obtain diffusion constant (13).

4. HOLE SCATTERING RATE

The scattering of a hole by bosonic excitations is a special case of the problem of dynamics of a mobile impurity in a Luttinger liquid [31]. At low temperatures, the leading scattering process involves two bosons moving in the opposite directions. By absorbing one boson and emitting the other, the impurity can scatter from a state Q to a new state Q' without violating the momentum and energy conservation (Fig. 2). The authors of Ref. [31] obtained the temperature dependence of the mobility of the impurity in a Luttinger liquid in this regime, $\mu \propto T^{-4}$. Using the expression $\mu = T/B$ for the mobility (see [39], § 21), we conclude that

$$B = \chi T^5. \tag{18}$$

The evaluation of the coefficient χ presents an interesting problem. Microscopic calculations can be performed in the special cases of either weak or strong interactions [23, 25]. Interestingly, a phenomenological expression for χ can also be obtained in terms of the spectrum of the mobile impurity (hole) in the Luttinger liquid [26, 27]. Here, we review the last approach.

The diffusion constant B in expression (17) for the equilibration rate should be evaluated at $Q = p_F$. On the other hand, it is instructive to consider a more general problem and study the hole scattering rate $W_{Q,Q+\delta Q}$ in Eq. (13) for arbitrary Q. This scattering rate can be found from Fermi's Golden rule

$$W_{Q,Q+\delta Q} = \frac{2\pi}{\hbar}\sum_{q_1,q_2}|t_{q_1,q_2}|^2 N_{q_1}(N_{q_2}+1)\,\delta_{q_1-q_2,\delta Q}\times \\ \times\,\delta(\varepsilon_Q - \varepsilon_{Q+\delta Q} + \hbar v|q_1| - \hbar v|q_2|), \quad (19)$$

where t_{q_1,q_2} is the matrix element of the process in which the hole absorbs the boson q_1 and emits the boson q_2 (Fig. 2). Since the typical energies of the bosons are of the order of temperature, we assume that $|\delta Q| \ll p_F$. In this case, we can easily obtain the momenta q_1 and q_2 from the conservation laws:

$$q_1 = \frac{1}{2}\delta Q + \frac{v_Q}{2v}|\delta Q|, \quad q_2 = -\frac{1}{2}\delta Q + \frac{v_Q}{2v}|\delta Q|. \quad (20)$$

Here, $v_Q = \varepsilon_Q'$ is the velocity of the hole with the momentum Q. Using Eq. (20), we easily express the scattering rate as

$$W_{Q,Q+\delta Q} = \frac{L}{\hbar^2 v}\,N_{q_1}(N_{q_2}+1)|t_{q_1,q_2}|^2. \quad (21)$$

To evaluate the matrix element t_{q_1,q_2}, we need to discuss the Hamiltonian of the Luttinger liquid in the presence of a mobile impurity.

We start by writing Hamiltonian (1) in the alternative form [9]

$$H_0 = \frac{\hbar v}{2\pi}\int dx[K(\nabla\theta)^2 + K^{-1}(\nabla\phi)^2], \quad (22)$$

where the two bosonic fields $\phi(x)$ and $\theta(x)$ satisfy the commutation relation

$$[\phi(x), \nabla\theta(x')] = i\pi\delta(x-x') \quad (23)$$

and the Luttinger-liquid parameter K depends on the interactions between particles. The case of noninteracting fermions corresponds to $K = 1$.

Hamiltonian (22) can be brought to form (1) with the help of the following expressions for the fields ϕ and θ in terms of the bosonic operators:

$$\nabla\phi(x) = -i\sum_q \sqrt{\frac{\pi K|q|}{2\hbar L}}\,\mathrm{sgn}(q)(b_q + b_{-q}^\dagger)e^{iqx/\hbar}, \quad (24)$$

$$\nabla\theta(x) = i\sum_q \sqrt{\frac{\pi|q|}{2\hbar KL}}\,(b_q - b_{-q}^\dagger)e^{iqx/\hbar}. \quad (25)$$

The advantage of the form (22) of the Hamiltonian is that the fields ϕ and θ have clear meanings in terms of the observables characterizing the quantum liquid. For instance, the field $\phi(x)$ accounts for fluctuations of the density of the liquid,

$$n(x) = n_0 + \frac{1}{\pi}\nabla\phi(x), \quad (26)$$

where $n_0 = N/L$ is the average density [9]. Similarly, the field θ is related to the momentum κ of the liquid per particle,

$$\kappa(x) = -\hbar\nabla\theta(x) \quad (27)$$

(see, e.g., Ref. [27]).

The coupling of the hole to bosonic excitations in the Luttinger liquid can now be obtained by considering the dependence $\varepsilon_Q(n,\kappa)$ of the energy of the hole on the density and momentum of the liquid. Using Eqs. (26) and (27), we expand $\varepsilon_Q(n,\kappa)$ in powers of the bosonic fields,

$$\varepsilon_Q(n,\kappa) = \varepsilon_Q(n_0,0) + \frac{1}{\pi}\partial_n\varepsilon_Q\nabla\phi - \hbar\,\partial_\kappa\varepsilon_Q\nabla\theta + \\ + \frac{1}{2\pi^2}\partial_n^2\varepsilon_Q(\nabla\phi)^2 + \frac{\hbar^2}{2}\partial_\kappa^2\varepsilon_Q(\nabla\theta)^2 - \\ - \frac{\hbar}{\pi}\partial_n\partial_\kappa\varepsilon_Q\nabla\phi\nabla\theta + \ldots \quad (28)$$

All derivatives of $\varepsilon_Q(n,\kappa)$ are here taken at $n = n_0$ and $\kappa = 0$. Taking into account Eqs. (24) and (25), we see that the second-order terms in Eq. (28) contain contributions in which a boson q_1 is absorbed and a boson q_2 on the opposite branch is emitted. The corresponding matrix element has the form

$$t^{(a)}_{q_1,q_2} = -\frac{\sqrt{|q_1 q_2|}}{2\pi\hbar L}\,\partial_{LR}^2\varepsilon_Q, \quad (29)$$

where we assume that the hole is at $x = 0$ and introduce the notation

$$\partial_{LR}^2 = K\partial_n^2 - \frac{(\pi\hbar)^2}{K}\partial_\kappa^2. \quad (30)$$

In addition to the terms coupling the hole to two bosons, Eq. (28) contains the contribution linear in bosonic operators:

$$i\partial_L\varepsilon_Q\sum_{q<0}\sqrt{\frac{|q|}{2\pi\hbar L}}\,(b_q - b_q^\dagger) - \\ - i\partial_R\varepsilon_Q\sum_{q>0}\sqrt{\frac{|q|}{2\pi\hbar L}}\,(b_q - b_q^\dagger), \quad (31)$$

where

$$\partial_L = \sqrt{K}\partial_n - \frac{\pi\hbar}{\sqrt{K}}\partial_\kappa, \quad \partial_R = \sqrt{K}\partial_n + \frac{\pi\hbar}{\sqrt{K}}\partial_\kappa. \quad (32)$$

Linear coupling terms (31) also contribute to the matrix element t_{q_1,q_2}, but in the second-order perturbation theory,

$$t^{(b)}_{q_1,q_2} = -\frac{\sqrt{|q_1q_2|}}{2\pi\hbar L}\left[\frac{\partial_L\varepsilon_{Q+q_1}\partial_R\varepsilon_Q}{\varepsilon_Q + vq_1 - \varepsilon_{Q+q_1}} + \right.$$
$$\left. + \frac{\partial_R\varepsilon_{Q-q_2}\partial_L\varepsilon_Q}{\varepsilon_Q - \varepsilon_{Q-q_2} - v|q_2|}\right], \quad (33)$$

where we assume that $q_1 > 0$ and $q_2 < 0$, i. e., positive δQ as in Fig. 2. It is important to account for the corrections to the momentum of the hole in the numerator, which occur because the two perturbations of form (31) act on the states of the hole with different values of Q.

Expression (33) can be simplified by using the smallness of $q_1 \sim q_2 \sim \delta Q \ll p_F$. The expression in the brackets appears to scale as $1/\delta Q$. This term is obtained by neglecting corrections to Q in the numerator and linearizing the denominators in q_1 and q_2. However, for the specific values (20) of the boson momenta, the two contributions in the brackets cancel each other. Evaluating the next order terms in q_1 and q_2, we find

$$t^{(b)}_{q_1,q_2} = -\frac{\sqrt{|q_1q_2|}}{2\pi\hbar L}\left[\frac{1}{m^*_Q}\frac{\partial_L\varepsilon_Q}{v+v_Q}\frac{\partial_R\varepsilon_Q}{v-v_Q} + \right.$$
$$\left. + \partial_L v_Q\frac{\partial_R\varepsilon_Q}{v-v_Q} - \partial_R v_Q\frac{\partial_L\varepsilon_Q}{v+v_Q}\right], \quad (34)$$

where the momentum-dependent effective mass of the hole is defined by $1/m^*_Q = -\varepsilon''_Q$. The first term in Eq. (34) originates from the expansion of the denominators in Eq. (33) to the second order in q_1 and q_2, whereas the remaining two terms are obtained by accounting for linear corrections in the numerators.

Finally, one more contribution to the scattering matrix element t_{q_1,q_2} is obtained when the hole couples to a single boson, Eq. (31), which in turn splits into two. The matrix element involves three bosons and should therefore originate from corrections to the Hamiltonian that are cubic in ϕ and θ. For a fluid at rest, the symmetry allows only for even powers of θ, and hence the correction must have the form

$$H_\alpha = \int dx\left[\alpha_\theta(\nabla\phi)(\nabla\theta)^2 + \alpha_\phi(\nabla\phi)^3\right]. \quad (35)$$

The values of the coefficients α_θ and α_ϕ can be related to the density dependences of the parameters v and K

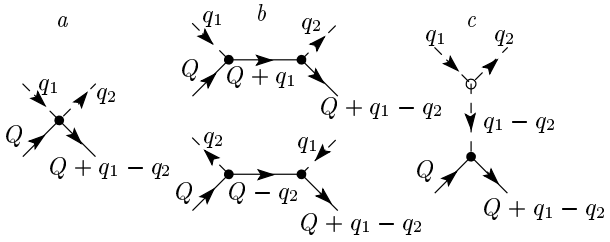

Fig. 3. The three types of processes contributing to the scattering matrix element t_{q_1,q_2}. *a*) The first-order process, in which the hole couples to two bosons. *b*) The second-order processes involving two perturbations, each coupling the hole to one of the two bosons. *c*) The second-order process where the hole couples to a virtual boson, which is separately coupled to bosons q_1 and q_2

of quadratic Hamiltonian (22) by considering the correction to the total Hamiltonian $H_0 + H_\alpha$ caused by a small change of the particle density δn. This yields [27]

$$\alpha_\theta = \frac{\hbar}{2\pi^2}\,\partial_n(vK), \quad \alpha_\phi = \frac{\hbar}{6\pi^2}\,\partial_n\left(\frac{v}{K}\right). \quad (36)$$

To find a contribution to t_{q_1,q_2}, we need the matrix element of H_α that absorbs a boson with the momentum q_1 on one branch and emits a boson q_2 on the other branch. Using Eq. (36), we obtain

$$\frac{i\,\mathrm{sgn}(q_1)}{\sqrt{2\pi\hbar L}}\,\frac{v\partial_n K}{\sqrt{K}}\sqrt{|q_1q_2(q_1-q_2)|}\,\times$$
$$\times\, b^\dagger_{q_2}b_{q_1}\left(b_{q_2-q_1} + b^\dagger_{q_1-q_2}\right). \quad (37)$$

The second-order calculation of the matrix element t_{q_1,q_2} with perturbations (37) and (31) yields

$$t^{(c)}_{q_1,q_2} = -\frac{\sqrt{|q_1q_2|}}{2\pi\hbar L}\,\frac{v\partial_n K}{\sqrt{K}}\left(\frac{\partial_R\varepsilon_Q}{v-v_Q} + \frac{\partial_L\varepsilon_Q}{v+v_Q}\right). \quad (38)$$

The three types of processes leading to hole scattering with absorption of the boson q_1 and emission of the boson q_2 are illustrated in Fig. 3. Their total is given by

$$t_{q_1,q_2} = -\frac{\sqrt{|q_1q_2|}}{2\pi\hbar L}\,Y_Q, \quad (39)$$

where

$$Y_Q = \partial^2_{LR}\varepsilon_Q + \frac{1}{m^*_Q}\frac{\partial_L\varepsilon_Q}{v+v_Q}\frac{\partial_R\varepsilon_Q}{v-v_Q} + \partial_L v_Q\frac{\partial_R\varepsilon_Q}{v-v_Q} -$$
$$- \partial_R v_Q\frac{\partial_L\varepsilon_Q}{v+v_Q} + \frac{v\partial_n K}{\sqrt{K}}\left(\frac{\partial_R\varepsilon_Q}{v-v_Q} + \frac{\partial_L\varepsilon_Q}{v+v_Q}\right). \quad (40)$$

An alternative way of evaluating the scattering matrix element involves performing a unitary transformation that eliminates the linear coupling (31) of the hole to the bosons [27]. After this transformation, only the quadratic coupling remains, which is then evaluated in the first order, similarly to Eq. (29). The resulting expression given by Eqs. (49) and (50) in Ref. [27] is equivalent to Eq. (39).

Using expression (39) for the scattering matrix element in combination with Eqs. (21) and (13), we easily recover the temperature dependence (18). The coefficient χ takes the form

$$\chi = \frac{4\pi Y_{p_F}^2}{15\hbar^5 v^6}. \tag{41}$$

Equations (17), (18), and (41) provide a complete expression for the equilibration rate of a one-dimensional quantum liquid in terms of the spectrum of hole excitations and its dependences on the particle density n and the momentum per particle κ.

5. DISCUSSION

In this paper, we discussed the equilibration of a one-dimensional quantum liquid of interacting fermions. The conventional Luttinger liquid theory [8, 9] of these systems neglects the processes of backscattering. In many cases, this is an excellent approximation because the corresponding scattering rates are exponentially small at low temperatures, Eq. (17). But the Luttinger liquid approximation does not allow treating a number of interesting phenomena in which the backscattering plays the crucial role.

One example is the conductance of a long uniform quantum wire. The Luttinger liquid theory predicts perfect conductance quantization in these devices, regardless of the interaction strength [19–21]. On the other hand, it is easy to show that at weak electron–electron interactions, a correction to the conductance appears due to the backscattering processes [22, 23]. Interestingly, an expression for the conductance of a quantum wire can be obtained for any interaction strength [24],

$$G = \frac{e^2}{h}\left(1 - \frac{\pi^2}{3}\frac{T^2}{v^2 p_F^2}\frac{L}{L+2v\tau}\right). \tag{42}$$

The backscattering gives rise to a negative correction to the quantized conductance, which grows with temperature and with the length of the wire L. In short wires, the correction $\delta G \propto \tau^{-1}$ is exponentially small, but it saturates at

$$\delta G \sim -\frac{e^2}{h}\left(\frac{T}{v p_F}\right)^2$$

in long wires.

Temperature-dependent corrections to conductance of quantum wire devices have been observed in multiple experiments [40–42]. The data shows excellent quantization of conductance at lowest temperatures and a negative correction developing as the temperature is increased. These observations are in qualitative agreement with Eq. (42). In comparing the data with theory, it is important to keep in mind that our discussion has ignored spins, which appear to play an important role in experiments. The result (42) can be generalized to include spins [24], but the evaluation of the equilibration rate of a system with spins is still an open problem. Another complication is that most experiments study rather short wires, which cannot be treated as uniform.

Our discussion of the equilibration rate did not assume the Galilean invariance of the system. On the other hand, momentum conservation was assumed. Hence, the results do not automatically apply to systems of interacting particles in periodic potentials, such as spin chains. In such systems, umklapp scattering by the external potential may facilitate equilibration. On the other hand, electrons in GaAs quantum wires have an essentially quadratic spectrum $\epsilon_p = p^2/2m$, where m is the effective mass of the electron in this material. Such an electron system is Galilei invariant, which leads to a few simplifications. First, the Luttinger liquid parameter in this case is determined by the velocity of the bosons, $K = \pi\hbar n/mv$. Second, the dependence of the excitation energy on the momentum κ has the simple form [43]

$$\varepsilon_Q(n,\kappa) = \varepsilon_Q(n) + Q\frac{\kappa}{m}. \tag{43}$$

For momenta Q in the vicinity of p_F, we can expand

$$\varepsilon_Q(n) = \Delta(n) - \frac{(Q-p_F)^2}{2m^*}$$

and find

$$Y_{p_F} = K\left(\Delta'' + \frac{\Delta'^2}{m^* v^2} - \frac{2v'}{v}\Delta'\right). \tag{44}$$

Substituting this in Eq. (41) recovers the results in Ref. [26] for the equilibration rate in Galilei-invariant systems obtained by a different technique. It is worth noting that in this case, the equilibration rate is fully determined by the density dependences of the velocity of bosonic excitations v and the maximum energy of the hole Δ.

Although our main focus was on interacting systems of fermions, the approach and the results should be equally applicable to systems of bosons. Similar techniques have been recently applied to dynamics of dark solitons and mobile impurities in bosonic fluids [37, 38]. Finally, it is worth mentioning that in integrable models, apart from energy and momentum, there are multiple additional conserved quantities, and one expects that no equilibration of the system should take place. In particular one should find $\tau^{-1} = 0$. This conjecture has been checked [26] for the Calogero–Sutherland [44] and Lieb–Liniger [45] models. More generally, one expects [27] that for integrable models, the quantity Y_Q given by Eq. (40) should vanish for any Q.

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