ЖЭТФ, 2014, том 145, вып. 6, стр. 991–1009

NEUTRINO MASS AND MIXING IN THE 3–3–1 MODEL AND S_3 FLAVOR SYMMETRY WITH MINIMAL HIGGS CONTENT

V. V. Vien[a*], **H. N. Long**[b**]

[a] *Department of Physics, Tay Nguyen University*
Buon Ma Thuot, DakLak, Vietnam

[b] *Institute of Physics, VAST*
Ba Dinh, Hanoi, Vietnam

Received December 20 2013

A new S_3 flavor model based on the $\mathrm{SU}(3)_C \otimes \mathrm{SU}(3)_L \otimes \mathrm{U}(1)_X$ gauge symmetry responsible for fermion masses and mixings different from our previous work [14, 17] is constructed. The new feature is a two-dimensional representation of a Higgs anti-sextet under S_3, which is responsible for neutrino masses and mixings. The neutrinos acquire small masses from only an anti-sextet of $\mathrm{SU}(3)$, which is in a doublet under S_3. If the difference of components of the anti-sextet is regarded as a small perturbation, S_3 is equivalently broken into identity, the corresponding neutrino mass mixing matrix acquires the most general form, and the model can fit the latest data on neutrino oscillations. This way of symmetry breaking helps us reduce a content in the Higgs sector, to only one anti-sextet instead of two as in our previous work [14]. Our results show that the neutrino masses are naturally small and a small deviation from the tri-bimaximal neutrino mixing form can be realized. The Higgs potential of the model as well as the minimization conditions and gauge boson masses and mixings are also considered.

DOI: 10.7868/S0044451014060044

1. INTRODUCTION

The experiments on neutrino oscillations have indicated that the neutrinos have small masses and mixings [1–4], and therefore the standard model of fundamental particles and interactions must be extended. Among this direction, there have been various models proposed, such as [5, 6] and others. An alternative is to extend the electroweak symmetry $\mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y$ to $\mathrm{SU}(3)_L \otimes \mathrm{U}(1)_X$, in which to complete the fundamental representations of $\mathrm{SU}(3)_L$ with the standard-model doublets so as to obtain the neutral fermions. This proposal, which has nice features and has been extensively studied over the last two decades, is called 3–3–1 models [7–9], with the number of fermion families having been proved to be three [7, 10].

The parameters of neutrino oscillations such as the squared mass differences and mixing angles are now very constrained. The data in Ref. [4] imply that

$$\begin{aligned}
&\sin^2(2\theta_{12}) = 0.857 \pm 0.024 \quad (t_{12} \approx 0.6717),\\
&\sin^2(2\theta_{13}) = 0.098 \pm 0.013 \quad (s_{13} \approx 0.1585),\\
&\sin^2(2\theta_{23}) > 0.95,\\
&\quad \Delta m^2_{21} = (7.50 \pm 0.20) \cdot 10^{-5}\ \mathrm{eV}^2,\\
&\quad \Delta m^2_{32} = (2.32^{+0.12}_{-0.08}) \cdot 10^{-3}\ \mathrm{eV}^2.
\end{aligned} \tag{1}$$

These large neutrino mixing angles are completely different from the quark mixing ones defined by the Cabibbo–Kobayashi–Maskawa (CKM) matrix. Therefore, it is very important to find a natural model that leads to these mixing patterns of quarks and leptons with good accuracy. Small non-Abelian discrete symmetries are considered to be the most attractive choice for the flavor sector [11–13]. The simplest explanation for these conclusions is probably due to an S_3 flavor symmetry, which is the smallest non-Abelian discrete group [14, 15]. In fact, there is an approximately maximal mixing of two flavors μ and τ as given above, which can be connected by the $\underline{2}$ irreducible representation of S_3. Besides the $\underline{2}$, the S_3 group can provide two inequivalent singlet representations $\underline{1}$ and $\underline{1}'$, which play

[*] E-mail: wvienk16@gmail.com
[**] E-mail: hnlong@iop.vast.ac.vn

a crucial role in reproducing consistent fermion masses and mixings [14]. The S_3 models have been studied extensively over the last decade [13]. In [14], we proposed two 3–3–1 models, with either neutral fermions or right-handed neutrinos, based on the S_3 flavor symmetry, in which a large number of Higgs triplets was required. In this paper, we propose a new S_3 flavor symmetry in the 3–3–1 model with neutral fermions, in which the number of Higgs triplets required is less and the Higgs potential of the model is therefore much simpler than the previous ones.

The motivation for extending the above application to the 3–3–1 models with the neutral fermions N_R is mentioned in [14, 16, 17]. In this paper, we investigate simpler choices for Higgs multiplets of S_3 in which the unique anti-sextet responsible for the neutrino mass and mixing lying in $\underline{2}$ under S_3 and the difference between two VEV components of the anti-sextet play the role of perturbation. It is also noted that the numbers of fermion families in the 3–3–1 models originate from the anomaly-free gauge symmetry and naturally meet our criteria on the dimensions of flavor group representations such as S_3, unlike the others in the literature, mostly imposed by hand [11–13].

The rest of this work is as follows. In Sec. 2, we present the necessary elements of the 3–3–1 model with neutral fermions N_R under the S_3 symmetry and introduce the necessary Higgs fields responsible for the charged-lepton and quark masses. Section 3 is devoted to the neutrino mass and mixing. In Sec. 4, we consider the Higgs potential and minimization conditions. We summarize our results and make conclusions in Sec. 6.

2. THE MODEL

The fermion content of the model is similar to that in [14]: the fermions in the model transform under the respective $[\mathrm{SU}(3)_L, \mathrm{U}(1)_X, \mathrm{U}(1)_\mathcal{L}, \underline{S}_3]$ symmetries as

$$\begin{aligned} &\psi_{1L} = (\nu_{1L}, l_{1L}, N^c_{1R})^T \sim [3, -1/3, 2/3, \underline{1}],\\ &l_{1R} \sim [1, -1, 1, \underline{1}],\\ &\psi_{\alpha L} = (\nu_{\alpha L}, l_{\alpha L}, N^c_{\alpha R})^T \sim [3, -1/3, 2/3, \underline{2}],\\ &l_{\alpha R} \sim [1, -1, 1, \underline{2}],\\ &Q_{1L} = (u_{1L}, d_{1L}, U_L)^T \sim [3, 1/3, -1/3, \underline{1}],\\ &u_{1R} \sim [1, 2/3, 0, \underline{1}], \quad d_{1R} \sim [1, -1/3, 0, \underline{1}],\\ &U_R \sim [1, 2/3, -1, \underline{1}],\\ &Q_{\alpha L} = (d_{\alpha L},\ -u_{\alpha L},\ D_{\alpha L})^T \sim [3^*, 0, 1/3, \underline{2}],\\ &u_{\alpha R} \sim [1, 2/3, 0, \underline{2}], \quad d_{\alpha R} \sim [1, -1/3, 0, \underline{2}],\\ &D_{\alpha R} \sim [1, -1/3, 1, \underline{2}], \end{aligned} \tag{2}$$

where $\alpha = 2, 3$ is a family index of the last two lepton and quark families, which are defined as the components of the $\underline{2}$ representations. We note that the $\underline{2}$ for quarks satisfies the requirement of anomaly cancellation, where the last two left-quark families are in 3^* and the first one as well as the leptons are in 3. All the $\mathcal{L}$ charges of the model multiplets are listed in square brackets. In what follows, we consider possibilities for generating the fermion masses. The scalar multiplets needed for this purpose are to be introduced accordingly.

To generate masses for the charged leptons, we introduce two $\mathrm{SU}(3)_L$ scalar triplets ϕ and ϕ' respectively lying in $\underline{1}$ and $\underline{1}'$ under S_3, with the VEVs $\langle\phi\rangle = (0\,v\,0)^T$ and $\langle\phi'\rangle = (0\,v'\,0)^T$ [14]. From the invariant Yukawa couplings for the charged leptons, we obtain $m_e = h_1 v$, $m_\mu = hv - h'v'$, $m_\tau = hv + h'v'$, and the mixing matrices of the left- and right-handed charged leptons are diagonal, $U_{lL} = U_{lR} = 1$. The charged leptons $l_{1,2,3}$ are therefore by themselves the physical mass eigenstates and the lepton mixing matrix depends on only that of the neutrinos, which is studied in the next section.

In similarity to the charged lepton sector, to generate the quark masses, we additionally introduce the three scalar Higgs triplets χ, η, and η' respectively lying in $\underline{1}$, $\underline{1}$, and $\underline{1}'$ under S_3. Quark masses can be derived from the invariant Yukawa interactions for quarks, assuming that the VEVs of η, η', and χ are $u = \langle\eta^0_1\rangle$, $u' = \langle\eta'^0_1\rangle$, and $w = \langle\chi^0_3\rangle$ and the other VEVs $\langle\eta^0_3\rangle$, $\langle\eta'^0_3\rangle$, and $\langle\chi^0_1\rangle$ vanish due to the lepton parity conservation. The exotic quarks therefore acquire the masses $m_U = f_1 w$ and $m_{D_{1,2}} = fw$. The masses of ordinary up-quarks and down-quarks are

$$m_u = h^u_1 u, \quad m_c = h^u v + h'^u v', \quad m_t = h^u v - h'^u v',$$

$$m_d = h^d_1 v, \quad m_s = h^d u + h'^d u', \quad m_b = h^d u - h'^d u'.$$

The unitary matrices that couple the left-handed quarks u_L and d_L to those in the mass bases are unit ones. The CKM quark mixing matrix at the tree level is then $U_{CKM} = U^\dagger_{dL} U_{uL} = 1$. The lepton parity breaking due to the odd VEVs $\langle\eta^0_3\rangle$, $\langle\eta'^0_3\rangle$, $\langle\chi^0_1\rangle$, or a violation of $\mathcal{L}$ and/or S_3 symmetry in terms of Yukawa interactions would disturb the tree-level matrix, resulting in a mixing between the SM and exotic quarks and/or possibly providing the desirable quark mixing pattern $\bar{Q}_{1L}\chi u_{1R}$, $\bar{Q}_L\chi^* d_R$, $\bar{Q}_{1L}\chi u_R$, with a mixing between SM and exotic quarks. To obtain a realistic pattern of the SM quark mixing, we should add radiative corrections or use the effective six-dimensional operators (see Ref. [18] for the details). However, we leave this problem for the future work. A detailed study of charged lepton and quark masses can be found in Ref. [14]. In this paper, we consider a new representation for the anti-sextet responsible for neutrino masses and mixings that are different from those in Ref. [14].

3. NEUTRINO MASSES AND MIXING

The neutrino masses arise from the couplings of $\bar{\psi}^c_{\alpha L}\psi_{\alpha L}$, $\bar{\psi}^c_{1L}\psi_{1L}$ and $\bar{\psi}^c_{1L}\psi_{\alpha L}$ to scalars, where $\bar{\psi}^c_{\alpha L}\psi_{\alpha L}$ transforms as $3^*\oplus 6$ under $\mathrm{SU}(3)_L$ and as $\underline{1}\oplus\underline{1}'\oplus\underline{2}$ under S_3; $\bar{\psi}^c_{1L}\psi_{1L}$ transforms as $3^*\oplus 6$ under $\mathrm{SU}(3)_L$ and as $\underline{1}$ under S_3, and $\bar{\psi}^c_{1L}\psi_{\alpha L}$ transforms as $3^*\oplus 6$ under $\mathrm{SU}(3)_L$ and as $\underline{2}$ under S_3. For the known scalar triplets $(\phi,\phi',\chi,\eta,\eta')$, the available interactions are only $(\bar{\psi}^c_{\alpha L}\psi_{\alpha L})\phi$ and $(\bar{\psi}^c_{\alpha L}\psi_{\alpha L})\phi'$, but are explicitly suppressed because of the $\mathcal{L}$-symmetry. We therefore propose a new $\mathrm{SU}(3)_L$ anti-sextet coupling to $\bar{\psi}^c_L\psi_L$ responsible for the neutrino masses lying in either $\underline{1}$, $\underline{1}'$, or $\underline{2}$ under S_3. To obtain a realistic neutrino spectrum with the minimal Higgs content, we introduce the Higgs anti-sextet

$$s_i = \begin{pmatrix} s^0_{11} & s^+_{12} & s^0_{13} \\ s^+_{12} & s^{++}_{22} & s^+_{23} \\ s^0_{13} & s^+_{23} & s^0_{33} \end{pmatrix}_i \sim [6^*, 2/3, -4/3, \underline{2}],$$
$$i = 1, 2,$$

where numerical subscripts on the component scalars are the $\mathrm{SU}(3)_L$ indices, whereas $i = 1, 2$ is that of S_3. The VEV of s is set as $(\langle s_1\rangle, \langle s_2\rangle)$ under S_3, with

$$\langle s_i\rangle = \begin{pmatrix} \lambda_i & 0 & v_i \\ 0 & 0 & 0 \\ v_i & 0 & \Lambda_i \end{pmatrix}, \quad i = 1, 2. \tag{3}$$

Following the potential minimization conditions, we have several VEV alignments. The first one is that $\langle s_1\rangle = \langle s_2\rangle$; then S_3 is broken into Z_2 consisting of the identity element and one transposition (out of the three) of S_3. The second one is that $\langle s_1\rangle \neq 0 = \langle s_2\rangle$ or $\langle s_1\rangle = 0 \neq \langle s_2\rangle$; then S_3 is broken to Z_3 as in the case of the charged lepton sector. The third one is that $\langle s_1\rangle \neq \langle s_2\rangle$; then S_3 is broken to the identity. In our previous work [14], we have argued that both breakings $S_3 \to Z_2$ and $S_3 \to Z_3$ must take place, and hence, to obtain a realistic neutrino spectrum, we additionally introduced a triplet (ρ) and an anti-sextet (s) that lie in $\underline{1}'$ and $\underline{2}$ under S_3. With these alignments, the number of Higgs multiplets required is eight. In this work, we propose that both the first and the third direction take place. The Yukawa interactions are

$$\begin{aligned} -\mathcal{L}_\nu &= \frac{x}{2}(\bar{\psi}^c_{1L}s)_{\underline{2}}\psi_{\alpha L} + \frac{y}{2}(\bar{\psi}^c_{\alpha L}s)_{\underline{2}}\psi_{\alpha L} + \mathrm{H.c.} = \\ &= \frac{x}{2}\bar{\psi}^c_{1L}(\psi_{2L}s_2 + \psi_{3L}s_1) + \\ &+ \frac{y}{2}(\bar{\psi}^c_{2L}\psi_{2L}s_1 + \bar{\psi}^c_{3L}\psi_{3L}s_2) + \mathrm{H.c.}, \end{aligned} \tag{4}$$

where the Yukawa coupling x is that of lepton-flavor-changing interactions. The mass Lagrangian for the neutrinos is given by

$$\begin{aligned} -\mathcal{L}^{mass}_\nu &= \frac{1}{2}x(\lambda_2\bar{\nu}^c_{1L}\nu_{2L} + v_2\bar{\nu}^c_{1L}N^c_{2R} + \\ &+ v_2\bar{N}_{1R}\nu_{2L} + \Lambda_2\bar{N}_{1R}N^c_{2R}) + \\ &+\frac{1}{2}x(\lambda_1\bar{\nu}^c_{1L}\nu_{3L} + v_1\bar{\nu}^c_{1L}N^c_{3R} + v_1\bar{N}_{1R}\nu_{3L} + \Lambda_1\bar{N}_{1R}N^c_{3R}) + \\ &+\frac{1}{2}y(\lambda_1\bar{\nu}^c_{2L}\nu_{2L} + v_1\bar{\nu}^c_{2L}N^c_{2R} + v_1\bar{N}_{2R}\nu_{2L} + \Lambda_1\bar{N}_{2R}N^c_{2R}) + \\ &+\frac{1}{2}y(\lambda_2\bar{\nu}^c_{3L}\nu_{3L} + v_2\bar{\nu}^c_{3L}N^c_{3R} + v_2\bar{N}_{3R}\nu_{3L} + \Lambda_2\bar{N}_{3R}N^c_{3R}) + \\ &+ \mathrm{H.c.}, \end{aligned} \tag{5}$$

and also by

$$-\mathcal{L}^{mass}_\nu = \frac{1}{2}\bar{\chi}^c_L M_\nu \chi_L + \mathrm{H.c.}, \quad \chi_L \equiv \begin{pmatrix} \nu_L \\ N^c_R \end{pmatrix}, \quad M_\nu \equiv \begin{pmatrix} M_L & M^T_D \\ M_D & M_R \end{pmatrix}, \tag{6}$$

where $\nu = (\nu_1, \nu_2, \nu_3)^T$ and $N = (N_1, N_2, N_3)^T$. The mass matrices are then obtained by

$$M_{L,R,D} = \begin{pmatrix} 0 & a_{L,R,D} & b_{L,R,D} \\ a_{L,R,D} & c_{L,R,D} & 0 \\ b_{L,R,D} & 0 & d_{L,R,D} \end{pmatrix}, \tag{7}$$

with

$$\begin{aligned} & a_L = \frac{x}{2}\lambda_s \equiv \frac{x}{2}\lambda_2, \quad a_D = \frac{x}{2}v_s \equiv \frac{x}{2}v_2, \\ & a_R = \frac{x}{2}\Lambda_s \equiv \frac{x}{2}\Lambda_2, \\ & b_L = \frac{x}{2}\lambda_1, \quad b_D = \frac{x}{2}v_1, \quad b_R = \frac{x}{2}\Lambda_1, \\ & c_L = y\lambda_1, \quad c_D = yv_1, \quad c_R = y\Lambda_1, \\ & d_L = y\lambda_s \equiv y\lambda_2, \quad d_D = yv_s \equiv yv_2, \\ & d_R = y\Lambda_s \equiv y\Lambda_2. \end{aligned} \tag{8}$$

In general, three active neutrinos therefore gain masses via a combination of type-I and type-II seesaw mechanisms, derived from (6) and (7) as

$$M_{eff} = M_L - M^T_D M^{-1}_R M_D = \begin{pmatrix} A & B_1 & B_2 \\ B_1 & C_1 & D \\ B_2 & D & C_2 \end{pmatrix}, \tag{9}$$

where

$$\begin{aligned}
A &= -\frac{(a_R b_D - a_D b_R)^2}{b_R^2 c_R + a_R^2 d_R},\\
B_1 &= \frac{b_R\left[a_R b_D c_D + a_L b_R c_R - a_D(b_R c_D + b_D c_R)\right] + a_R(a_L a_R - a_D^2) d_R}{b_R^2 c_R + a_R^2 d_R},\\
B_2 &= \frac{-b_D^2 b_R c_R + b_L b_R^2 c_R + a_D a_R b_R d_D + a_R^2 b_L d_R - a_R b_D(a_R d_D + a_D d_R)}{b_R^2 c_R + a_R^2 d_R},\\
C_1 &= \frac{b_R^2(c_L c_R - c_D^2) + (a_R^2 c_L + a_D^2 c_R - 2a_D a_R c_D) d_R}{b_R^2 c_R + a_R^2 d_R},\\
C_2 &= \frac{-2b_D b_R c_R d_D + b_R^2 c_R d_L + b_D^2 c_R d_R + a_R^2(d_L d_R - d_D^2)}{b_R^2 c_R + a_R^2 d_R},\\
D &= \frac{(a_R c_D - a_D c_R)(b_R d_D - b_D d_R)}{b_R^2 c_R + a_R^2 d_R}.
\end{aligned} \tag{10}$$

The neutrino mass matrix in (9) is similar to the one in Ref. [14] but the broken symmetry directions are different. Indeed, in this model, there are two broken symmetry directions as follows.

1. If S_3 is broken to Z_2 (the subgroup Z_2 is unbroken), then we have $A = D = 0$, $B_1 = B_2$ and $C_1 = C_2$.

2. If $S_3 \to \{\text{identity}\}$ (or, equivalently, $Z_2 \to \{\text{identity}\}$), then we have $A \neq 0$, $D \neq 0$, $B_1 \neq B_2$, and $C_1 \neq C_2$, but A and D are close to zero, and B_1, B_2, C_1, C_2 are kept close to each other in pairs. In this case, the disparity between $\langle s_1\rangle$ and $\langle s_2\rangle$ is very small and can be regarded as a small perturbation.

We next divide our considerations into two cases to fit the data: the first case is where only S_3 is broken to Z_2, and the second case is a combination of both $S_3 \to Z_2$ and $Z_2 \to \{\text{identity}\}$.

3.1. Experimental constraints under $S_3 \to Z_2$

In the case $S_3 \to Z_2$, $\lambda_1 = \lambda_2 \equiv \lambda_s$, $v_1 = v_2 \equiv v_s$, $\Lambda_1 = \Lambda_2 \equiv \Lambda_s$, we have $A = D = 0$, $B_1 = B_2 \equiv B$, $C_1 = C_2 \equiv C$, and M_{eff} in (9) reduces to

$$M_{eff} = \begin{pmatrix} 0 & B & B \\ B & C & 0 \\ B & 0 & C \end{pmatrix}, \tag{11}$$

with

$$B = \left(\lambda_s - \frac{v_s^2}{\Lambda_s}\right)\frac{x}{2}, \quad C = \left(\lambda_s - \frac{v_s^2}{\Lambda_s}\right) y. \tag{12}$$

We can diagonalize the matrix M_{eff} in (11) as

$$U^T M_{eff} U = \mathrm{diag}(m_1, m_2, m_3),$$

where

$$\begin{aligned}
m_1 &= \frac{1}{2}\left(C - \sqrt{C^2 + 8B^2}\right) = \\
&= \left(\lambda_s - \frac{v_s^2}{\Lambda_s}\right)\frac{y + \sqrt{y^2 + 2x^2}}{2},\\
m_2 &= \frac{1}{2}\left(C + \sqrt{C^2 + 8B^2}\right) = \\
&= \left(\lambda_s - \frac{v_s^2}{\Lambda_s}\right)\frac{y - \sqrt{y^2 + 2x^2}}{2},\\
m_3 &= C = \left(\lambda_s - \frac{v_s^2}{\Lambda_s}\right) y,
\end{aligned} \tag{13}$$

and the neutrino mixing matrix takes the form

$$U_0 = \\ = \begin{pmatrix} \frac{|K|}{\sqrt{|K|^2+2}} & -\frac{\sqrt{2}}{\sqrt{|K|^2+2}} & 0 \\ \frac{1}{\sqrt{|K|^2+2}} & \frac{1}{\sqrt{2}}\frac{|K|}{\sqrt{|K|^2+2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{|K|^2+2}} & \frac{1}{\sqrt{2}}\frac{|K|}{\sqrt{|K|^2+2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \tag{14}$$

$$K = -\frac{C + \sqrt{C^2 + 8B^2}}{2B}.$$

We note that $m_1 m_2 = -2B^2$. This matrix can be parameterized by three Euler's angles, which implies

$$\theta_{13} = 0, \quad \theta_{23} = \pi/4, \quad \mathrm{tg}\,\theta_{12} = \frac{\sqrt{2}}{|K|}. \tag{15}$$

This case coincides with the data because $\sin^2(2\theta_{13}) < < 0.15$ and $\sin^2(2\theta_{23}) > 0.92$ [3]. For the remaining constraints, taking the central values from the data [3],

$$\sin^2(2\theta_{12}) \approx 0.87, \quad s_{12}^2 = 0.32,$$

$$\Delta m_{21}^2 = 7.59\cdot 10^{-5}\ \text{eV}^2, \quad \Delta m_{32}^2 = 2.43\cdot 10^{-3}\ \text{eV}^2,$$

we have a solution

$$m_1 = 0.0280284\ \text{eV}, \quad m_2 = 0.0293347\ \text{eV}, \quad m_3 = 0.0573631\ \text{eV}, \tag{16}$$

and $B = -0.0202757i$ eV, $C = 0.0573631$ eV, $K = 1.44667$, and $|x/y| = 0.707087$. It follows that $\operatorname{tg}\theta_{12} = 0.977565$ ($\theta_{12} \approx 44.35^\circ$), and the neutrino mixing matrix form is very close to that of the bi-maximal mixing pattern mentioned in Ref. [19]:

$$U = \begin{pmatrix} 0.715083 & -0.69904 & 0 \\ 0.494296 & 0.50564 & -\frac{1}{\sqrt{2}} \\ 0.494296 & 0.50564 & \frac{1}{\sqrt{2}} \end{pmatrix} \approx \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{17}$$

Now, it is natural to choose λ_s and v_s^2/Λ_s in eV, and suppose that $\lambda_s > v_s^2/\Lambda_s$. We assume that $\lambda_s - v_s^2/\Lambda_s = 0.1$, we have $x = -0.573631$ and $y = 0.399403i$.

It was assumed in recent analyses that $\theta_{13} \neq 0$, but is small, as in Ref. [4]. If this is correct, then that case will fail. But the direction of the breakings $S_3 \to \{\text{identity}\}$ can improve this.

3.2. Experimental constraints under $S_3 \to \{\text{identity}\}$

If both $S_3 \to Z_2$ and $Z_2 \to \{\text{identity}\}$ directions are realized, then $\lambda_1 \neq \lambda_2 \equiv \lambda_s$, $v_1 \neq v_2 \equiv v_s$, and $\Lambda_1 \neq \Lambda_2 \equiv \Lambda_s$ and, consequently, $A \approx 0$, $D_1 \approx 0$, $B_1 \approx B_2$, and $C_1 \approx C_2$, and the general neutrino mass matrix in (9) can be rewritten in the form

$$M_{eff} = \begin{pmatrix} 0 & B & B \\ B & C & 0 \\ B & 0 & C \end{pmatrix} + \begin{pmatrix} r_1 & p_1 & p_2 \\ p_1 & q_1 & r_2 \\ p_2 & r_2 & q_2 \end{pmatrix}, \tag{18}$$

where B and C are given by (12), to match the case $S_2 \to Z_2$ as in (11). The last matrix in (18) is a deviation from the contribution due to the disparity of $\langle s_1 \rangle$ and $\langle s_2 \rangle$, namely,

$$p_1 = B_1 - B, \quad B_2 - B = p_1, \quad C_1 - C = q_1, \quad C_2 - C = q_2, \quad r_1 = A, \quad r_2 = D. \tag{19}$$

With A, D and $B_{1,2}$, $C_{1,2}$ defined in (10), it corresponds to $S_3 \to \{\text{identity}\}$. Substituting (10) and (12) in (19) with the help of (8), we obtain

$$r_1 = -\frac{(\Lambda_s v_1 - \Lambda_1 v_s)^2}{\Lambda_1^3 + \Lambda_s^3}\frac{x^2}{4y} = -\frac{\Lambda_1^2\Lambda_s^2}{\Lambda_1^3 + \Lambda_s^3}\left(\frac{v_1}{\Lambda_1} - \frac{v_s}{\Lambda_s}\right)^2 \frac{x^2}{4y}, \tag{20}$$

$$r_2 = -\frac{(\Lambda_s v_1 - \Lambda_1 v_s)^2 y}{\Lambda_1^3 + \Lambda_s^3} = -\frac{\Lambda_1^2\Lambda_s^2}{\Lambda_1^3 + \Lambda_s^3}\left(\frac{v_1}{\Lambda_1} - \frac{v_s}{\Lambda_s}\right)^2 y, \tag{21}$$

$$p_1 = \frac{\Lambda_1(\Lambda_s v_1 - \Lambda_1 v_s)^2 x}{2\Lambda_s(\Lambda_1^3 + \Lambda_s^3)} = \frac{\Lambda_1}{\Lambda_s}\frac{\Lambda_1^2\Lambda_s^2}{\Lambda_1^3 + \Lambda_s^3}\left(\frac{v_1}{\Lambda_1} - \frac{v_s}{\Lambda_s}\right)^2 \frac{x}{2}, \tag{22}$$

$$p_2 = \frac{(\lambda_1 - \lambda_s)\Lambda_s(\Lambda_1^3 + \Lambda_s^3) - \Lambda_1^2\Lambda_s v_1^2 - 2\Lambda_s^3 v_1 v_s + (\Lambda_1^3 + \Lambda_1\Lambda_s^2 + \Lambda_s^3)}{\Lambda_s(\Lambda_1^3 + \Lambda_s^3)v_s^2}\frac{x}{2}, \tag{23}$$

$$q_1 = \frac{[(\lambda_1 - \lambda_s)\Lambda_s(\Lambda_1^3 + \Lambda_s^3) - \Lambda_1^2\Lambda_s v_1^2 - 2\Lambda_s^3 v_1 v_s + (\Lambda_1^3 + \Lambda_1\Lambda_s^2 + \Lambda_s^3)v_s^2]y}{\Lambda_s(\Lambda_1^3 + \Lambda_s^3)} = (\lambda_1 - \lambda_s)y - \frac{(\Lambda_1^2\Lambda_s v_1^2 + 2\Lambda_s^3 v_1 v_s)y}{\Lambda_s(\Lambda_1^3 + \Lambda_s^3)} + \frac{(\Lambda_1^3 + \Lambda_1\Lambda_s^2 + \Lambda_s^3)v_s^2 y}{\Lambda_s(\Lambda_1^3 + \Lambda_s^3)} = (\lambda_1 - \lambda_s)y - \frac{\left(v_1^2 + 2\frac{\Lambda_s^2}{\Lambda_1^2}v_1 v_s\right)y}{\Lambda_1 + \frac{\Lambda_s^2}{\Lambda_1^2}\Lambda_s} + \frac{\left(\frac{\Lambda_1}{\Lambda_s} + \frac{\Lambda_s}{\Lambda_1} + \frac{\Lambda_s^2}{\Lambda_1^2}\right)v_s^2 y}{\Lambda_1 + \frac{\Lambda_s^2}{\Lambda_1^2}\Lambda_s}, \tag{24}$$

$$q_2 = \frac{\Lambda_1(\Lambda_s v_1 - \Lambda_1 v_s)^2 y}{\Lambda_s(\Lambda_1^3 + \Lambda_s^3)} = \\ = \frac{\Lambda_1}{\Lambda_s}\frac{\Lambda_1^2\Lambda_s^2}{\Lambda_1^3+\Lambda_s^3}\left(\frac{v_1}{\Lambda_1} - \frac{v_s}{\Lambda_s}\right)^2 y. \quad (25)$$

Indeed, if $S_3 \to Z_2$, then the deviations p_i, q_i, and r_i $(i = 1, 2)$ vanish, and therefore the mass matrix M_{eff} in (9) reduces to its first term coinciding with (11). The first term in (18) provides a bi-maximal mixing pattern with $\theta_{13} = 0$, as shown in Sec. 3.1. The others, proportional to p_i, q_i, r_i due to the contribution of the disparity of $\langle s_1\rangle$ and $\langle s_2\rangle$, take the role of perturbation for such a deviation of θ_{13}. Hence, in this work we consider the disparity of the $\langle s_1\rangle$ and $\langle s_2\rangle$ contributions as a small perturbation and truncate the theory at the first order.

In Ref. [20], we considered the case of $S_4 \to K_4$ breaking corresponding to $S_3 \to$ {identity} in this paper, with $\lambda_1 \neq \lambda_s$, but $v_1 = v_s$ and $\Lambda_1 = \Lambda_s$. Then

$$r_1 = r_2 = p_1 = q_2 = 0, \quad p_2 = \frac{x}{2y}q_1,$$

$$q_1 = (\lambda_1 - \lambda_s)y \equiv \epsilon y$$

with $\epsilon = \lambda_1 - \lambda_s$ being a small parameter that plays the role of a perturbation. In this paper, we consider the more general case, in which all elements of $\langle s_1\rangle$ and $\langle s_2\rangle$ are different from each other.

If $\langle|s_1 - s_2|\rangle \ll \langle s_1\rangle \sim \langle s_2\rangle$ and $\frac{v_1}{\Lambda_1} \sim \frac{v_s}{\Lambda_s} \ll 1$, then we can evaluate r_1, r_2, p_1, $q_2 \ll 1$, which are of the second order in the perturbation and are therefore ignored. The remaining parameters p_2 and q_1 are easily obtained as

$$p_2 = \alpha\frac{x}{2}, \quad q_1 = \alpha y, \quad (26)$$

where

$$\alpha = \lambda_1 - \lambda_s - \frac{v_1^2 + 2\frac{\Lambda_s^2}{\Lambda_1^2}v_1 v_s}{\Lambda_1 + \frac{\Lambda_s^2}{\Lambda_1^2}\Lambda_s} + \\ + \frac{\left(\frac{\Lambda_1}{\Lambda_s} + \frac{\Lambda_s}{\Lambda_1} + \frac{\Lambda_s^2}{\Lambda_1^2}\right)v_s^2}{\Lambda_1 + \frac{\Lambda_s^2}{\Lambda_1^2}\Lambda_s}. \quad (27)$$

The matrix M_{eff} in (18) thus reduces to

$$M_{eff} = \begin{pmatrix} 0 & B & B \\ B & C & 0 \\ B & 0 & C \end{pmatrix} + \alpha\begin{pmatrix} 0 & 0 & x/2 \\ 0 & y & 0 \\ x/2 & 0 & 0 \end{pmatrix} \equiv \\ \equiv M_{eff}^0 + \alpha M^{(1)}. \quad (28)$$

Evaluating α shows that it is a small parameter, which can be regarded as a small perturbation. Within the perturbation theory up to the first order of α, the physical neutrino masses are obtained as

$$\begin{aligned} m_1' &= \lambda_1 = m_1 + \alpha\left(\frac{Kx + y}{K^2 + 2}\right), \\ m_2' &= \lambda_2 = m_2 + \frac{\alpha K(Ky - 2x)}{2(K^2 + 2)}, \\ m_3' &= \lambda_3 = m_3 + \alpha\frac{y}{2}, \end{aligned} \quad (29)$$

where $m_{1,2,3}$ are the mass values as in the case $S_3 \to Z_2$ given by (16). For the corresponding perturbed eigenstates, we set

$$U \to U' = U + \Delta U,$$

where U is defined by (14), and

$$\Delta U = \begin{pmatrix} \Delta U_{11} & \Delta U_{12} & \Delta U_{13} \\ \Delta U_{21} & \Delta U_{22} & \Delta U_{23} \\ \Delta U_{31} & \Delta U_{32} & \Delta U_{33} \end{pmatrix}, \quad (30)$$

where

$$\begin{aligned} \Delta U_{11} &= -\alpha\frac{\left(K^2 - 2\right)x + 2Ky}{2(K^2 + 2)^{3/2}(m_1 - m_2)}, \\ \Delta U_{21} &= -\alpha\frac{Kx - 2y}{4\sqrt{K^2 + 2}(m_1 - m_3)} + \\ &+ \alpha\frac{K[(K^2 - 2)x + 2Ky]}{4(K^2 + 2)^{3/2}(m_1 - m_2)}, \\ \Delta U_{31} &= \alpha\frac{Kx - 2y}{4\sqrt{K^2 + 2}(m_1 - m_3)} + \\ &+ \alpha\frac{K[(K^2 - 2)x + 2Ky]}{4(K^2 + 2)^{3/2}(m_1 - m_2)}, \\ \Delta U_{12} &= -\alpha\frac{K[(K^2 - 2)x + 2Ky]}{2\sqrt{2}(K^2 + 2)^{3/2}(m_1 - m_2)}, \\ \Delta U_{22} &= \frac{\alpha}{2\sqrt{2}}\frac{Ky + x}{\sqrt{K^2 + 2}(m_2 - m_3)} - \\ &- \frac{\alpha}{2\sqrt{2}}\frac{(K^2 - 2)x + 2Ky}{(K^2 + 2)^{3/2}(m_1 - m_2)}, \\ \Delta U_{32} &= -\frac{\alpha}{2\sqrt{2}}\frac{Ky + x}{\sqrt{K^2 + 2}(m_2 - m_3)} - \\ &- \frac{\alpha}{2\sqrt{2}}\frac{(K^2 - 2)x + 2Ky}{(K^2 + 2)^{3/2}(m_1 - m_2)}, \\ \Delta U_{13} &= -\frac{\alpha}{2\sqrt{2}}\frac{K(Kx - 2y)}{(K^2 + 2)(m_1 - m_3)} - \\ &- \frac{\alpha}{\sqrt{2}}\frac{Ky + x}{(K^2 + 2)(m_2 - m_3)}, \\ \Delta U_{23} &= \Delta U_{33} = -\frac{\alpha}{2\sqrt{2}}\frac{Kx - 2y}{(K^2 + 2)(m_1 - m_3)} + \\ &+ \frac{\alpha}{2\sqrt{2}}\frac{K(Ky + x)}{(K^2 + 2)(m_2 - m_3)}. \end{aligned} \quad (31)$$

In this case, the lepton mixing matrix U' can still be parameterized in terms of three new Euler's angles θ'_{ij}, which are also a perturbation from the θ_{ij} in case 1, defined by

$$s'_{13} = -U'_{13} = \Delta U_{13} = -\frac{\alpha y}{2\sqrt{2}B},$$

$$\begin{aligned} t'_{12} = -\frac{U'_{12}}{U'_{11}} = \\ = -\Big[4\alpha B^2 C x + \alpha C^2 \left(C + \sqrt{C^2+8B^2}\right) x + \\ + 2BC\left(C + \sqrt{C^2+8B^2}\right)(2C - \alpha y) + \\ + 8B^3\left(4C + 4\sqrt{C^2+8B^2} - \alpha y\right)\Big] \times \\ \times \Big\{\sqrt{2}\Big[64B^4 + 2C^3\left(C + \sqrt{C^2+8B^2}\right) - \\ - \alpha BC\left(C + \sqrt{C^2+8B^2}\right) x + \\ + 2B^2\left(12C^2 + 8C\sqrt{C^2+8B^2}\right) + \\ + \alpha C y + \alpha y \sqrt{C^2+8B^2}\Big]\Big\}^{-1}, \end{aligned}$$

$$t'_{23} = -\frac{U'_{23}}{U'_{33}} = \frac{4B^2 + \alpha(Bx - Cy)}{4B^2 - \alpha(Bx - Cy)}.$$

It is easily shown that our model is consistent because the five experimental constraints on the mixing angles and squared mass differences of neutrinos can be respectively fitted with two Yukawa coupling parameters x, y of the anti-sextet scalar s, if the VEVs are previously given. Indeed, taking the data in (1), we obtain $\alpha \approx 0.0692$, $x \approx 0.0728$, $y \approx -0.1562$, and $B \approx -0.0241$, $C = 0.022$, $K = 1.943$, and $t'_{23} = 0.9045$ ($\theta'_{23} \approx 42.13°$, $\sin^2(2\theta'_{23}) = 0.98999$ satisfying the condition $\sin^2(2\theta'_{23}) > 0.95$ as in (1)). The neutrino masses are explicitly given as $m'_1 \approx -0.02737$ eV, $m'_2 \approx -0.02870$ eV, and $m'_3 \approx -0.05607$ eV, which are in a normal ordering. The neutrino mixing matrix then takes the form

$$U = \begin{pmatrix} 0.8251 & -0.5657 & -0.1585 \\ 0.3302 & 0.6781 & -0.6716 \\ 0.4697 & 0.4888 & 0.7426 \end{pmatrix}. \qquad (32)$$

4. SCALAR POTENTIAL

To be complete, we write the scalar potentials of both the models mentioned. It is also noted that

$$(\mathrm{Tr}\, A)(\mathrm{Tr}\, B) = \mathrm{Tr}\,(A\, \mathrm{Tr}\, B)$$

and

$$V(X \to X', Y \to Y', \ldots) \equiv V(X, Y, \ldots)|_{X=X', Y=Y', \ldots}.$$

The general potential invariant under any subgroup takes the form

$$V_{total} = V_{tri} + V_{sext} + V_{tri-sext}, \qquad (33)$$

where V_{tri} comes from only contributions of $\mathrm{SU(3)}_L$ triplets given as a sum of the following terms:

$$V(\chi) = \mu_\chi^2 \chi^\dagger \chi + \lambda^\chi (\chi^\dagger \chi)^2, \qquad (34)$$

$$\begin{aligned} V(\phi) = V(\chi \to \phi), \quad V(\phi') = V(\chi \to \phi'), \\ V(\eta) = V(\chi \to \eta), \quad V(\eta') = V(\chi \to \eta'), \end{aligned} \qquad (35)$$

$$\begin{aligned} V(\phi, \chi) = \lambda_1^{\phi\chi}(\phi^\dagger\phi)(\chi^\dagger\chi) + \lambda_2^{\phi\chi}(\phi^\dagger\chi)(\chi^\dagger\phi), \\ V(\phi', \chi) = V(\phi \to \phi', \chi), \quad V(\chi, \eta) = V(\chi, \phi \to \eta), \\ V(\chi, \eta') = V(\chi, \phi \to \eta'), \end{aligned}$$

$$\begin{aligned} V(\phi, \phi') = V(\phi, \chi \to \phi') + \lambda_3^{\phi\phi'}(\phi^\dagger\phi')(\phi^\dagger\phi') + \\ + \lambda_4^{\phi\phi'}(\phi'^\dagger\phi)(\phi'^\dagger\phi), \end{aligned}$$

$$\begin{aligned} V(\phi, \eta) = V(\phi, \chi \to \eta), \quad V(\phi, \eta') = V(\phi, \chi \to \eta'), \\ V(\phi', \eta) = V(\phi \to \phi', \chi \to \eta), \\ V(\phi', \eta') = V(\phi \to \phi', \chi \to \eta'), \end{aligned}$$

$$\begin{aligned} V(\eta, \eta') = V(\phi \to \eta, \chi \to \eta') + \lambda_3^{\eta\eta'}(\eta^\dagger\eta')(\eta^\dagger\eta') + \\ + \lambda_4^{\eta\eta'}(\eta'^\dagger\eta)(\eta'^\dagger\eta), \end{aligned}$$

$$\begin{aligned} V_{\chi\phi\phi'\eta\eta'} = \mu_1 \chi\phi\eta + \mu'_1 \chi\phi'\eta' + \\ + \lambda_1^1(\phi^\dagger\phi')(\eta^\dagger\eta') + \lambda_1^2(\phi^\dagger\phi')(\eta'^\dagger\eta) + \\ + \lambda_1^3(\phi^\dagger\eta)(\eta'^\dagger\phi') + \lambda_1^4(\phi^\dagger\eta')(\eta^\dagger\phi') + \mathrm{H.c.} \end{aligned} \qquad (36)$$

The V_{sext} is given by only $V(s)$:

$$\begin{aligned} V(s) = \mu_s^2\,\mathrm{Tr}(s^\dagger s) + \lambda_1^s\,\mathrm{Tr}[(s^\dagger s)_{\underline{1}}(s^\dagger s)_{\underline{1}}] + \\ + \lambda_2^s\,\mathrm{Tr}[(s^\dagger s)_{\underline{1}'}(s^\dagger s)_{\underline{1}'}] + \lambda_3^s\,\mathrm{Tr}[(s^\dagger s)_{\underline{2}}(s^\dagger s)_{\underline{2}}] + \\ + \lambda_4^s\,\mathrm{Tr}(s^\dagger s)_{\underline{1}}\,\mathrm{Tr}(s^\dagger s)_{\underline{1}} + \lambda_5^s\,\mathrm{Tr}(s^\dagger s)_{\underline{1}'}\,\mathrm{Tr}(s^\dagger s)_{\underline{1}'} + \\ + \lambda_6^s\,\mathrm{Tr}(s^\dagger s)_{\underline{2}}\,\mathrm{Tr}(s^\dagger s)_{\underline{2}}. \end{aligned} \qquad (37)$$

Next, $V_{tri-sext}$ is a sum of all the terms connecting both the sectors:

$$V(\phi, s) = \lambda_1^{\phi s}(\phi^\dagger\phi)\,\mathrm{Tr}(s^\dagger s)_{\underline{1}} + \lambda_2^{\phi s}[(\phi^\dagger s^\dagger)(s\phi)]_{\underline{1}},$$

$$V(\phi', s) = V(\phi \to \phi', s), \quad V(\chi, s) = V(\phi \to \chi, s),$$

$$V(\eta, s) = V(\phi \to \eta, s), \quad V(\eta', s) = V(\phi \to \eta', s),$$

$$V_{s\chi\phi\phi'\eta\eta'} = (\lambda'_1 \phi^\dagger \phi' + \lambda'_2 \eta^\dagger \eta') \operatorname{Tr}(s^\dagger s)_{\underline{1}'} + \\ + \lambda'_3 [(\phi^\dagger s^\dagger)(s\phi')]_{\underline{1}} + \lambda'_4 [(\eta^\dagger s^\dagger)(s\eta')]_{\underline{1}} + \text{H.c.}$$

To provide the Majorana masses for the neutrinos, the lepton number must be broken. This can be achieved via the scalar potential violating $U(1)_{\mathcal{L}}$, but the other symmetries must be conserved. The $\mathcal{L}$ violating potential is given by

$$\bar{V} = [\bar{\lambda}_1 \operatorname{Tr}(s^\dagger s)_{\underline{1}} + \bar{\lambda}_2 \eta^\dagger \chi + \bar{\lambda}_3 \eta^\dagger \eta + \bar{\lambda}_4 \eta'^\dagger \eta' + \\ + \bar{\lambda}_5 \eta^\dagger \eta' + \bar{\lambda}_6 \eta'^\dagger \eta + \bar{\lambda}_7 \phi^\dagger \phi + \\ + \bar{\lambda}_8 \phi'^\dagger \phi' + \bar{\lambda}_9 \phi^\dagger \phi' + \bar{\lambda}_{10} \phi'^\dagger \phi] \eta^\dagger \chi + \\ + [\bar{\lambda}_{11} \operatorname{Tr}(s^\dagger s)_{\underline{1}'} + \bar{\lambda}_{12} \eta'^\dagger \chi + \bar{\lambda}_{13} \eta^\dagger \eta + \\ + \bar{\lambda}_{14} \eta'^\dagger \eta' + \bar{\lambda}_{15} \eta^\dagger \eta' + \bar{\lambda}_{16} \eta'^\dagger \eta + \bar{\lambda}_{17} \phi^\dagger \phi + \bar{\lambda}_{18} \phi'^\dagger \phi' + \\ + \bar{\lambda}_{19} \phi^\dagger \phi' + \bar{\lambda}_{20} \phi'^\dagger \phi]_{\underline{1}'} \eta'^\dagger \chi + \bar{\lambda}_{21} (\eta^\dagger \phi)(\phi^\dagger \chi) + \\ + \bar{\lambda}_{22} (\eta^\dagger \phi')_{\underline{1}'} (\phi'^\dagger \chi)_{\underline{1}'} + \bar{\lambda}_{23} (\eta'^\dagger \phi)_{\underline{1}'} (\phi'^\dagger \chi)_{\underline{1}'} + \\ + \bar{\lambda}_{24} (\eta'^\dagger \phi')_{\underline{1}} (\phi^\dagger \chi)_{\underline{1}} + \bar{\lambda}_{25} (\eta^\dagger s^\dagger)_{\underline{2}} (s\chi)_{\underline{2}} + \\ + \bar{\lambda}_{26} (\eta'^\dagger s^\dagger)_{\underline{2}} (s\chi)_{\underline{2}} + \text{H.c.} \quad (38)$$

We have not pointed it out, but there must additionally exist the terms in $\bar{V}$ explicitly violating only the S_3 symmetry or both the S_3 and $\mathcal{L}$-charge. In what follows, most of them are omitted, and only the terms of interest to us are provided.

We now consider the potential V_{tri}. The flavons χ, ϕ, ϕ', η, η' with their VEVs aligned in the same direction (all of them being singlets) are an automatic solution of the minimization conditions for V_{tri}. To explicitly see this, in the system of equations for minimization, we set $v^* = v$, $v'^* = v'$, $u^* = u$, $u'^* = u'$, and $v^*_\chi = v_\chi$. Then the potential minimization conditions for triplets reduces to

$$\frac{\partial V_{tri}}{\partial \omega} = 4\lambda^\chi \omega^3 + 2\left(\mu_\chi^2 + \lambda_1^{\chi\eta} u^2 + \lambda_1^{\chi\eta'} u'^2 + \\ + \lambda_1^{\chi\phi} v^2 + \lambda_1^{\chi\phi'} v'^2\right)\omega - \mu_1 uv - \mu'_1 u'v' = 0, \quad (39)$$

$$\frac{\partial V_{tri}}{\partial v} = 4\lambda^\phi v^3 + 2\left[\mu_\phi^2 + \lambda_1^{\phi\eta} u^2 + \lambda_1^{\phi\eta'} u'^2 + \\ + (\lambda_1^{\phi\phi'} + \lambda_2^{\phi\phi'} + \lambda_3^{\phi\phi'} + \lambda_4^{\phi\phi'}) v'^2 + \omega^2 \lambda_1^{\phi\chi}\right] v + \\ + (\lambda_1^1 + \lambda_1^2) uu'v' - \mu_1 \omega u = 0, \quad (40)$$

$$\frac{\partial V_{tri}}{\partial v'} = 4\lambda^{\phi'} v'^3 + 2\left[\mu_{\phi'}^2 + \lambda_1^{\phi'\eta} u^2 + \lambda_1^{\phi'\eta'} u'^2 + \\ + (\lambda_1^{\phi\phi'} + \lambda_2^{\phi\phi'} + \lambda_3^{\phi\phi'} + \lambda_4^{\phi\phi'}) v^2 + \omega^2 \lambda_1^{\phi'\chi}\right] v' + \\ + (\lambda_1^1 + \lambda_1^2) uu'v - \mu'_1 \omega u' = 0, \quad (41)$$

$$\frac{\partial V_{tri}}{\partial u} = 4\lambda^\eta u^3 + \\ + 2\left[\mu_\eta^2 + (\lambda_1^{\eta\eta'} + \lambda_2^{\eta\eta'} + \lambda_3^{\eta\eta'} + \lambda_4^{\eta\eta'}) u'^2 + \lambda_1^{\phi\eta} v^2 + \\ + \lambda_1^{\phi'\eta} v'^2 + \omega^2 \lambda_1^{\eta\chi}\right] u + (\lambda_1^1 + \lambda_1^2) u'vv' - \mu_1 \omega v = 0, \quad (42)$$

$$\frac{\partial V_{tri}}{\partial u'} = 4\lambda^{\eta'} u'^3 + 2\left[\mu_{\eta'}^2 + (\lambda_1^{\eta\eta'} + \lambda_2^{\eta\eta'} + \lambda_3^{\eta\eta'} + \\ + \lambda_4^{\eta\eta'}) u^2 + \lambda_1^{\phi\eta'} v^2 \lambda_1^{\phi'\eta'} v'^2 + \omega^2 \lambda_1^{\eta'\chi}\right] u' + \\ + (\lambda_1^1 + \lambda_1^2) uvv' - \mu'_1 \omega v' = 0. \quad (43)$$

It is easy to see that the derivatives of V_{tri} with respect to the variables u, u', v, v' shown in (40), (41), (42), and (43) are symmetric with respect to one another. System of equations (39)–(43) always has the solution (u, v, u', v') as expected, even though it is complicated. We also note that the above alignment is only one of the conditions to be imposed to have the desirable results. We have evaluated that Eqs. (40)–(43) have the same structure of solutions. Consequently, to have a simple solution, we can assume that $u = u' = v = v'$. In this case, Eqs. (40)–(43) reduce to a single equation, and system of equations (39)–(43) becomes

$$\frac{\partial V_{tri}}{\partial \omega} = 4\lambda^\chi \omega^3 + \\ + 2\omega[\mu_\chi^2 + (2\lambda_1^{\chi\eta} + 2\lambda_1^{\chi\phi}) v^2] - 2\mu_1 v^2 = 0, \quad (44)$$

$$\frac{\partial V_{tri}}{\partial v} = 2v\left[2\omega^2 (\lambda_1^{\chi\eta} + \lambda_1^{\chi\phi}) + 2(\mu_\eta^2 + \mu_\phi^2) + \\ + 2\left(\lambda_1^1 + \lambda_1^2 + 4\lambda_1^{\phi\eta} + \lambda_1^{\eta\eta'} + \lambda_2^{\eta\eta'} + \lambda_3^{\eta\eta'} + \lambda_4^{\eta\eta'} + \\ + \lambda_1^{\phi\phi'} + \lambda_2^{\phi\phi'} + \lambda_3^{\phi\phi'} + \lambda_4^{\phi\phi'} + 2\lambda^\phi + 2\lambda^\eta\right) v^2 - \\ - 2\mu_1 \omega\right] = 0. \quad (45)$$

This system has the solution

$$u = u' = v' = v = \pm \sqrt{\frac{\omega(\mu_\chi^2 + \lambda^\chi \omega^2)}{\mu_1 - 2\omega(\lambda_1^{\chi\eta} + \lambda_1^{\chi\phi})}},$$

$$\omega = \frac{\alpha\mu_1}{2(\alpha^2 - \beta\lambda^\chi)} - \\ - \frac{\Omega}{3 \cdot 2^{2/3} (\alpha^2 - \beta\lambda^\chi) \left(\Gamma + \sqrt{\Gamma^2 + 4\Omega^3}\right)^{1/3}} + \\ + \frac{\left(\Gamma + \sqrt{\Gamma^2 + 4\Omega^3}\right)^{1/3}}{6 \cdot 2^{1/3} (\alpha^2 - \beta\lambda^\chi)},$$

where

$$\Gamma = 54\alpha\beta\mu_1(\lambda^\chi\mu_1^2 + \alpha^2\mu_\chi^2 - \beta\lambda^\chi\mu_\chi^2) - - 108\lambda^\chi\mu_1\beta\gamma(\alpha^2 - \lambda^\chi\beta), \quad (46)$$

$$\Omega = 6(\alpha^2 - \beta\lambda^\chi)(2\alpha\gamma + \mu_1^2 - \beta\mu_\chi^2) - 9\alpha^2\mu_1^2, \quad (47)$$

$$\alpha = \lambda_1^{\chi\eta} + \lambda_1^{\chi\phi}, \quad (48)$$

$$\beta = \lambda_1^1 + \lambda_1^2 + 4\lambda_1^{\phi\eta} + \lambda^{\phi\phi'} + \lambda^{\eta\eta'} + 2(\lambda^\eta + \lambda^\phi), \quad (49)$$

$$\lambda^{\phi\phi'} = \lambda_1^{\phi\phi'} + \lambda_2^{\phi\phi'} + \lambda_3^{\phi\phi'} + \lambda_4^{\phi\phi'},$$

$$\lambda^{\eta\eta'} = \lambda_1^{\eta\eta'} + \lambda_2^{\eta\eta'} + \lambda_3^{\eta\eta'} + \lambda_4^{\eta\eta'}.$$

We next consider the potentials V_{sext} and $V_{tri-sext}$. By imposing the conditions

$$\begin{aligned} &\lambda_1^* = \lambda_1, \quad \lambda_2^* = \lambda_2, \quad v_1^* = v_1, \quad v_2^* = v_2, \\ &\Lambda_1^* = \Lambda_1, \quad \Lambda_2^* = \Lambda_2, \\ &v^* = v, \quad v'^* = v', \quad u^* = u, u'^* = u', \\ &v_\chi^* = v_\chi, \quad v_\rho^* = v_\rho, \end{aligned} \quad (50)$$

we obtain a system of equations of the potential minimization for anti-sextets:

$$\begin{aligned} \frac{\partial V_1}{\partial \lambda_1} &= 2\left\{\lambda_2\left[\lambda_1^{\chi s}\omega^2 + \mu_s^2 + (\lambda_1^{\eta s} + \lambda_2^{\eta s} + \lambda_3^{\eta s})u^2 + \right.\right. \\ &+ (\lambda_2' + \lambda_4')uu' + (\lambda_1^{\eta' s} + \lambda_2^{\eta' s} + \lambda_3^{\eta' s})u'^2 + \lambda_1' vv' + \\ &+ \lambda_1^{\phi s}v^2 + \lambda_1^{\phi' s}v'^2 + 4\lambda_4^s\Lambda_1\Lambda_2 + 2(3\lambda_1^s + \lambda_2^s + \\ &+ \lambda_3^s + 4\lambda_4^s)v_1v_2] + 2\Lambda_2(\lambda_1^s - \lambda_2^s + \lambda_3^s)v_1v_2 + \\ &+ 2\Lambda_1(\lambda_1^s + \lambda_2^s)v_2^2 + 2\lambda_1\left[\lambda_6^s\Lambda_2^2 + \lambda_2^2(2\lambda_1^s + \lambda_3^s + \right. \\ &\left.\left. + 2\lambda_4^s + \lambda_6^s) + (\lambda_1^s - \lambda_2^s + \lambda_3^s + 2\lambda_6^s)v_2^2\right]\right\}, \quad (51) \end{aligned}$$

$$\begin{aligned} \frac{\partial V_1}{\partial \lambda_2} &= 2\left\{\lambda_1\left[\lambda_1^{\chi s}\omega^2 + \mu_s^2 + (\lambda_1^{\eta s} + \lambda_2^{\eta s} + \lambda_3^{\eta s})u^2 + \right.\right. \\ &+ (\lambda_2' + \lambda_4')uu' + (\lambda_1^{\eta' s} + \lambda_2^{\eta' s} + \lambda_3^{\eta' s})u'^2 + \lambda_1' vv' + \\ &+ \lambda_1^{\phi s}v^2 + \lambda_1^{\phi' s}v'^2 + 4\lambda_4^s\Lambda_1\Lambda_2 + \\ &+ 2(3\lambda_1^s + \lambda_2^s + \lambda_3^s + 4\lambda_4^s)v_1v_2] + \\ &+ 2\Lambda_1(\lambda_1^s - \lambda_2^s + \lambda_3^s)v_1v_2 + 2\Lambda_2(\lambda_1^s + \lambda_2^s)v_1^2 + \\ &+ 2\lambda_2\left[\lambda_6^s\Lambda_1^2 + \lambda_1^2(2\lambda_1^s + \lambda_3^s + 2\lambda_4^s + \lambda_6^s) + \right. \\ &\left.\left. + (\lambda_1^s - \lambda_2^s + \lambda_3^s + 2\lambda_6^s)v_1^2\right]\right\}, \quad (52) \end{aligned}$$

$$\begin{aligned} \frac{\partial V_1}{\partial v_1} &= 2\left\{v_2\left[(2\lambda_1^{\chi s} + \lambda_2^{\chi s} + \lambda_3^{\chi s})\omega^2 + 2\mu_s^2 + \right.\right. \\ &+ (2\lambda_1^{\eta s} + \lambda_2^{\eta s} + \lambda_3^{\eta s})u^2 + (2\lambda_2' + \lambda_4')uu' + \\ &+ (2\lambda_1^{\eta' s} + \lambda_2^{\eta' s} + \lambda_3^{\eta' s})u'^2 + 2\lambda_1^{\phi s}v^2 + 2\lambda_1' vv' + \\ &+ 2\lambda_1^{\phi' s}v'^2 + 2(\lambda_1\Lambda_2 + \lambda_2\Lambda_1)(\lambda_1^s - \lambda_2^s + \lambda_3^s) + \\ &+ 2(\lambda_1\lambda_2 + \Lambda_1\Lambda_2)(3\lambda_1^s + \lambda_2^s + \lambda_3^s + 4\lambda_4^s)] + \\ &+ 2\left[2\lambda_2\Lambda_2(\lambda_1^s + \lambda_2^s) + (\lambda_2^2 + \Lambda_2^2)(\lambda_1^s - \lambda_2^s + \right. \\ &\left.\left. + \lambda_3^s + 2\lambda_6^s)\right]v_1 + 4(2\lambda_1^s + \lambda_3^s + 4\lambda_4^s + 2\lambda_6^s)v_1v_2^2\right\}, \quad (53) \end{aligned}$$

$$\begin{aligned} \frac{\partial V_1}{\partial v_2} &= 2\left\{v_1\left[(2\lambda_1^{\chi s} + \lambda_2^{\chi s} + \lambda_3^{\chi s})\omega^2 + 2\mu_s^2 + \right.\right. \\ &+ (2\lambda_1^{\eta s} + \lambda_2^{\eta s} + \lambda_3^{\eta s})u^2 + (2\lambda_2' + \lambda_4')uu' + \\ &+ (2\lambda_1^{\eta' s} + \lambda_2^{\eta' s} + \lambda_3^{\eta' s})u'^2 + 2\lambda_1^{\phi s}v^2 + 2\lambda_1' vv' + \\ &+ 2\lambda_1^{\phi' s}v'^2 + 2(\lambda_1\Lambda_2 + \lambda_2\Lambda_1)(\lambda_1^s - \lambda_2^s + \lambda_3^s) + \\ &+ 2(\lambda_1\lambda_2 + \Lambda_1\Lambda_2)(3\lambda_1^s + \lambda_2^s + \lambda_3^s + 4\lambda_4^s)] + \\ &+ 2\left[2\lambda_1\Lambda_1(\lambda_1^s + \lambda_2^s) + (\lambda_1^2 + \Lambda_1^2)(\lambda_1^s - \lambda_2^s + \right. \\ &\left.\left. + \lambda_3^s + 2\lambda_6^s)\right]v_2 + 4(2\lambda_1^s + \lambda_3^s + 4\lambda_4^s + 2\lambda_6^s)v_2v_1^2\right\}, \quad (54) \end{aligned}$$

$$\begin{aligned} \frac{\partial V_1}{\partial \Lambda_1} &= 2\left\{\Lambda_2\left[(\lambda_1^{\chi s} + \lambda_2^{\chi s} + \lambda_3^{\chi s})\omega^2 + \mu_s^2 + \right.\right. \\ &+ \lambda_1^{\eta s}u^2 + \lambda_2' uu' + \lambda_1^{\eta' s}u'^2 + \lambda_1' vv' + \lambda_1^{\phi s}v^2 + \lambda_1^{\phi' s}v'^2 + \\ &+ 4\lambda_4^s\lambda_1\lambda_2 + 2(3\lambda_1^s + \lambda_2^s + \lambda_3^s + 4\lambda_4^s)v_1v_2] + \\ &+ 2\lambda_2(\lambda_1^s - \lambda_2^s + \lambda_3^s)v_1v_2 + 2\lambda_1(\lambda_1^s + \lambda_2^s)v_2^2 + \\ &+ 2\Lambda_1\left[\lambda_6^s\lambda_2^2 + \Lambda_2^2(2\lambda_1^s + \lambda_3^s + 2\lambda_4^s + \lambda_6^s) + \right. \\ &\left.\left. + (\lambda_1^s - \lambda_2^s + \lambda_3^s + 2\lambda_6^s)v_2^2\right]\right\}, \quad (55) \end{aligned}$$

$$\begin{aligned} \frac{\partial V_1}{\partial \Lambda_2} &= 2\left\{\Lambda_1\left[(\lambda_1^{\chi s} + \lambda_2^{\chi s} + \lambda_3^{\chi s})\omega^2 + \mu_s^2 + \right.\right. \\ &+ \lambda_1^{\eta s}u^2 + \lambda_2' uu' + \lambda_1^{\eta' s}u'^2 + \\ &+ \lambda_1' vv' + \lambda_1^{\phi s}v^2 + \lambda_1^{\phi' s}v'^2 + \\ &+ 4\lambda_4^s\lambda_1\lambda_2 + 2(3\lambda_1^s + \lambda_2^s + \lambda_3^s + 4\lambda_4^s)v_1v_2] + \\ &+ 2\lambda_1(\lambda_1^s - \lambda_2^s + \lambda_3^s)v_1v_2 + 2\lambda_2(\lambda_1^s + \lambda_2^s)v_1^2 + \\ &+ 2\Lambda_2\left[\lambda_6^s\lambda_1^2 + \Lambda_1^2(2\lambda_1^s + \lambda_3^s + 2\lambda_4^s + \lambda_6^s) + \right. \\ &\left.\left. + (\lambda_1^s - \lambda_2^s + \lambda_3^s + 2\lambda_6^s)v_1^2\right]\right\}, \quad (56) \end{aligned}$$

where

$$V_1 = V_{sext} + V_{tri-sext}.$$

It is easy to see that Eqs. (51)–(56) take the same form pairwise. This system of equations yields the relations

$$\lambda_1 = \kappa\lambda_2, \quad v_1 = \kappa v_2, \quad \Lambda_1 = \kappa\Lambda_2, \quad (57)$$

with κ is a constant. This means that there are several alignments for VEVs. In this paper, to obtain the desired results, we impose the two directions for breaking $S_3 \to Z_2$ and $Z_2 \to \{\text{identity}\}$ as mentioned, in which $\kappa = 1$ and $\kappa \neq 1$ but approaches to the unit. In the case where $\kappa = 1$ or $\lambda_1 = \lambda_2 = \lambda_s$, $v_1 = v_2 = v_s$, and $\Lambda_1 = \Lambda_2 = \Lambda_s$, system of equations (51)–(56) reduces to a system for the potential minimal consisting of three equations:

$$\lambda_s \left[A_\omega + \mu_s^2 + 2A_s\Lambda_s^2 + 2(A_s + B_s)\lambda_s^2 + A_v + \right. \\ \left. + 4(A_s + B_s)v_s^2\right] + 2B_s\Lambda_s v_s^2 = 0, \quad (58)$$

$$2(A_\omega + B_\omega) + 2\mu_s^2 + A_v + A'_v + \\ + 4B_s\lambda_s\Lambda_s + 4(A_s + B_s)(\lambda_s^2 + v_s^2 + \Lambda_s^2) = 0, \quad (59)$$

$$\Lambda_s \left[A_\omega + B_\omega + \mu_s^2 + 2A_s\lambda_s^2 + 2(A_s + B_s)\Lambda_s^2 + \right. \\ \left. + A'_v + 4(A_s + B_s)v_s^2\right] + 2B_s\lambda_s v_s^2 = 0, \quad (60)$$

where

$$A_\omega = \lambda_1^{\chi s}\omega^2, \quad B_\omega = (\lambda_2^{\chi s} + \lambda_3^{\chi s})\omega^2,$$

$$A_s = 2\lambda_4^s + \lambda_6^s, \quad B_s = 2\lambda_1^s + \lambda_3^s,$$

$$A_v = (\lambda'_1 + \lambda'_2 + \lambda'_4 + \lambda_1^{\phi s} + \lambda_1^{\phi' s} + \lambda_1^{\eta s} + \lambda_2^{\eta s} + \\ + \lambda_3^{\eta s} + \lambda_1^{\eta' s} + \lambda_2^{\eta' s} + \lambda_3^{\eta' s})v^2,$$

$$A'_v = (\lambda'_1 + \lambda'_2 + \lambda_1^{\phi s} + \lambda_1^{\phi' s} + \lambda_1^{\eta s} + \lambda_1^{\eta' s})v^2.$$

System of equations (58)–(60) always has the solution $(\lambda_s, v_s, \Lambda_s)$ as expected, even though it is complicated. We also note that the above alignment is only one of the conditions to be imposed to have the desired results.

5. GAUGE BOSONS

The covariant derivative of the triplet is given by

$$D_\mu = \partial_\mu - ig\frac{\lambda_a}{2}W_{\mu a} - ig_X X\frac{\lambda_9}{2}B_\mu = \partial_\mu - iP_\mu, \quad (61)$$

where $\lambda_9 = \sqrt{2/3}\,\text{diag}(1,1,1)$ and $\lambda_a (a = 1, 2, \ldots, 8)$ are Gell-Mann matrices that satisfy the relations $\mathrm{Tr}\,\lambda_a\lambda_b = 2\delta_{ab}$ and $\mathrm{Tr}\,\lambda_9\lambda_9 = 2$, and X is the $U(1)_X$-charge of Higgs triplets.

We can rewrite P_μ in a convenient form as follows:

$$\frac{g}{2}\begin{pmatrix} W_{\mu 3} + \frac{W_{\mu 8}}{\sqrt{3}} + t\sqrt{\frac{2}{3}}XB_\mu & \sqrt{2}W'^+_\mu & \sqrt{2}X'^0_\mu, \\ \sqrt{2}W'^-_\mu & -W_{\mu 3} + \frac{W_{\mu 8}}{\sqrt{3}} + t\sqrt{\frac{2}{3}}XB_\mu & \sqrt{2}Y'^-_\mu \\ \sqrt{2}X'^{0*}_\mu & \sqrt{2}Y'^+_\mu & -\frac{2}{\sqrt{3}}W_{\mu 8} + t\sqrt{\frac{2}{3}}XB_\mu \end{pmatrix}, \quad (62)$$

where we set

$$W'^+_\mu = \frac{W_{\mu 1} - iW_{\mu 2}}{\sqrt{2}}, \quad X'^0_\mu = \frac{W_{\mu 4} - iW_{\mu 5}}{\sqrt{2}}, \\ Y'^-_\mu = \frac{W_{\mu 6} - iW_{\mu 7}}{\sqrt{2}}, \quad W'^-_\mu = (W'^+_\mu)^*, \\ Y'^+_\mu = (Y'^-_\mu)^*, \quad (63)$$

and $t = g_X/g$. We note that W_4 and W_5 are respectively purely real and imaginary parts of X^0 and X^{0*}. The covariant derivative for the anti-sextet with a VEV part is [21, 22]

$$D_\mu\langle s_i\rangle = \frac{ig}{2}\{W_\mu^a\lambda_a^*\langle s_i\rangle + \langle s_i\rangle W_\mu^a\lambda_a^{*T}\} - \\ - ig_X T_9 X B_\mu\langle s_i\rangle. \quad (64)$$

Covariant derivative (64) acting on the anti-sextet VEVs is given by

$$[D_\mu\langle s_i\rangle]_{11} = ig\left(\lambda_i W_{\mu 3} + \frac{\lambda_i}{\sqrt{3}}W_{\mu 8} + \right. \\ \left. + \sqrt{\frac{2}{3}}\frac{1}{3}t\lambda_i B_\mu + \sqrt{2}v_i X'^{0*}\right),$$

$$[D_\mu\langle s_i\rangle]_{12} = \frac{ig}{\sqrt{2}}\left(\lambda_i W'^+_\mu + v_i Y'^+_\mu\right),$$

$$[D_\mu\langle s_i\rangle]_{13} = \frac{ig}{2}\left(v_i W_{\mu 3} - \frac{v_i}{\sqrt{3}}W_{\mu 8} + \right. \\ \left. + \frac{2}{3}\sqrt{\frac{2}{3}}tv_i B_\mu + \sqrt{2}\lambda_i X'^0_\mu + \sqrt{2}\Lambda_i X'^{0*}_\mu\right),$$

$$[D_\mu\langle s_i\rangle]_{21} = [D_\mu\langle s_i\rangle]_{12},$$

$$[D_\mu\langle s_i\rangle]_{22}=0,$$
$$[D_\mu\langle s_i\rangle]_{23}=\frac{ig}{\sqrt{2}}\left(v_iW'^+_\mu+\Lambda_iY'^+_\mu\right),$$
$$[D_\mu\langle s_i\rangle]_{31}=[D_\mu\langle s_i\rangle]_{13},$$
$$[D_\mu\langle s_i\rangle]_{32}=[D_\mu\langle s_i\rangle]_{23},$$
$$[D_\mu\langle s_i\rangle]_{33}=ig\left(-\frac{2}{\sqrt{3}}\Lambda_iW_{\mu8}+\sqrt{\frac{2}{3}}\frac{1}{3}t\Lambda_iB_\mu+\sqrt{2}v_iX'^0_\mu\right).$$

The masses of gauge bosons in this model are defined as

$$\mathcal{L}^{GB}_{mass}=(D_\mu\langle\phi\rangle)^+(D^\mu\langle\phi\rangle)+(D_\mu\langle\phi'\rangle)^+(D^\mu\langle\phi'\rangle)+(D_\mu\langle\chi\rangle)^+(D^\mu\langle\chi\rangle)+(D_\mu\langle\eta\rangle)^+(D^\mu\langle\eta\rangle)+(D_\mu\langle\eta'\rangle)^+(D^\mu\langle\eta'\rangle)+\mathrm{Tr}[(D_\mu\langle s_1\rangle)^+(D^\mu\langle s_1\rangle)]+\mathrm{Tr}[(D_\mu\langle s_2\rangle)^+(D^\mu\langle s_2\rangle)]. \quad (65)$$

Substituting the Higgs VEVs of the model in (65) yields

$$\begin{aligned}\mathcal{L}^{GB}_{mass}&=\frac{v^2}{324}\Big[81g^2(W^2_{\mu1}+W^2_{\mu2})+81g^2(W^2_{\mu6}+W^2_{\mu7})+(-9gW_{\mu3}+3\sqrt{3}gW_{\mu8}+2\sqrt{6}g_XB_\mu)^2\Big]+\\&+\frac{v'^2}{324}\Big[81g^2(W^2_{\mu1}+W^2_{\mu2})+81g^2(W^2_{\mu6}+W^2_{\mu7})+(-9gW_{\mu3}+3\sqrt{3}gW_{\mu8}+2\sqrt{6}g_XB_\mu)^2\Big]+\\&+\frac{\omega^2}{108}\Big[27g^2(W^2_{\mu4}+W^2_{\mu5})+27g^2(W^2_{\mu6}+W^2_{\mu7})+36g^2W^2_{\mu8}+12\sqrt{2}gg_xW_{\mu8}B_\mu+2g^2_XB^2_\mu\Big]+\\&+\frac{u^2}{324}\Big[81g^2(W^2_{\mu1}+W^2_{\mu2})+81g^2(W^2_{\mu4}+W^2_{\mu5})+(-9gW_{\mu3}-3\sqrt{3}gW_{\mu8}+\sqrt{6}g_XB_\mu)^2\Big]+\\&+\frac{u'^2}{324}\Big[81g^2(W^2_{\mu1}+W^2_{\mu2})+81g^2(W^2_{\mu4}+W^2_{\mu5})+(-9gW_{\mu3}-3\sqrt{3}gW_{\mu8}+\sqrt{6}g_XB_\mu)^2\Big]+\\&+\frac{g^2}{6}\Big[2(\Lambda_1v_1+\Lambda_2v_2)\Big(3W_{\mu3}W_{\mu4}+3W_{\mu1}W_{\mu6}-3W_{\mu2}W_{\mu7}-5\sqrt{3}W_{\mu4}W_{\mu8}\Big)+\\&+3(v_1^2+v_2^2+\lambda_1^2+\lambda_2^2)W^2_{\mu1}+3(v_1^2+v_2^2+\lambda_1^2+\lambda_2^2)W^2_{\mu2}+3(v_1^2+v_2^2+2\lambda_1^2+2\lambda_2^2)W^2_{\mu3}+\\&+3(4v_1^2+4v_2^2+\lambda_1^2+\lambda_2^2+\Lambda_1^2+\Lambda_2^2+2\Lambda_1\lambda_1+2\Lambda_2\lambda_2)W^2_{\mu4}+3\left(4v_1^2+4v_2^2+\lambda_1^2+\lambda_2^2+\Lambda_1^2+\Lambda_2^2-2\Lambda_1\lambda_1-2\Lambda_2\lambda_2\right)W^2_{\mu5}+3(v_1^2+v_2^2+\Lambda_1^2+\Lambda_2^2)W^2_{\mu6}+\\&+3(v_1^2+v_2^2+\Lambda_1^2+\Lambda_2^2)W^2_{\mu7}+2\sqrt{3}(-v_1^2-v_2^2+2\lambda_1^2+2\lambda_2^2)W_{\mu3}W_{\mu8}+(v_1^2+v_2^2+2\lambda_1^2+2\lambda_2^2+8\Lambda_1^2+8\Lambda_2^2)W^2_{\mu8}+\\&+18(\lambda_1v_1+\lambda_2v_2)W_{\mu3}W_{\mu4}+6(\lambda_1v_1+\lambda_2v_2)W_{\mu1}W_{\mu6}-6(\lambda_1v_1+\lambda_2v_2)W_{\mu2}W_{\mu7}+2\sqrt{3}(\lambda_1v_1+\lambda_2v_2)W_{\mu4}W_{\mu8}\Big]+\\&+\frac{2}{27}t^2g^2(\lambda_1^2+\lambda_2^2+\Lambda_1^2+\Lambda_2^2+2v_1^2+2v_2^2)B^2_\mu-\frac{2}{3}\sqrt{\frac{2}{3}}tg^2(\lambda_1^2+\lambda_2^2+v_1^2+v_2^2)W_{\mu3}B_\mu-\\&-\frac{4}{3}\sqrt{\frac{2}{3}}tg^2\left[(\lambda_1+\Lambda_1)v_1+(\lambda_2+\Lambda_2)v_2\right]W_{\mu4}B_\mu-\frac{2\sqrt{2}}{9}tg^2(\lambda_1^2+\lambda_2^2-v_1^2-v_2^2-2\Lambda_1^2-2\Lambda_2^2)W_{\mu8}B_\mu. \quad (66)\end{aligned}$$

We can split $\mathcal{L}^{GB}_{mass}$ in (66) as

$$\mathcal{L}^{GB}_{mass}=\mathcal{L}^{W_5}_{mass}+\mathcal{L}^{CGB}_{mix}+\mathcal{L}^{NGB}_{mix}, \quad (67)$$

where $\mathcal{L}^{W_5}_{mass}$ is the Lagrangian part of the imaginary part W_5. This boson is decoupled, with its mass given by

$$\mathcal{L}^{W_5}=\frac{g^2}{4}\left(\omega^2+u^2+u'^2+8v_1^2+8v_2^2+2\lambda_1^2+2\lambda_2^2+2\Lambda_1^2+2\Lambda_2^2-4\Lambda_1\lambda_1-4\Lambda_2\lambda_2\right)W^2_{\mu5}.$$

Hence,

$$M^2_{W_5}=\frac{g^2}{2}\left(\omega^2+u^2+u'^2+8v_1^2+8v_2^2+2\lambda_1^2+2\lambda_2^2+2\Lambda_1^2+2\Lambda_2^2-4\Lambda_1\lambda_1-4\Lambda_2\lambda_2\right). \quad (68)$$

In the limit λ_1, λ_2, v_1, $v_2\to0$, we have

$$M^2_{W_5}=\frac{g^2}{2}\left(\omega^2+u^2+u'^2+2\Lambda_1^2+2\Lambda_2^2\right). \quad (69)$$

Next,

$$\begin{aligned}\mathcal{L}^{CGB}_{mix}&=\frac{g^2}{4}\left[v^2+v'^2+u^2+u'^2+2(v_1^2+v_2^2+\lambda_1^2+\lambda_2^2)\right](W^2_{\mu1}+W^2_{\mu2})+\\&+\frac{g^2}{4}\left[v^2+v'^2+\omega^2+2(v_1^2+v_2^2+\Lambda_1^2+\Lambda_2^2)\right]\times\\&\times(W^2_{\mu6}+W^2_{\mu7})+g^2(\Lambda_1v_1+\lambda_1v_1+\Lambda_2v_2+\lambda_2v_2)\times\\&\times(W_{\mu1}W_{\mu6}-W_{\mu2}W_{\mu7}) \quad (70)\end{aligned}$$

is the Lagrangian part of the charged gauge bosons W and Y, which can be rewritten in matrix form as

$$\mathcal{L}^{CGB}_{mix}=\frac{g^2}{4}(W'^-_\mu Y'^-_\mu)M^2_{WY}\left(W'^+_\mu Y'^+_\mu\right)^T,$$

where

$$M_{WY}^2 = 2\begin{pmatrix} v^2+v'^2+u^2+u'^2+2(v_1^2+v_2^2+\lambda_1^2+\lambda_2^2) & 2(\Lambda_1 v_1+\lambda_1 v_1+\Lambda_2 v_2+\lambda_2 v_2) \\ 2(\Lambda_1 v_1+\lambda_1 v_1+\Lambda_2 v_2+\lambda_2 v_2) & v^2+v'^2+\omega^2+2(v_1^2+v_2^2+\Lambda_1^2+\Lambda_2^2) \end{pmatrix}. \quad (71)$$

The matrix M_{WY}^2 in (71) can be diagonalized as

$$U_2^T M_{WY}^2 U_2 = \mathrm{diag}(M_W^2, M_Y^2),$$

where

$$M_W^2 = \frac{g^2}{4}\Big\{2(\lambda_1^2+\lambda_2^2+2v_1^2+2v_2^2+\Lambda_1^2+\Lambda_2^2) + \omega^2+u^2+u'^2+2(v^2+v'^2)-\sqrt{\Gamma}\Big\},$$
$$M_Y^2 = \frac{g^2}{4}\Big\{2(\lambda_1^2+\lambda_2^2+2v_1^2+2v_2^2+\Lambda_1^2+\Lambda_2^2) + \omega^2+u^2+u'^2+2(v^2+v'^2)+\sqrt{\Gamma}\Big\}, \quad (72)$$

with

$$\Gamma = 4\lambda_1^4+4\Lambda_1^4+(2\lambda_2^2-2\Lambda_2^2-\omega^2+u^2+u'^2)^2 - 4\lambda_1^2\left(2\Lambda_1^2-2\lambda_2^2+2\Lambda_2^2+\omega^2-u^2-u'^2-4v_1^2\right) - 4\Lambda_1^2(2\lambda_2^2-2\Lambda_2^2-\omega^2+u^2+u'^2-4v_1^2) + 32\Lambda_1(\lambda_2+\Lambda_2)v_1v_2+16(\lambda_2+\Lambda_2)^2v_2^2 + 32\lambda_1 v_1(\Lambda_1 v_1+\lambda_2 v_2+\Lambda_2 v_2). \quad (73)$$

In our model, the following limits are often used:

$$\lambda_{1,2}^2, v_{1,2}^2 \ll u^2, u'^2, v^2, v'^2, \quad (74)$$

$$u^2, u'^2, v^2, v'^2 \ll \omega^2 \sim \Lambda_{1,2}^2. \quad (75)$$

With the help of (74), Γ in (73) becomes

$$\Gamma \approx 2\Lambda_1^2+2\Lambda_2^2+\omega^2-u^2-u'^2 + \frac{16\Lambda_1\Lambda_2 v_1 v_2+8\Lambda_2^2 v_2^2}{2\Lambda_1^2+2\Lambda_2^2+\omega^2-u^2-u'^2}. \quad (76)$$

It then follows that

$$M_W^2 \approx \frac{g^2}{2}\left(u^2+u'^2+v^2+v'^2\right) - \frac{g^2}{2}\Delta_{M_W^2}, \quad (77)$$

with

$$\Delta_{M_W^2} = \frac{4(2\Lambda_1\Lambda_2 v_1 v_2+\Lambda_2^2 v_2^2)}{2\Lambda_1^2+2\Lambda_2^2+\omega^2-u^2-u'^2}. \quad (78)$$

The corresponding eigenstates are arranged into the charged gauge boson mixing matrix

$$U_2 = \begin{pmatrix} \frac{\mathcal{R}}{\sqrt{\mathcal{R}^2+1}} & -\frac{1}{\sqrt{\mathcal{R}^2+1}} \\ \frac{1}{\sqrt{\mathcal{R}^2+1}} & \frac{\mathcal{R}}{\sqrt{\mathcal{R}^2+1}} \end{pmatrix} \equiv \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

where

$$\mathcal{R} = \frac{2\lambda_1^2-2\Lambda_1^2+2\lambda_2^2-2\Lambda_2^2-\omega^2+u^2+u'^2-\sqrt{\Gamma}}{4(\lambda_1+\Lambda_1)v_1+4(\lambda_2+\Lambda_2)v_2}.$$

The physical charged gauge bosons is defined as

$$W_\mu^- = W'^-_\mu\cos\theta + Y'^-_\mu\sin\theta,$$
$$Y_\mu^- = -W'^-_\mu\sin\theta + Y'^-_\mu\cos\theta.$$

The mixing angle θ is given by

$$\mathrm{tg}\,\theta = \frac{1}{\mathcal{R}} = \frac{4(\lambda_1+\Lambda_1)v_1+4(\lambda_2+\Lambda_2)v_2}{2\lambda_1^2-2\Lambda_1^2+2\lambda_2^2-2\Lambda_2^2-\omega^2+u^2+u'^2-\sqrt{\Gamma}} \approx \frac{4\Lambda_1 v_1+4\Lambda_2 v_2}{-2\Lambda_1^2-2\Lambda_2^2-\omega^2-2(\Lambda_1^2+\Lambda_2^2)} \sim \frac{v_i}{\Lambda_i}, \quad i=1,2. \quad (79)$$

We note that in the limit $v_{1,2}\to 0$, the mixing angle θ tends to zero,

$$\Gamma = 2\Lambda_1^2+2\Lambda_2^2+\omega^2-u^2-u'^2,$$

and we have

$$M_W^2 = \frac{g^2}{2}\left(u^2+u'^2+v^2+v'^2\right),$$
$$M_Y^2 = \frac{g^2}{2}\left(2\Lambda_1^2+2\Lambda_2^2+\omega^2+v^2+v'^2\right). \quad (80)$$

There is a mixing among the neutral gauge bosons W_3, W_8, B, and W_4. The mass Lagrangian in this case has the form

$$\mathcal{L}_{mix}^{NGB} = \frac{v^2}{324}\Big(81g^2W_{\mu3}^2+27g^2W_{\mu8}^2+24g_X^2B_\mu^2 - 54\sqrt{3}g^2W_{\mu3}W_{\mu8}-36\sqrt{6}\,gg_XW_{\mu3}B_\mu + 36\sqrt{2}\,gg_XW_{\mu8}B_\mu\Big) + \frac{v'^2}{324}\Big(81g^2W_{\mu3}^2+27g^2W_{\mu8}^2 + 24g_X^2B_\mu^2-54\sqrt{3}g^2W_{\mu3}W_{\mu8} - 36\sqrt{6}\,gg_XW_{\mu3}B_\mu+36\sqrt{2}\,gg_XW_{\mu8}B_\mu\Big) + \frac{\omega^2}{108}\Big(27g^2W_{\mu4}^2+36g^2W_{\mu8}^2 + 12\sqrt{2}\,gg_xW_{\mu8}B_\mu+2g_X^2B_\mu^2\Big) + \frac{u^2}{324}\Big(81g^2W_{\mu4}^2+81g^2W_{\mu3}^2+27g^2W_{\mu8}^2+6g_X^2B_\mu^2 + 54\sqrt{3}g^2W_{\mu3}W_{\mu8}-18\sqrt{6}\,gg_XW_{\mu3}B_\mu-18\sqrt{2}W_{\mu8}B_\mu\Big) + \frac{u'^2}{324}\Big(81g^2W_{\mu4}^2+81g^2W_{\mu3}^2+$$

$$+ 27g^2W_{\mu8}^2 + 6g_X^2B_\mu^2 + 54\sqrt{3}g^2W_{\mu3}W_{\mu8} - - 18\sqrt{6}\,gg_XW_{\mu3}B_\mu - 18\sqrt{2}W_{\mu8}B_\mu\Big) + + \frac{g^2}{6}\Big[2(\Lambda_1v_1+\Lambda_2v_2)\left(3W_{\mu3}W_{\mu4} - 5\sqrt{3}W_{\mu4}W_{\mu8}\right) + + 3(v_1^2+v_2^2+2\lambda_1^2+2\lambda_2^2)W_{\mu3}^2 + + 3(4v_1^2+4v_2^2+\lambda_1^2+\lambda_2^2+\Lambda_1^2+\Lambda_2^2+2\Lambda_1\lambda_1 + +2\Lambda_2\lambda_2)W_{\mu4}^2 + 2\sqrt{3}(-v_1^2-v_2^2+2\lambda_1^2+2\lambda_2^2)W_{\mu3}W_{\mu8} + + (v_1^2+v_2^2+2\lambda_1^2+2\lambda_2^2+8\Lambda_1^2+8\Lambda_2^2)W_{\mu8}^2 + + 18(\lambda_1v_1+\lambda_2v_2)W_{\mu3}W_{\mu4} + + 2\sqrt{3}(\lambda_1v_1+\lambda_2v_2)W_{\mu4}W_{\mu8}\Big] + + \frac{2}{27}t^2g^2(\lambda_1^2+\lambda_2^2+\Lambda_1^2+\Lambda_2^2+2v_1^2+2v_2^2)B_\mu^2 - - \frac{2}{3}\sqrt{\frac{2}{3}}tg^2(\lambda_1^2+\lambda_2^2+v_1^2+v_2^2)W_{\mu3}B_\mu - - \frac{4}{3}\sqrt{\frac{2}{3}}tg^2\left[(\lambda_1+\Lambda_1)v_1+(\lambda_2+\Lambda_2)v_2\right]W_{\mu4}B_\mu - - \frac{2\sqrt{2}}{9}tg^2(\lambda_1^2+\lambda_2^2-v_1^2-v_2^2-2\Lambda_1^2-2\Lambda_2^2)W_{\mu8}B_\mu. \quad (81)$$

In the basis ($W_{\mu3}$, $W_{\mu8}$, B_μ, $W_{\mu4}$), $\mathcal{L}_{mix}^{NGB}$ can be rewritten in matrix form:

$$\mathcal{L}_{mix}^{NGB} \equiv \frac{1}{2}V^TM^2V,$$

where

$$V^T = (W_{\mu3}, W_{\mu8}, B_\mu, W_{\mu4}),$$

$$M^2 = \frac{g^2}{4}\begin{pmatrix} M_{11}^2 & M_{12}^2 & M_{13}^2 & M_{14}^2 \\ M_{12}^2 & M_{22}^2 & M_{23}^2 & M_{24}^2 \\ M_{13}^2 & M_{23}^2 & M_{33}^2 & M_{34}^2 \\ M_{14}^2 & M_{24}^2 & M_{34}^2 & M_{44}^2 \end{pmatrix}, \quad (82)$$

with

$$M_{11}^2 = 2(v^2+v'^2+u^2+u'^2+2v_1^2+2v_2^2+4\lambda_1^2+4\lambda_2^2),$$
$$M_{12}^2 = -\frac{2\sqrt{3}}{3}\left(v^2+v'^2-u^2-u'^2+2v_1^2 + 2v_2^2-4\lambda_1^2-4\lambda_2^2\right),$$
$$M_{13}^2 = -\frac{2}{3}\sqrt{\frac{2}{3}}t\left(2v^2+2v'^2+u^2+u'^2+4\lambda_1^2 + 4\lambda_2^2+4v_1^2+4v_2^2\right),$$
$$M_{14}^2 = 4(\Lambda_1v_1+\Lambda_2v_2)+12(\lambda_1v_1+\lambda_2v_2),$$
$$M_{22}^2 = \frac{2}{3}\left(v^2+v'^2+4\omega^2+u^2+u'^2 + 2v_1^2+2v_2^2+4\lambda_1^2+4\lambda_2^2+16\Lambda_1^2+16\Lambda_2^2\right),$$
$$M_{23}^2 = \frac{2\sqrt{2}t}{9}\left(2v^2+2v'^2+2\omega^2-u^2-u'^2 - 4\lambda_1^2-4\lambda_2^2+4v_1^2+4v_2^2+8\Lambda_1^2+8\Lambda_2^2\right),$$
$$M_{24}^2 = \frac{4}{\sqrt{3}}\left[\lambda_1v_1+\lambda_2v_2-5(\Lambda_1v_1+\Lambda_2v_2)\right], \quad (83)$$
$$M_{33}^2 = \frac{4t^2}{27}\left(4v^2+4v'^2+\omega^2+u^2+u'^2 + 4\lambda_1^2+4\lambda_2^2+4\Lambda_1^2+4\Lambda_2^2+8v_1^2+8v_2^2\right),$$
$$M_{34}^2 = -\frac{16}{3}\sqrt{\frac{2}{3}}t(\lambda_1v_1+\Lambda_1v_1+\lambda_2v_2+\Lambda_2v_2),$$
$$M_{44}^2 = 2(\omega^2+u^2+u'^2+8v_1^2+8v_2^2+2\lambda_1^2 + 2\lambda_2^2+2\Lambda_1^2+2\Lambda_2^2+4\Lambda_1\lambda_1+4\Lambda_2\lambda_2).$$

The matrix M^2 in (82) with the elements in (83) has one exact eigenvalue, which is identified with the photon mass,

$$M_\gamma^2 = 0. \quad (84)$$

The corresponding eigenvector of M_γ^2 is

$$A_\mu = = \left(\frac{\sqrt{3}t}{\sqrt{4t^2+18}}\ -\frac{t}{\sqrt{4t^2+18}}\ \frac{3\sqrt{2}}{\sqrt{4t^2+18}}\ 0\right)^T. \quad (85)$$

We note that in the limit $\lambda_{1,2}, v_{1,2} \to 0$, $M_{14}^2 = M_{24}^2 = M_{34}^2 = 0$ and W_4 does not mix with $W_{3\mu}$, $W_{8\mu}$, and B_μ. In the general case $\lambda_{1,2}, v_{1,2} \neq 0$, the mass matrix in (82) contains one exact eigenvalue as in (84) with the corresponding eigenstate defined in (85).

The diagonalization of the mass matrix M^2 in (82) is done in two steps. In the first step, the basis ($W_{\mu3}$, $W_{\mu8}$, B'_μ, $W_{4\mu}$) is transformed into the basis (A_μ, Z_μ, Z'_μ, $W_{4\mu}$) by the matrix

$$U_{NGB} = \begin{pmatrix} s_W & -c_W & 0 & 0 \\ -\frac{c_Wt_W}{\sqrt{3}} & -\frac{s_Wt_W}{\sqrt{3}} & \sqrt{1-\frac{t_W^2}{3}} & 0 \\ c_W\sqrt{1-\frac{t_W^2}{3}} & s_W\sqrt{1-\frac{t_W^2}{3}} & \frac{t_W}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (86)$$

The eigenstates are defined as

$$\begin{aligned} A_\mu &= s_W W_{3\mu} + \\ &+ c_W \left(-\frac{t_W}{\sqrt{3}} W_{8\mu} + \sqrt{1-\frac{t_W^2}{3}}\, B_\mu \right), \\ Z_\mu &= -c_W W_{3\mu} + \\ &+ s_W \left(-\frac{t_W}{\sqrt{3}} W_{8\mu} + \sqrt{1-\frac{t_W^2}{3}}\, B_\mu \right), \\ Z'_\mu &= \sqrt{1-\frac{t_W^2}{3}} W_{8\mu} + \frac{t_W}{\sqrt{3}} B_\mu. \end{aligned} \quad (87)$$

To obtain (86) and (87), we used the continuation of the $SU(3)_L$ gauge coupling constant g to the spontaneous symmetry breaking point, where

$$t = \frac{3\sqrt{2}s_W}{\sqrt{3-4s_W^2}}. \quad (88)$$

In this basis, the mass matrix M^2 in (82) becomes

$$M'^2 = U^+_{NGB} M^2 U_{NGB} = \\ = \frac{g^2}{4} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & M'^2_{22} & M'^2_{23} & M'^2_{24} \\ 0 & M'^2_{23} & M'^2_{33} & M'^2_{34} \\ 0 & M'^2_{24} & M'^2_{34} & M'^2_{44} \end{pmatrix}, \quad (89)$$

where

$$\begin{aligned} M'^2_{22} &= \frac{4(2t^2+9)}{t^2+18} \left(u^2+u'^2+v^2+v'^2+4\lambda_1^2 + \right. \\ &\left. + 4\lambda_2^2+2v_1^2+2v_2^2\right) = \frac{2}{c_W^2} \left(u^2+u'^2+v^2 + \right. \\ &\left. + v'^2 + 4\lambda_1^2 + 4\lambda_2^2 + 2v_1^2 + 2v_2^2\right), \\ M'^2_{23} &= \frac{4}{3\sqrt{3}} \frac{\sqrt{2t^2+9}}{t^2+18} \times \\ &\times \left[(t^2-9)(4\lambda_1^2+4\lambda_2^2+u^2+u'^2) + \right. \\ &\left. + (2t^2+9)(v^2+v'^2+2v_1^2+2v_2^2)\right] = \\ &= \frac{2}{c_W^2} \left[(1-2c_W^2)(u^2+u'^2+4\lambda_1^2+4\lambda_2^2) + \right. \\ &\left. + v^2+v'^2+v_1^2+v_2^2\right] \sqrt{\alpha_0}, \\ M'^2_{24} &= -4\sqrt{2}\sqrt{\frac{2t^2+9}{t^2+18}} \times \\ &\times (\Lambda_1 v_1 + 3\lambda_1 v_1 + \Lambda_2 v_2 + 3\lambda_2 v_2) = \\ &= -\frac{4}{c_W} (\Lambda_1 v_1 + \Lambda_2 v_2 + 3\lambda_1 v_1 + 3\lambda_2 v_2), \\ M'^2_{33} &= \frac{4}{27(t^2+18)} \left[4\lambda_1^2(t^2-9)^2 + 4\Lambda_1^2(t^2+18)^2 + \right. \\ &+ 81\left(4\lambda_2^2+16\Lambda_2^2+4\omega^2+u^2+u'^2 + \right. \\ &\left. + v^2+v'^2+2v_1^2+2v_2^2\right) + \\ &+ 18t^2\left(8\Lambda_2^2+2\omega^2-u^2 - \right. \\ &\left. - u'^2 + 2v^2 + 2v'^2 + 4v_1^2 + 4v_2^2\right) + \\ &+ 4\lambda_2^2 t^2(t^2-18) + t^4\left(4\Lambda_2^2+\omega^2+u^2+u'^2 + \right. \\ &\left.\left. + 4v^2 + 4v'^2 + 8v_1^2 + 8v_2^2\right)\right] = \\ &= 32(\Lambda_1^2+\Lambda_2^2)c_W^2\alpha_0 + \\ &+ 8\omega^2 c_W^2\alpha_0 + \frac{2}{c_W^2}(v^2+v'^2+2v_1^2+2v_2^2)\alpha_0 + \\ &+ \frac{2}{c_W^2}(2c_W^2-1)^2(u^2+u'^2)\alpha_0 + \\ &+ \frac{8(2c_W^2-1)^2}{c_W^2}(\lambda_1^2+\lambda_2^2)\alpha_0, \quad (90) \\ M'^2_{34} &= -\frac{4\sqrt{2}}{3\sqrt{3}} \frac{1}{\sqrt{t^2+18}} \left[(4t^2-9)(\lambda_1 v_1+\lambda_2 v_2) + \right. \\ &\left. + (4t^2+45)(\Lambda_1 v_1 + \Lambda_2 v_2)\right] = -\frac{4\sqrt{\alpha}}{c_W} \times \\ &\times \left[x_0(\Lambda_1 v_1 + \Lambda_2 v_2) + \left(2 - \frac{1}{\alpha_0}\right) \times \right. \\ &\left. \times (\lambda_1 v_1 + \lambda_2 v_2)\right], \\ M'^2_{44} &= 2\left[2(\lambda_1+\Lambda_1)^2 + 2(\lambda_2+\Lambda_2)^2 + \omega^2 + \right. \\ &\left. + u^2 + u'^2 + 8v_1^2 + 8v_2^2\right] = 2\left(u^2+u'^2 + \right. \\ &+ \omega^2 + 2\lambda_1^2 + 2\lambda_2^2 + 2\Lambda_1^2 + 2\Lambda_2^2 + \\ &\left. + 4\lambda_1\Lambda_1 + 4\lambda_2\Lambda_2 + 8v_1^2 + 8v_2^2\right). \end{aligned}$$

In the approximation $\lambda_{1,2}^2, v_{1,2}^2 \ll \Lambda_{1,2}^2 \sim \omega^2$, we have

$$\begin{aligned} M'^2_{22} &= \frac{2}{c_W^2}\left(u^2+u'^2+v^2+v'^2\right), \\ M'^2_{23} &= \frac{2}{c_W^2}\left[(1-2c_W^2)(u^2+u'^2)+v^2+v'^2\right]\sqrt{\alpha_0}, \\ M'^2_{24} &= -\frac{4}{c_W}\left(\Lambda_1 v_1 + \Lambda_2 v_2\right), \\ M'^2_{33} &= 32(\Lambda_1^2+\Lambda_2^2)c_W^2\alpha_0 + 8\omega^2 c_W^2\alpha_0 + \\ &+ \frac{2}{c_W^2}(v^2+v'^2)\alpha_0 + \\ &+ \frac{2}{c_W^2}(2c_W^2-1)^2(u^2+u'^2)\alpha_0, \\ M'^2_{34} &= -\frac{4x_0\sqrt{\alpha}}{c_W}\left(\Lambda_1 v_1 + \Lambda_2 v_2\right), \\ M'^2_{44} &= 2\left(u^2+u'^2+\omega^2+2\Lambda_1^2 + \right. \\ &\left. + 2\Lambda_2^2 + 4\lambda_1\Lambda_1 + 4\lambda_2\Lambda_2\right), \end{aligned} \quad (91)$$

with

$$s_W = \sin\theta_W, \quad c_W = \cos\theta_W, \quad t_W = \operatorname{tg}\theta_W, \\ x_0 = 4c_W^2 + 1, \quad \alpha_0 = (4c_W^2-1)^{-1}. \quad (92)$$

From (89), there exist mixings between Z_μ, Z'_μ and $W_{\mu 4}$. It is noteworthy that in the limit $v_{1,2}=0$, the elements M'^2_{24} and M'^2_{34} vanish, and there is no mixing between W_4 and Z_μ, Z'_μ.

In the second step, three remaining neutral gauge bosons gain masses via the seesaw mechanism:

$$M_Z^2 = \frac{g^2}{4}\left[M'^2_{22} - (M^{off})^T (M'^2_{2\times 2})^{-1} M^{off}\right], \quad (93)$$

where

$$M^{off} = \begin{pmatrix} M'^2_{23} \\ M'^2_{24} \end{pmatrix}, \quad M'^2_{2\times 2} = \begin{pmatrix} M'^2_{33} & M'^2_{34} \\ M'^2_{34} & M'^2_{44} \end{pmatrix}. \quad (94)$$

Combining (93) and (94) yields

$$M_Z^2 = \frac{g^2}{4}\left(M'^2_{22} + \frac{(M'^2_{24})^2 M'^2_{33} - 2M'^2_{23}M'^2_{24}M'^2_{34} + (M'^2_{23})^2 M'^2_{44}}{(M'^2_{34})^2 - M'^2_{33}M'^2_{44}}\right) = \frac{g^2\left(u^2+u'^2+v^2+v'^2\right)}{2c_W^2} - \frac{g^2}{2c_W^2}\Delta_{M_Z^2},$$

where

$$\Delta_{M_Z^2} = \frac{4\Delta_Z^2\left(4c_W^4 x_3 - x_0x_1 + x_4\right) + x_1\left[x_2x_1 - 4\Delta_Z^2 x_0\right]}{x_2(x_4+4c_W^4x_3) - 4\Delta_Z^2x_0^2} = \frac{4\Delta_Z^2\left(4c_W^4x_3 - 2x_0x_1 + x_4\right) + x_1^2x_2}{x_2(x_4+4c_W^4x_3) - 4\Delta_Z^2 x_0^2}, \quad (95)$$

with

$$x_1 = (1-2c_W^2)(u^2+u'^2) + v^2 + v'^2,$$
$$x_2 = 2\Lambda_1(2\lambda_1+\Lambda_1) + 2\Lambda_2(2\lambda_2+\Lambda_2) + \omega^2 + u^2 + u'^2,$$
$$x_3 = 4\Lambda_1^2 + 4\Lambda_2^2 + \omega^2 + u^2 + u'^2,$$
$$x_4 = (1-4c^2)(u^2+u'^2) + v^2 + v'^2,$$
$$\Delta_Z = \Lambda_1 v_1 + \Lambda_2 v_2.$$

The ρ parameter in our model is given by

$$\rho = \frac{M_W^2}{M_Z^2\cos^2\theta_W} = 1 + \delta_{tree}, \quad (96)$$

where

$$\delta_{tree} = \frac{\delta_{WZ}}{M_Z^2}, \quad \delta_{WZ} = \frac{g^2}{2c_W^2}\left(\Delta_{M_Z^2} - \Delta_{M_W^2}\right). \quad (97)$$

Using approximations (74) and (75), we have

$$\Delta_{M_Z^2} - \Delta_{M_W^2} \approx 8(\Lambda_1 v_1 + \Lambda_2 v_2) \times \left\{-\frac{\Lambda_2 v_2}{2\Lambda_1^2+2\Lambda_2^2+\omega^2} + \frac{(4\Lambda_1^2+4\Lambda_2^2+\omega^2)(4c_W^2-1)c_W^2(\Lambda_1v_1+\Lambda_2v_2)}{2(4c_W^2-1)\left[(2\Lambda_1^2+2\Lambda_2^2+\omega^2)(4\Lambda_1^2+4\Lambda_2^2+\omega^2)c_W^4 - (4c_W^2+1)^2(\Lambda_1v_1+\Lambda_2v_2)^2\right]}\right\}. \quad (98)$$

We assume relations (57) and $v_2 \equiv v_s$, $\omega = \Lambda_2 \equiv \Lambda_s$; then

$$\Delta_{M_Z^2} - \Delta_{M_W^2} \approx 8(k^2+1)\Lambda_s v_s\left[-\frac{v_s}{(2k^2+3)\Lambda_s} + \frac{(k^2+1)(4k^2+5)c_W^2\Lambda_s v_s}{2[(8k^4+22k^2+15)c_W^4\Lambda_s^2 - (k^2+1)^2(4c_W^2+1)^2v_s^2]}\right] \approx$$
$$\approx -\frac{8(k^2+1)v_s^2}{2k^2+3} + \frac{8(k^2+1)^2(4k^2+5)c_W^2v_s^2}{2(2k^2+3)(4k^2+5)c_W^4} \approx -\frac{8(k^2+1)v_s^2}{2k^2+3} + \frac{8(k^2+1)^2v_s^2}{2(2k^2+3)c_W^2} = \frac{8(k^2+1)v_s^2}{2k^2+3}\left(\frac{k^2+1}{2c_W^2} - 1\right), \quad (99)$$

$$\Delta_{M_Z^2} - \Delta_{M_W^2} \approx \frac{8(k^2+1)v_s^2}{2k^2+3}\left(\frac{k^2+1}{2c_W^2} - 1\right). \quad (100)$$

From (97) and (100), we have

$$\delta_{tree} = \frac{g^2}{2c_W^2}\frac{1}{M_Z^2}\frac{8(k^2+1)v_s^2}{2k^2+3}\left(\frac{k^2+1}{2c_W^2} - 1\right). \quad (101)$$

The experimental value of the ρ parameter and M_W are given in Ref. [4]:

$$\rho = 1.0004^{+0.0003}_{-0.0004} \quad (\delta_{tree} = 0.0004^{+0.0003}_{-0.0004}),$$
$$s_W^2 = 0.23116 \pm 0.00012, \quad (102)$$
$$M_W = 80.358 \pm 0.015 \text{ GeV}.$$

Hence,

$$0 \le \delta_{tree} \le 0.0007. \quad (103)$$

From (102) and (103), we can deduce the relations between v, g, and k. Indeed,

$$v = \pm\frac{c_W^2\sqrt{\delta_{tree}}\sqrt{2k^2+3}M_Z}{g\sqrt{2k^2+2}\sqrt{k^2+1-2c_W^2}}.$$

Figure 1 gives the relation between v_s and g, k with $g = 0.5$ and $k \in (0.9, 1.1)$, for $|v_s| \in (0, 8)$ GeV. Conditions (74) and (75) are then satisfied. Figure 2 gives the relation between g and δ_{tree}, v_s with $k = 1$ and $\delta_{tree} \in (0, 0.0007)$, $v_s \in (0, 8)$ GeV, for $|g| \in (0, 2)$ GeV. Conditions (74) and (75) are then satisfied. Figure 3 gives the relation between k and g, v_s with $\delta_{tree} = 0.0005$ and $g \in (0.4, 0.6)$, $v_s \in (0, 8)$ GeV, for $k \in (1, 3)$ GeV (k is a real number, Fig. 3*a*) or $k = ik_1$, $k_1 \in (-1.2, -1.05)$ GeV (k is a purely complex number, Fig. 3*b*). Conditions (74) and (75) are then satisfied. From Fig. 3, we see that many values of k that are different from close to unity still can fit the recent experimental data [4]. This means that the difference of $\langle s_1 \rangle$ and $\langle s_1 \rangle$ as mentioned in this work is necessary.

Diagonalizing the mass matrix $M'^2_{2\times 2}$, we obtain two new physical gauge bosons

$$\begin{aligned} Z''_\mu &= Z'_\mu \cos\phi + W_{\mu 4} \sin\phi, \\ W'_{\mu 4} &= -Z'_\mu \sin\phi + W_{\mu 4}\cos\phi. \end{aligned} \tag{104}$$

The mixing angle ϕ is given by

$$\operatorname{tg}\phi = \frac{4\sqrt{\alpha_0}c_W(\Lambda_1 v_1 + \Lambda_2 v_2)x_0}{-4c_W^4\alpha_0 x_3 + c_W^2 x_2 - \alpha_0 x_4 + \sqrt{F}}, \tag{105}$$

where

$$\begin{aligned} F = \left(4c_W^4\alpha_0 x_3 - c_W^2 x_2 + \alpha_0 x_4\right)^2 + \\ + 16\alpha_0 c_W^2(\Lambda_1 v_1 + \Lambda_2 v_2)^2 x_0^2. \end{aligned}$$

If $\lambda_{1,2}^2, v_{1,2}^2, u^2, u'^2, v^2, v'^2 \ll \omega^2 \sim \Lambda_s^2 \sim \Lambda_\sigma^2$ then

$$\sqrt{F} \approx c_W^2[2\Lambda_1^2 + 2\Lambda_2^2 + \omega^2 - 4\alpha c_W^2(4\Lambda_1^2 + 4\Lambda_2^2 + \omega^2)],$$

and we can evaluate

$$\begin{aligned} &\operatorname{tg}\phi \approx \\ &\approx -\frac{2\sqrt{\alpha_0}(\Lambda_1 v_1 + \Lambda_2 v_2)x_0}{c_W\left[2(8\alpha_0 c_W^2 - 1)(\Lambda_1^2 + \Lambda_2^2) + (4\alpha_0 c_W^2 - 1)\omega^2\right]} \sim \\ &\sim \frac{v_i}{\Lambda_i}, \quad i = 1, 2. \end{aligned} \tag{106}$$

The physical mass eigenvalues are defined by

$$\begin{aligned} M^2_{Z''_\mu, W'_{\mu 4}} = \frac{g^2}{4c_W^2} \times \\ \times \left\{4\alpha_0 c_W^4 x_3 + c_W^2 x_2 + \alpha_0 x_4 \pm \sqrt{F}\right\}. \end{aligned} \tag{107}$$

In the limit $\lambda_{1,2}, v_{1,2} \to 0$, the mixing angle ϕ tends to zero, and $M^2_{Z''_\mu, W'_{\mu 4}}$ in (107) reduces to

$$\begin{aligned} M^2_{Z''_\mu} &= \frac{g^2}{2c_W^2}\left[c_{2W}^2(u^2 + u'^2) + v^2 + v'^2 + \right. \\ &\quad \left. + 4c_W^4(4\Lambda_1^2 + 4\Lambda_2^2 + \omega^2)\right]\alpha_0, \\ M^2_{W'_{\mu 4}} &= \frac{g^2}{2}\left(u^2 + u'^2 + \omega^2 + 2\Lambda_1^2 + 2\Lambda_2^2\right). \end{aligned} \tag{108}$$

From (69) and (108), the $W'_{\mu 4}$ and $W_{\mu 5}$ components have the same mass, and hence, in this approximation we should identify the linear combination

$$\sqrt{2}X_\mu^0 = W'_{\mu 4} - iW_{\mu 5} \tag{109}$$

as a physical neutral non-Hermitian gauge boson. The subscript "0" indicates the neutrality of the gauge boson X_μ. We note that the identification in (109) can only be acceptable in the limit $\lambda_{1,2}, v_{1,2} \to 0$. In general, it is not true because of the difference in masses of $W'_{\mu 4}$ and $W_{\mu 5}$ as in (68) and (107).

Expressions (79) and (106) show that, in the limits (74) and (75), the mixings between the charged gauge bosons W–Y and the neutral ones Z'–W_4 are of the same order because they are proportional to v_i/Λ_i $i = 1, 2$. In addition, from (108),

$$M^2_{Z''_\mu} \approx g^2(4\Lambda_1^2 + 4\Lambda_2^2 + \omega^2)$$

is somewhat bigger than

$$M^2_{W'_{\mu 4}} \approx \frac{g^2}{2}\left(\omega^2 + 2\Lambda_1^2 + 2\Lambda_2^2\right)$$

(or $M^2_{X_\mu^0}$), and

$$|M_Y^2 - M^2_{X_\mu^0}| = \frac{g^2}{2}(u^2 + u'^2 - v^2 - v'^2)$$

is slightly smaller than

$$M_W^2 = \frac{g^2}{2}(u^2 + u'^2 + v^2 + v'^2).$$

In that limit, the masses of X_μ^0 and Y degenerate.

6. CONCLUSIONS

We have studied new features of the 3–3–1 model with a neutral fermion based on the S_3 flavor symmetry in which the anti-sextet responsible for the neutrino mass and mixing lies in the $\underline{2}$ representation under S_3 and the number of Higgs multiplets required is reduced. If the S_3 symmetry is violated as a perturbation by the difference in components of the anti-sextet, S_3 is equivalently broken into identity, the corresponding neutrino mass mixing matrix acquires the most general form. This way of symmetry breaking helps us reduce

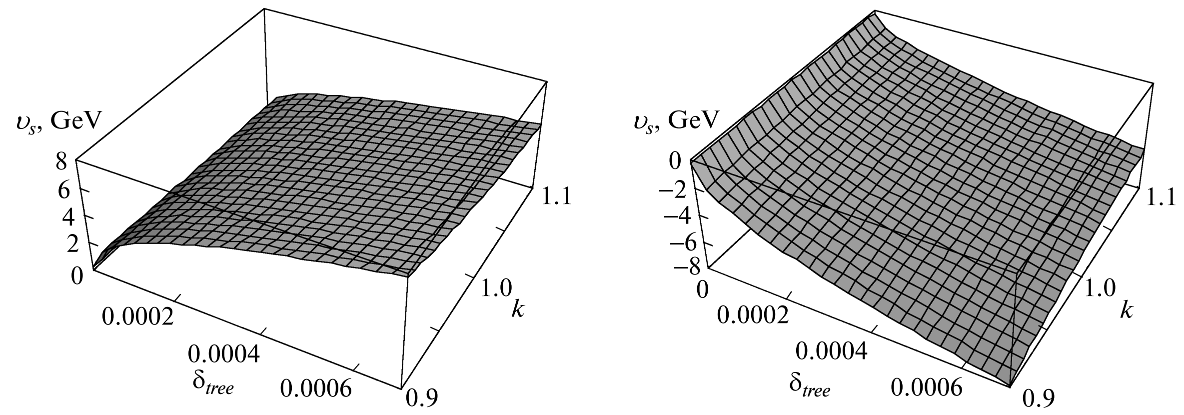

Fig. 1. The relation between v_s and g, k with $g = 0.5$ and $k \in (0.9, 1.1)$

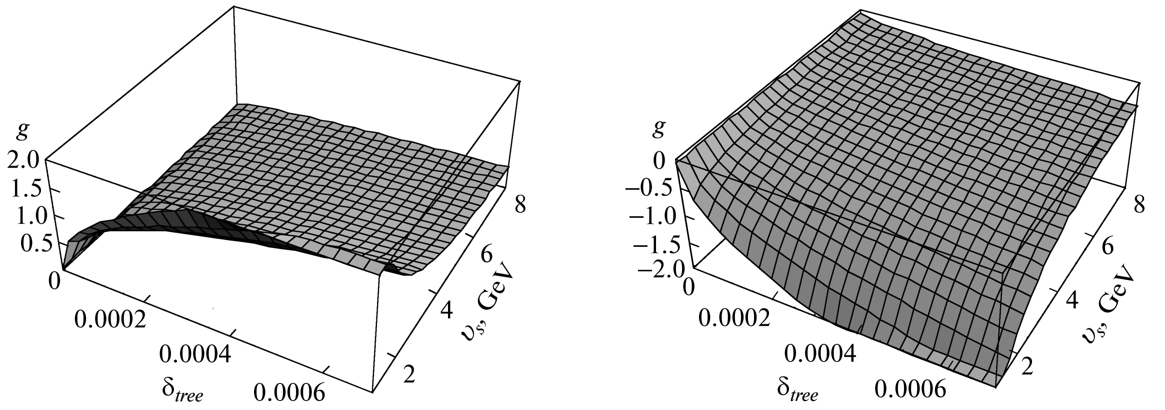

Fig. 2. The relation between g and δ_{tree}, v_s with $k = 1$ and $\delta_{tree} \in (0, 0.0007)$, $v_s \in (0, 8)$ GeV

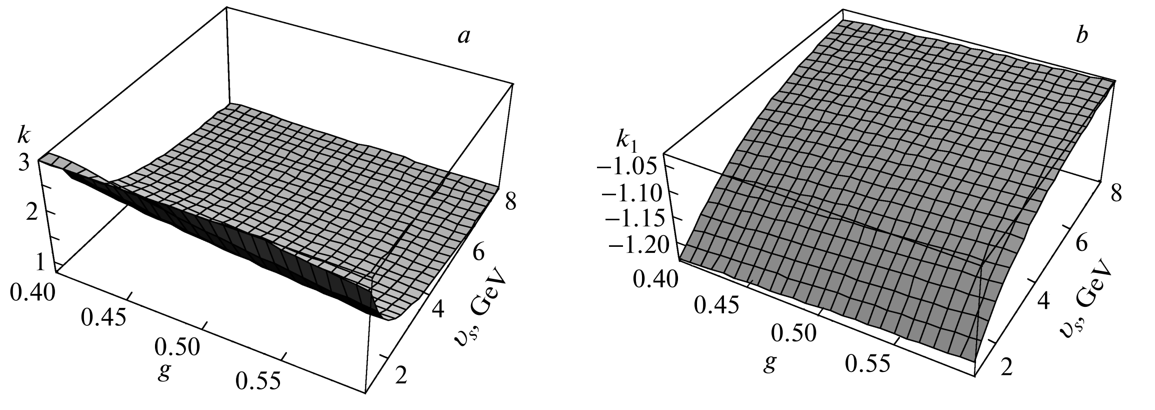

Fig. 3. The relation between k and g, v_s with $\delta_{tree} = 0.0005$ and $g \in (0.4, 0.6)$, $v_s \in (0, 8)$ GeV

the content in the Higgs sector: only one anti-sextet instead of three multiplets (two anti-sextets and one triplet) as in our previous work. By assuming that the VEVs of the anti-sextets differ from each other and regarding the difference between these VEVs as a small perturbation, we can make the model fit the latest data on neutrino oscillations. Our results show that the neutrino masses are naturally small and a deviation from the tri-bimaximal neutrino mixing form can be realized. The Higgs potential of the model and minimization conditions are also considered.

This work was supported in part by the National Foundation for Science and Technology Development of Vietnam (NAFOSTED) under the grant 103.01-2011.63.

REFERENCES

1. Super-Kamiokande Collaboration, Y. Fukuda et al., Phys. Rev. Lett. **81**, 1562 (1998); arXiv:9807003 [hep-ex]; Super-Kamiokande Collaboration, S. Fukuda et al., Phys. Rev. Lett. **86**, 5651 (2001); arXiv: 0103032[hep-ex]; Super-Kamiokande Collaboration, S. Fukuda et al., Phys. Rev. Lett. **86**, 5656 (2001); arXiv:0103033[hep-ex]; Super-Kamiokande Collaboration, S. Fukuda et al., Phys. Lett. B **539**, 179 (2002); arXiv:0205075[hep-ex].

2. Super-Kamiokande Collaboration, Y. Ashie et al., Phys. Rev. Lett. **93**, 101801 (2004); arXiv:0404034 [hep-ex]; Super-Kamiokande Collaboration, Y. Ashie et al., Phys. Rev. D **71**, 112005 (2005); arXiv:0501064 [hep-ex].

3. K. Nakamura et al. (Particle Data Group), J. Phys. G **37**, 075021 (2010).

4. J. Beringer et al. (Particle Data Group), Phys. Rev. D **86**, 010001 (2012); T. Schwetz, M. Tortola, and J. Valle, New J. Phys. **13**, 109401 (2011); arXiv:131108.1376[hep-ph]; K. Abe et al. (T2K Collaboration), Phys. Rev. Lett. **107**, 041801 (2011); P. Adamson et al. (MINOS Collaboration), Phys. Rev. Lett. **107**, 181802 (2011); arXiv:1108.0015[hep-ex]; G. L. Fogli et al., Phys. Rev. D **84**, 053007 (2011); arXiv:1106.6028[hep-ph].

5. A. Zee, Phys. Lett. B **93**, 389 (1980); P. H. Frampton and S. L. Glashow, Phys. Lett. B **461**, 95 (1999); A. S. Joshipura and S. D. Rindani, Phys. Lett. B **464**, 239 (1999); Y. Koide, Nucl. Phys. B (Proc. Suppl.) **111**, 294 (2002); K. Hasegawa, C. S. Lim, and K. Ogure, Phys. Rev. D **68**, 053006 (2003); arXiv: 0303252[hep-ph].

6. K. S. Babu, Phys. Lett. B **203**, 132 (1988).

7. F. Pisano and V. Pleitez, Phys. Rev. D **46**, 410 (1992); P. H. Frampton, Phys. Rev. Lett. **69**, 2889 (1992); R. Foot, O. F. Hernandez, F. Pisano, and V. Pleitez, Phys. Rev. D **47**, 4158 (1993).

8. W. A. Ponce, Y. Giraldo, and L. A. Sanchez, Phys. Rev. D **67**, 075001 (2003); P. V. Dong, H. N. Long, D. T. Nhung, and D. V. Soa, Phys. Rev. D **73**, 035004 (2006); P. V. Dong and H. N. Long, Adv. High Energy Phys. **2008**, 739492 (2008).

9. M. Singer, J. W. F. Valle, and J. Schechter, Phys. Rev. D **22**, 738 (1980); J. C. Montero, F. Pisano, and V. Pleitez, Phys. Rev. D **47**, 2918 (1993); R. Foot, H. N. Long, and Tuan A. Tran, Phys. Rev. D **50**, 34(R) (1994); H. N. Long, Phys. Rev. D **53**, 437 (1996); **54**, 4691 (1996).

10. A. Doff and F. Pisano, Mod. Phys. Lett. A **15**, 1471 (2000); P. V. Dong and H. N. Long, Int. J. Mod. Phys. A **21**, 6677 (2006).

11. E. Ma and G. Rajasekaran, Phys. Rev. D **64**, 113012 (2001); K. S. Babu, E. Ma, and J. W. F. Valle, Phys. Lett. B **552**, 207 (2003); G. Altarelli and F. Feruglio, Nucl. Phys. B **720**, 64 (2005); E. Ma, Phys. Rev. D **73**, 057304 (2006); X. G. He, Y. Y. Keum, and R. R. Volkas, JHEP **0604**, 039 (2006); S. Morisi, M. Picariello, and E. Torrente-Lujan, Phys. Rev. D **75**, 075015 (2007); C. S. Lam, Phys. Lett. B **656**, 193 (2007); F. Bazzocchi, S. Kaneko, and S. Morisi, JHEP **0803**, 063 (2008); A. Blum, C. Hagedorn, and M. Lindner, Phys. Rev. D **77**, 076004 (2008); F. Bazzochi, M. Frigerio, and S. Morisi, Phys. Rev. D **78**, 116018 (2008); G. Altarelli, F. Feruglio, and C. Hagedorn, JHEP **0803**, 052 (2008); M. Hirsch, S. Morisi, and J. W. F. Valle, Phys. Rev. D **78**, 093007 (2008); E. Ma, Phys. Lett. B **671**, 366 (2009); G. Altarelli and D. Meloni, J. Phys. G **36**, 085005 (2009); Y. Lin, Nucl. Phys. B **813**, 91 (2009); Y. H. Ahn and C. S. Chen, Phys. Rev. D **81**, 105013 (2010); J. Barry and W. Rodejohanny, Phys. Rev. D **81**, 093002 (2010); 119901(E) (2010); G. J. Ding and D. Meloni, Nucl. Phys. B **855**, 21 (2012).

12. R. N. Mohapatra, M. K. Parida, and G. Rajasekaran, Phys. Rev. D **69**, 053007 (2004); C. Hagedorn, M. Lindner, and R. N. Mohapatra, JHEP **0606**, 042 (2006); E. Ma, Phys. Lett. B **632**, 352 (2006); H. Zhang, Phys. Lett. B **655**, 132 (2007); Y. Koide, JHEP **0708**, 086 (2007); H. Ishimori, Y. Shimizu, and M. Tanimoto, Progr. Theor. Phys. **121**, 769 (2009); F. Bazzocchi, L. Merlo, and S. Morisi, Nucl. Phys. B **816**, 204 (2009); F. Bazzocchi and S. Morisi, Phys. Rev. D **80** 096005 (2009); G. Altarelli and F. Fergulio, Rev. Mod. Phys. **82**, 2701 (2010);

G. J. Ding, Nucl. Phys. B **827**, 82 (2010); Y. H. Ahn, S. K. Kang, C. S. Kim, and T. P. Nguyen, Phys. Rev. D **82**, 093005 (2010); Y. Daikoku and H. Okada, arXiv:1008.0914[hep-ph]; H. Ishimori, Y. Shimizu, M. Tanimoto, and A. Watanabe, Phys. Rev. D **83** 033004 (2011); H. Ishimori and M. Tanimoto, Progr. Theor. Phys. **125**, 653 (2011); R. Z. Yang and H. Zhang, Phys. Lett. B **700**, 316 (2011); S. Morisi and E. Peinado, Phys. Lett. B **701**, 451 (2011); S. Morisi, K. M. Patel, and E. Peinado, Phys. Rev. D **84**, 053002 (2011).

13. L. Wolfenstein, Phys. Rev. D **18**, 958 (1978); S. Pakvasa and H. Sugawara, Phys. Lett. B **73**, 61 (1978); **82**, 105 (1979); E. Durman and H. S. Tsao, Phys. Rev. D **20**, 1207 (1979); Y. Yamanaka, H. Sugawara, and S. Pakvasa, Phys. Rev. D **25**, 1895 (1982); K. Kang, J. E. Kim, and P. Ko, Z. Phys. C **72**, 671 (1996); H. Fritzsch and Z. Z. Xing, Phys. Lett. B **372**, 265 (1996); K. Kang, S. K. Kang, J. E. Kim, and P. Ko, Mod. Phys. Lett. A **12**, 1175 (1997); M. Fukugita, M. Tanimoto, and T. Yanagida, Phys. Rev. D **57**, 4429 (1998); H. Fritzsch and Z. Z. Xing, Phys. Lett. B **440**, 313 (1998); Y. Koide, Phys. Rev. D **60**, 077301 (1999); H. Fritzsch and Z. Z. Xing, Phys. Rev. D **61**, 073016 (2000); M. Tanimoto, Phys. Lett. B **483**, 417 (2000); G. C. Branco and J. I. Silva-Marcos, Phys. Lett. B **526**, 104 (2002); M. Fujii, K. Hamaguchi, and T. Yanagida, Phys. Rev. D **65**, 115012 (2002); J. Kubo, A. Mondragon, M. Mondragon, and E. Rodriguez-Jauregui, Progr. Theor. Phys. **109**, 795 (2003); **114**, 287(E) (2005); P. F. Harrison and W. G. Scott, Phys. Lett. B **557**, 76 (2003); S.-L. Chen, M. Frigerio, and E. Ma, Phys. Rev. D **70**, 073008 (2004); **70**, 079905(E) (2004); H. Fritzsch and Z. Z. Xing, Phys. Lett. B **598**, 237 (2004); F. Caravaglios and S. Morisi, arXiv:hep-ph/0503234; W. Grimus and L. Lavoura, JHEP **0508**, 013 (2005); R. N. Mohapatra, S. Nasri, and H. B. Yu, Phys. Lett. B **639**, 318 (2006); R. Jora, S. Nasri, and J. Schechter, Int. J. Mod. Phys. A **21**, 5875 (2006); J. E. Kim and J.-C. Park, JHEP **0605**, 017 (2006); Y. Koide, Europ. Phys. J. C **50**, 809 (2007); A. Mondragon, M. Mondragon, and E. Peinado, Phys. Rev. D **76**, 076003 (2007); AIP Conf. Proc. **1026**, 164 (2008); M. Picariello, Int. J. Mod. Phys. A **23**, 4435 (2008); C. Y. Chen and L. Wolfenstein, Phys. Rev. D **77**, 093009 (2008); R. Jora, J. Schechter, and M. Naeem Shahid, Phys. Rev. D **80**, 093007 (2009); **82**, 079902(E) (2010); D. A. Dicus, S. F. Ge, and W. W. Repko, Phys. Rev. D **82**, 033005 (2010); Z. Z. Xing, D. Yang, and S. Zhou, Phys. Lett. B **690**, 304 (2010); R. Jora, J. Schechter, and M. N. Shahid, Phys. Rev. D **82**, 053006 (2010); S. Dev, S. Gupta, and R. R. Gautam, Phys. Lett. B **702**, 28 (2011); D. Meloni, S. Morisi, and E. Peinado, J. Phys. G **38**, 015003 (2011); G. Bhattacharyya, P. Leser, and H. Pas, Phys. Rev. D **83**, 011701(R) (2011); T. Kaneko and H. Sugawara, Phys. Lett. B **697**, 329 (2011); S. Zhou, Phys. Lett. B **704**, 291 (2011).

14. P. V. Dong, H. N. Long, C. H. Nam, and V. V. Vien, Phys. Rev. D **85**, 053001 (2012); arXiv:1111.6360[hep-ph].

15. E. Ma, arXiv:hep-ph/0409075; H. Ishimori et al., Progr. Theor. Phys. Suppl. **183**, 1 (2010).

16. P. V. Dong, L. T. Hue, H. N. Long, and D. V. Soa, Phys. Rev. D **81**, 053004 (2010).

17. P. V. Dong, H. N. Long, D. V. Soa, and V. V. Vien, Europ. Phys. J. C **71**, 1544 (2011).

18. A. E. C. Hernadez, R. Martinez, and J. Nisperuza, arXiv:1401.0937[hep-ph].

19. V. D. Barger, S. Pakvasa, T. J. Weiler, and K. Whisnant, Phys. Lett. B **437**, 107 (1998); arXiv: 9806387[hep-ph]; I. Stancu and D. V. Ahluwalia, Phys. Lett. B **460**, 431 (1999); arXiv:9903408[hep-ph]; H. Georgi and S. L. Glashow, Phys. Rev. D **61**, 097301 (2000); arXiv:9808293[hep-ph]; N. Li and B.-Q. Ma, Phys. Lett. B **600**, 248 (2004); arXiv:0408235[hep-ph].

20. V. V. Vien and H. N. Long, Int. J. Mod. Phys. A **28**, 1350159 (2013); arXiv:1312.5034[hep-ph].

21. P. V. Dong, V. T. N. Huyen, H. N. Long, and H. V. Thuy, Adv. High Energy Phys. **2012**, 715038 (2012).

22. P. V. Dong and H. N. Long, Europ. Phys. J. C **42**, 325 (2005).