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ENERGY SPECTRUM OF THE ENSEMBLE OF WEAKLY NONLINEAR GRAVITY–CAPILLARY WAVES ON A FLUID SURFACE

E. Tobisch[*]

Institute for Analysis, Johannes Kepler University
4040, Linz, Austria

We consider nonlinear gravity–capillary waves with the nonlinearity parameter $\varepsilon \sim 0.1$–0.25. For this nonlinearity, time scale separation does not occur and the kinetic wave equation does not hold. An energy cascade in this case is built at the dynamic time scale (D-cascade) and is computed by the increment chain equation method first introduced in [15]. We for the first time compute an analytic expression for the energy spectrum of nonlinear gravity–capillary waves as an explicit function of the ratio of surface tension to the gravity acceleration. We show that its two limits — pure capillary and pure gravity waves on a fluid surface — coincide with the previously obtained results. We also discuss relations of the D-cascade model with a few known models used in the theory of nonlinear waves such as Zakharov's equation, resonance of modes with nonlinear Stokes-corrected frequencies, and the Benjamin–Feir index. These connections are crucial in understanding and forecasting specifics of the energy transport in a variety of multicomponent wave dynamics, from oceanography to optics, from plasma physics to acoustics.

1. INTRODUCTION

Until recently, the notion of an "energy cascade" in a weakly nonlinear wave system was traditionally associated with kinetic wave turbulence theory (WTT), where the energy spectrum is a stationary solution of the wave kinetic equation. The wave kinetic equation was first introduced in 1962 by Hasselmann [1], and its first stationary solution (for capillary waves) was found in 1967 by Zakharov and Filonenko [2]. Subsequently, their method of finding stationary solutions was generalized to various weakly nonlinear wave systems with dispersion [3].

The kinetic equation can be solved numerically for any nontrivial dispersion function ω with a given dispersion relation $\omega = \omega(\mathbf{k})$, where $\mathbf{k}$ is the wave vector. In the particular case of the dispersion function of the form

$$\omega \sim k^{\alpha}, \quad k = |\mathbf{k}|, \quad \alpha > 1,$$

kinetic WTT gives an analytic prediction for the energy spectrum in the power-law form $\sim k^{-\beta}$, $\beta > 0$, where β is different for wave systems with different disper-

sion functions but does not depend on the excitation parameters.

The prediction holds in the so-called inertial interval, where forcing and dissipation are balanced such that energy is conserved within this interval (it is assumed that pumping and dissipation are spaced far apart in Fourier space). The basic physical mechanism leading to the formation of a kinetic energy cascade (K-cascade) is the s-wave resonance interactions of linear Fourier modes

$$A(t/\varepsilon^{s-2}) \exp i[kx - \omega t] \qquad (1)$$

with slowly changing amplitudes. The s-wave resonances occur independently at different time scales t/ε^{s-2}, $s \geq 3$, where $0 < \varepsilon \ll 1$ is a small parameter, e. g., $\varepsilon \sim 10^{-2}$ for water waves.

Rapid technological progress in the field of measurement methods and measuring techniques allowed a systematic study of the spectrum in various fluid systems in the past two decades. The experimental data turned out to be rather contradictory, including chains such as: the energy spectrum is not formed, and energy exchange within a small set of Fourier modes occurs instead; the energy spectrum and a power law are observed, but the exponent differs from the one predicted by kinetic WTT; the exponent depends on the param-

[*]E-mail: Elena.Tobisch@jku.at

eters of the initial excitation; the inertial interval does not exist, and so on. Without claiming to be exhaustive, we give a few references to the most thorough and credible recent experiments [4–8]. A very respectable list of references can be found in a recent review by Newell and Rumpf [9].

Some of these effects have found their explanation in the framework of the discrete WTT [10, 11]; for instance, the absence of the inertial interval is due to the nonlocality of resonance interaction, for some types of dispersion functions. The locality of interaction in kinetic WTT is understood as follows: only the interaction of waves with wavelengths of the same order is allowed. However, it has been known for more than 20 years [12, 13] that, say, capillary waves with wavelengths of the orders k and k^3 can interact directly, i. e., build a joint resonance triad; more examples can be found in [14].

The model of the energy spectra formation in wave systems with weak and moderate nonlinearity allowing the observed experimental shape of the energy spectrum to be reconciled with the predictions made for the K-cascade was first proposed in 2012 by Kartashova [15]. In this model, the triggering physical mechanism for an energy cascade formation is the modulation instability (MI), and the corresponding energy cascade is called a dynamical cascade (D-cascade); a D-cascade is a sequence of distinct modes in Fourier space. The use of the specially developed increment chain equation method (ICEM) allows computing the energy spectrum of a D-cascade.

The energy spectrum in the D-model is a solution of the so-called chain equation. It connects frequencies and amplitudes of two adjacent modes in D-cascade. Energy spectra for capillary and surface water waves (with the respective dispersion functions $\omega^2 = \sigma k^3$ and $\omega^2 = gk$) are computed in [15] for different values of a small parameter $\varepsilon \sim 0.1$–0.4 chosen as the ratio of the wave amplitude to the wave length.

Here, we sketch the ICEM and compute the energy spectrum of an ensemble of weakly nonlinear gravity–capillary waves with the dispersion function

$$\omega^2 = gk + \sigma k^3.$$

We also demonstrate intrinsic mathematical connections between the D-model and other models describing nonlinear wave interaction at the same temporal and spatial scales: Zakharov's equation, resonances of nonlinear Stokes waves, and the Benjamin–Feir index.

2. INCREMENT CHAIN EQUATION METHOD (ICEM)

The physical mechanism underlying the formation of a D-cascade is modulation instability, which can be described as the decay of a carrier wave ω_0 into two side bands ω_1 and ω_2:

$$\omega_1 + \omega_2 = 2\omega_0, \quad \vec{k}_1 + \vec{k}_2 = 2\vec{k}_0 + \Theta, \tag{2}$$

$$\omega_1 = \omega_0 + \Delta\omega, \quad \omega_2 = \omega_0 - \Delta\omega, \quad 0 < \Delta\omega \ll 1. \tag{3}$$

A wave train with the initial real amplitude A, wavenumber $k = |\vec{k}|$, and frequency ω is modulationally unstable if

$$0 \leq \Delta\omega/Ak\omega \leq \sqrt{2}. \tag{4}$$

Equation (4) describes an instability interval for the wave systems with a small nonlinearity of the order of $\varepsilon \sim 0.1$ to 0.2, first obtained in [16]. It is also established for gravity surface waves that the most unstable modes in this interval satisfy the condition

$$\Delta\omega/Ak\omega = 1. \tag{5}$$

The essence of the ICEM is the use of (5) for computing the frequencies of the cascading modes. At the first step of the D-cascade, a carrier mode has a frequency ω_0 and the distance to the next cascading mode

$$(\Delta\omega)_1 = |\omega_0 - \omega_1|$$

with the frequency ω_0 chosen such that condition (5) is satisfied, i. e.,

$$|\omega_0 - \omega_1| = A_0 k_0 \omega_0.$$

At the second step of the D-cascade, a carrier mode has the frequency ω_1, the distance to the next cascading mode

$$(\Delta\omega)_2 = |\omega_1 - \omega_2|$$

is chosen such that

$$|\omega_1 - \omega_2| = A_1 k_1 \omega_1,$$

and so on.

In this way, a recursive relation for the cascading modes can easily be obtained:

$$\sqrt{p_n} A_n = A(\omega_n \pm \omega_n A_n k_n). \tag{6}$$

Here, we let p_n denote the fraction of energy transported from the cascading mode

$$A_n = A(\omega_n)$$

to the cascading mode

$$A_{n+1} = A(\omega_{n+1}),$$

i. e., $A_{n+1} = \sqrt{p_n}A_n$. This fraction p is called the cascade intensity [15].

Equation (6) describes two chain equations: one chain equation with the plus sign for a direct D-cascade with $\omega_n < \omega_{n+1}$ and another chain equation for the inverse D-cascade with $\omega_n > \omega_{n+1}$.

Speaking generally, the cascade intensity

$$p_n = p_n(A_0, \omega_0, n)$$

might be a function of the excitation parameters A_0, ω_0 and the step n. But because numerous experiments have established that p_n depends only on the excitation parameters and does not depend on the step n, all the formulas below are given for a constant cascade intensity. Accordingly, the notation p is used instead of p_n. This means in particular that

$$A_{n+1} = \sqrt{p}A_n = p^{n/2}A_0,$$

and because the energy behaves as

$$E_n \sim A_n^2,$$

it follows that

$$E_n \sim p^n A_0^2,$$

i. e., the energy spectrum of the D-cascade amplitudes has an exponential form.

Taking a Taylor expansion of the right-hand side of the chain equation and retaining only the first two terms of the resulting series, we can derive an ordinary differential equation describing stationary amplitudes of the cascading modes. The consequent steps of the ICEM are given below.

1) Relation between neighboring amplitudes:

$$A_{n+1} = \sqrt{p}A_n. \tag{7}$$

2) Condition for the maximal increment:

$$\frac{|\omega_{n+1} - \omega_n|}{f(\omega_n A_n k_n)} = 1, \tag{8}$$

where $f(\omega_n A_n k_n)$ is a known function of the product $\omega_n A_n k_n$. For instance,

$$f(\omega_n A_n k_n) = \omega_n A_n k_n$$

for gravity surface waves with the small parameter of the order of 0.1–0.2. Examples for bigger nonlinearities and also for other wave types can be found in [17, 18].

3) Chain equations:

$$A_{n+1} \equiv A(\omega_{n+1}) =$$
$$= A(\omega_n \pm f(\omega_n A_n k_n)) = \sqrt{p}A(\omega_n), \tag{9}$$

where the plus sign should be taken for the direct cascade and the minus for the inverse cascade.

4) Approximate ODE(s) for the amplitude A_n:

$$\sqrt{p}A_n \approx A_n \pm A_n' f(\omega_n A_n k_n). \tag{10}$$

5) Discrete energy spectrum $E_n \sim A_n^2$.
6) Spectral density

$$S(\omega) = \left| \lim_{n \to \infty} \frac{dE_n}{d\omega_n} \right|. \tag{11}$$

In particular, the formula below gives an explicit expression for wave amplitudes (for the direct cascade) in the case of a small initial nonlinearity $\varepsilon \sim 0.1$–0.25:

$$A(\omega_n) = (\sqrt{p} - 1) \int \frac{d\omega_n}{\omega_n k(\omega_n)}. \tag{12}$$

Let us remark, that it is known that if the autocorrelation function involves temporal measurements at a single point, then the power spectrum has the units m^2/Hz. It is easy to verify that the spectral density $S(\omega)$ has the correct units. Indeed, as we compute the amplitudes at single points, the amplitudes $A(\omega_n)$ have the units m, then their squares $A(\omega_n)^2$ have the units m^2, and the spectral density $S(\omega)$ has the units m^2/Hz.

We illustrate the method described above with an example of gravity surface waves, with the weak nonlinearity $\varepsilon \sim 0.1$–0.25, first computed in [15].

Example: *Gravity surface waves.* In this case,

$$\omega^2 = gk,$$

and

$$f(\omega_n A_n k_n) = \omega_n A_n k_n$$

(see [16]), which yields

$$\frac{|(\Delta \omega)_n|}{\omega_n A_n k_n} = 1. \tag{13}$$

Then

$$\sqrt{p} - 1 \approx \pm A_n' \omega_n^3, \tag{14}$$

and we obtain

$$A_n^{(Dir)} = g \frac{\left(1 - \sqrt{p}\right)}{2} \omega_n^{-2} + C^{(Dir)}, \tag{15}$$

$$A_n^{(Inv)} = -g \frac{\left(1 - \sqrt{p}\right)}{2} \omega_n^{-2} + C^{(Inv)}, \tag{16}$$

$$E(\omega_n)^{(Dir)} \sim \left[g \, \frac{\left(1 - \sqrt{p}\right)}{2} \omega_n^{-2} + C^{(Dir)} \right]^2, \quad (17)$$

$$E(\omega_n)^{(Inv)} \sim \left[-g \, \frac{\left(1 - \sqrt{p}\right)}{2} \omega_n^{-2} + C^{(Inv)} \right]^2, \quad (18)$$

with

$$C^{(Dir)} = A_0 - g \, \frac{\left(1 - \sqrt{p}\right)}{2} \omega_0^{-2}, \quad (19)$$

$$C^{(Inv)} = A_0 + g \, \frac{\left(1 - \sqrt{p}\right)}{2} \omega_0^{-2}. \quad (20)$$

Because we assume that $p = \text{const}$, the specific choice of the excitation parameters allows computing the cascade intensity p explicitly as a function of the excitation parameters A_0, ω_0. Indeed, for example, we consider the case of a direct cascade and the choose the excitation parameter

$$C^{(Dir)} = 0.$$

We obtain

$$A_0 - g \, \frac{\left(1 - \sqrt{p}\right)}{2} \omega_0^{-2} = 0,$$

then

$$p = \left(1 - \frac{2A_0\omega_0^2}{g} \right)^2. \quad (21)$$

Accordingly,

$$E(\omega_n)^{(Dir)} = (A_0\omega_0\omega_n^{-2})^2 \propto \omega_n^{-4}$$

and the spectral density is given by

$$S(\omega)_{grav} \propto \omega^{-5}.$$

This corresponds to the celebrated Phillips spectrum [19], and also to the real oceanic measurements coined the JONSWAP wave spectrum for wind-generated waves. The JONSWAP spectrum is the standard wave spectrum input used in practical engineering, e. g., for practical fatigue calculation for off-shore structures.

A similar computation for capillary waves with a small nonlinearity yields

$$E(\omega_n)^{(Dir)} \propto \omega_n^{-4/3}, \quad S(\omega)_{cap} \propto \omega^{-7/3}.$$

3. D-SPECTRA OF GRAVITY–CAPILLARY WAVES

3.1. Computation of the spectrum

In this case, computation of $A_n = A(\omega_n)$ is very tedious and is omitted here. Instead, we compute $A_n = A(k_n)$ by changing the integration variable in (12); some preliminary computations are necessary:

$$\omega(k) = \sqrt{g\,k + \sigma k^3}, \quad (22)$$

$$\omega'_k = \frac{g + 3\sigma k^2}{2\sqrt{g\,k + \sigma k^3}}, \quad (23)$$

$$\frac{\omega'_k}{\omega(k)k} = \frac{g + 3\sigma k^2}{\left(2\sqrt{g\,k + \sigma k^3}\right)\left(\sqrt{g\,k + \sigma k^3}\right)k} = \quad (24)$$

$$= \frac{g + 3\sigma k^2}{2k(g\,k + \sigma k^3)}, \quad (25)$$

$$A_{gr-cap}(k) = (\sqrt{p} - 1) \int \frac{\omega'_k \, dk}{\omega(k)k} = \quad (26)$$

$$= \frac{\sqrt{p} - 1}{2} \int \frac{(g + 3\sigma k^2)\, dk}{k(g\,k + \sigma k^3)}. \quad (27)$$

This indefinite integral can be computed explicitly,

$$\int \frac{(g + 3\sigma k^2)\, dk}{k(gk + \sigma k^3)} =$$

$$= 2\sqrt{\frac{\sigma}{g}} \arctan\left(\sqrt{\frac{\sigma}{g}}k\right) - k^{-1} + \text{const}, \quad (28)$$

which yields (for the direct cascade)

$$A_{gr-cap}(k_n) =$$

$$= \frac{1 - \sqrt{p}}{2}\left[k_n^{-1} - 2\sqrt{\frac{\sigma}{g}}\arctan\left(\sqrt{\frac{\sigma}{g}}k_n\right)\right] -$$

$$- \frac{1 - \sqrt{p}}{2}\left[k_0^{-1} - 2\sqrt{\frac{\sigma}{g}}\arctan\left(\sqrt{\frac{\sigma}{g}}k_0\right)\right] + A_0. \quad (29)$$

Keeping in mind that the cascade intensity p is constant, we obtain

$$A_{gr-cap}(k_n) \sim \left[k_n^{-1} - 2\sqrt{a}\arctan\left(\sqrt{a}k_n\right)\right], \quad (30)$$

$$E_{gr-cap}(k_n) \sim \left[k_n^{-1} - 2\sqrt{a}\arctan\left(\sqrt{a}k_n\right)\right]^2, \quad (31)$$

$$S(k)_{gr-cap} \sim$$

$$\sim \left| \left(\frac{1}{k^2} + \frac{2a^2}{1 + a^2k^2} \right)\left(\frac{1}{k} - 2a\arctan(a\,k) \right) \right|, \quad (32)$$

where the notation $a = \sigma/g$ is used.

The D-cascade among gravity–capillary water waves with $\varepsilon \sim 0.1$ is formed on the time scale of the order of dozens of seconds [20]; for instance, for a wave with the wavelength 10 cm, the corresponding time scale is 25 seconds, and the D-cascade would be easy to observe in a laboratory experiment.

3.2. Consistency check

Energy spectra for pure gravity and pure capillary waves were obtained in [15] in the form $A = A(\omega)$. To check the consistency of (30) with the results obtained above for gravity–capillary waves, we have to rewrite them in the form $A = A(k)$ as follows:

$$A(k) = (\sqrt{p} - 1) \int \frac{\omega'_k dk}{\omega(k)k}. \tag{33}$$

For surface gravity waves, this yields

$$A_{grav}(k) = (\sqrt{p} - 1) \int \frac{\omega'_k dk}{\omega(k)k} =$$
$$= (\sqrt{p} - 1) \int \left(\frac{\sqrt{g}}{2k^{1/2}} \frac{1}{\sqrt{g}\, k^{1/2}k} \right) dk =$$
$$= \frac{\sqrt{p} - 1}{2} \int \frac{dk}{k^2},$$
$$A_{grav}(k) = \frac{1 - \sqrt{p}}{2}(k^{-1} - k_0^{-1}) + A_0, \tag{34}$$

and for capillary waves, we have

$$A_{cap}(k) = (\sqrt{p} - 1) \int \frac{\omega'_k dk}{\omega(k)k} =$$
$$= (\sqrt{p} - 1) \int \left(\frac{3\sqrt{\sigma}\, k^{1/2}}{2} \frac{1}{\sqrt{\sigma}k^{3/2}\, k} \right) dk =$$
$$= \frac{3\left(\sqrt{p} - 1\right)}{2} \int \frac{dk}{k^2},$$
$$A_{cap}(k) = \frac{3\left(1 - \sqrt{p}\right)}{2}(k^{-1} - k_0^{-1}) + A_0. \tag{35}$$

To avoid tedious calculations, we consider a special choice of the excitation parameters, such that $A(k) \propto \propto k^{-1}$; then

$$E_{grav}(k) \propto k^{-2}, \quad S_{grav}(k) \propto k^{-3}, \tag{36}$$

$$E_{cap}(k) \propto k^{-2}, \quad S_{cap}(k) \propto k^{-3}. \tag{37}$$

We now discuss the expression in the integrand in (27) in two limit cases: (a) $\sigma \to 0$ and (b) $g \to 0$.

In the first case, integral (27) transforms into $\int k^{-2}dk$, and consequently

$$A_{gr-cap}(k) = \frac{1 - \sqrt{p}}{2}\big[k^{-1} - k_0^{-1}\big] + A_0 =$$
$$= A_{grav}(k). \tag{38}$$

In the second case, integral (27) transforms into $\int 3k^{-2}dk$, and consequently

$$A_{gr-cap}(k) = \frac{3\left(1 - \sqrt{p}\right)}{2}\big[k^{-1} - k_0^{-1}\big] + A_0 =$$
$$= A_{cap}(k). \tag{39}$$

This means that the expression for the energy spectrum of gravity–capillary waves is consistent with the previously obtained results for pure gravity and pure capillary waves, i. e., the D-model itself is consistent.

Another important check follows from the standard relation

$$\int S(\omega)\, d\omega = \int S(k)\, dk$$

(in one spatial dimension). Rewriting it as

$$S(k) = S(\omega)\frac{d\omega}{dk},$$

we can compute $S(k)_{grav}$ and $S(k)_{cap}$:

$$S(k)_{grav} \propto k^{-1/2}k^{-5/2} = k^{-3}, \tag{40}$$

and

$$S(k)_{cap} \propto k^{-7/2}k^{1/2} = k^{-3}, \tag{41}$$

which is in accordance with formulas (36) and (37).

4. CONNECTION OF THE D-MODEL WITH OTHER MODELS

On different scales in time and space, there are many models describing various phenomena and processes in nonlinear wave interaction. Some of these models have the same time and space scale as the D-cascade. In this section we show that a direct mathematical relation between the D-model and a few other known models exists.

(I) The computation of the D-cascade spectra demonstrated above and in [21] has been performed in the framework of the nonlinear Schrödinger equation or its modifications. As modulation instability exists in other evolutionary dispersive nonlinear partial differential equations, e. g., in generalized versions of the Korteweg–de Vries equation [22, 23], Hasegawa–Mima

equation [24], and others; the ICEM can also be directly applied for these equations. All the difference between different equation would be "hidden" in the form of the chain equation.

(II) The computation kindly provided to us by Miguel Onorato in the general discussion at the Workshop "Wave Turbulence" (Ecole de Physique, Les Houches, France, 2012) shows a connection between the D-cascade and Zakharov's equation.

We first rewrite (8) as

$$\tilde{\omega}_R = \omega_0 + \omega_0 A_0 k_0, \tag{42}$$

$$\tilde{\omega}_L = \omega_0 - \omega_0 A_0 k_0 \tag{43}$$

and consider a system of three discrete waves using the decomposition

$$a_k = b_0 \delta_k^0 + b_L \delta_k^L + b_R \delta_k^R. \tag{44}$$

Here, b_0 is the carrier wave and b_L and b_R are the left and right sidebands;

$$L = k_0 + \Delta k, \quad R = k_0 - \Delta k.$$

Assuming that b_L and b_R are small compared with b_0 and neglecting nonlinear terms in the sidebands amplitude, after substituting (44) into Zakharov's equation

$$\frac{\partial a_1}{\partial t} + i\omega_1 a_1 = -i \int dk_{2,3,4} T_{1,2,3,4} a_2^* a_3 a_4 \delta_{12}^{34}, \tag{45}$$

we obtain

$$\frac{db_0}{dt} + i\omega_0 b_0 = -iT_{0,0,0,0}|b_0|^2 b_0,$$
$$\frac{db_L}{dt} + i\omega_L b_L =$$
$$= -i2T_{L,0,L,0}|b_0|^2 b_L - iT_{L,R,0,0} b_0^2 b_R^*, \tag{46}$$
$$\frac{db_R}{dt} + i\omega_R b_R =$$
$$= -i2T_{R,0,R,0}|b_0|^2 b_R - iT_{R,L,0,0} b_0^2 b_L^*.$$

If we are interested only in the interaction of each sideband with the carrier wave, independently of the other, then we can easily find the dispersion relation for b_L and b_R. The solution of the first equation is straightforward:

$$b_0 = \tilde{b}_0 \exp\left[-i(\omega_0 + T_{0,0,0,0}|\tilde{b}_0|^2)t\right], \tag{47}$$

whence

$$\tilde{\omega}_0 = \omega_0 + T_{0,0,0,0}|\tilde{b}_0|^2, \tag{48}$$

with

$$T_{0,0,0,0} = k^3,$$

but we must recall that the Zakharov equation is written for the wave action variable that is related to the surface elevation η_0 as

$$\eta_0 = \sqrt{2k_0/\omega_0}\, b_0,$$

and therefore the dispersion relation is

$$\tilde{\omega}_0 = \omega_0 \left(1 + \frac{1}{2}k_0^2 \eta_0^2\right). \tag{49}$$

Neglecting the interaction between the two sidebands, we obtain

$$\frac{db_L}{dt} + i\omega_L b_L = -i2T_{L,0,0,L}|b_0|^2 b_L,$$
$$\frac{db_R}{dt} + i\omega_R b_R = -i2T_{R,0,0,R}|b_0|^2 b_R. \tag{50}$$

The nonlinear dispersion relation for the sidebands is

$$\tilde{\omega}_L = \omega_L + 2T_{L,0,0,L}|\tilde{b}_0|^2,$$
$$\tilde{\omega}_R = \omega_R + 2T_{R,0,0,R}|\tilde{b}_0|^2. \tag{51}$$

The diagonal part of the coupling coefficient has the form

$$T_{1,2,1,2} = k_1 k_2 \min(k_1, k_2),$$

and therefore

$$T_{L,0,0,L} = k_0 k_L^2 = k_0 (k_0 - \Delta k)^2,$$
$$T_{R,0,0,R} = k_0^2 k_R = k_0^2 (k_0 + \Delta k). \tag{52}$$

Hence,

$$\tilde{\omega}_L = \sqrt{g(k_0 - \Delta k)} + \omega_0 (k_0 - \Delta k)^2 \eta_0^2,$$
$$\tilde{\omega}_R = \sqrt{g(k_0 + \Delta k)} + \omega_0 k_0 (k_0 + \Delta k)\eta_0^2. \tag{53}$$

Taylor expanding in Δk and neglecting nonlinear dispersive terms of the cubic order, we write the resonant condition as

$$2\tilde{\omega}_0 - \tilde{\omega}_L - \tilde{\omega}_R = -\eta_0^2 k_0^2 \omega_0 + \frac{\omega_0 \Delta k^2}{4k_0^2} = 0, \tag{54}$$

then

$$\frac{\Delta k}{k_0} = 2k_0 \eta_0 = 2\varepsilon, \tag{55}$$

where ε is the steepness of the carrier wave. Because

$$2\Delta k/k_0 = \Delta\omega/\omega_0,$$

Eq. (55) can be rewritten as

$$\frac{\Delta\omega}{\omega_0} = \varepsilon. \tag{56}$$

Equation (56) coincides with (5) and therefore corresponds to the instability maximum in the Benjamin–Feir instability curve.

Thus, we have shown that each step of the D-cascade, Eq. (9), describes an exact 4-wave resonance in the Zakharov's equation between the modes with nonlinear Stokes-corrected frequencies.

(III) The D-cascade as a whole can be regarded as a resonance cluster formed by a few connected exact resonances of nonlinear Stokes modes. This means that in order to deduce the chain equation, we do not need the modulation instability; in fact, the MI is just a suitable mathematical language for describing an energy cascade in the focusing evolutionary NPDEs. The general mathematical object describing energy cascades both for focusing and nonfocusing NPDEs is a resonance cluster formed by a few connected exact resonances of nonlinear Stokes modes.

Objects of this type are studied by the homotopy analysis method [25]. Recent application of this method allowed describing a steady-state resonance of multiple wave interactions in deep water [26]; numerical simulation with Zakharov's equations demonstrates qualitative agreement with the results obtained by the homotopy analysis method.

In contrast to perturbative methods usually applied for studying nonlinear problems, this method does not introduce a small parameter and works in realistic physical setups.

(IV) Freak or rogue waves are a quite popular subject in the last few decades, with different models describing their appearance having been proposed. Three main types of models are used for describing rogue wave formation: linear (spatial focusing or focusing due to dispersion), weakly nonlinear (focusing due to modulation instability), and essentially nonlinear wave interaction [27]. In the real numerical models for weather and ocean wave field prediction, the so-called Benjamin–Feir index (BFI) is successfully used for characterizing the probability of the freak wave appearance.

The BFI is defined as the ratio of the wave steepness $k_0 A$ to the spectrum width $\Delta\omega/\omega_0$ and the probability of high waves occurrence is nonzero if BFI = 1 or bigger, i. e., beginning with

$$\text{BFI} = \frac{k_0 A}{\Delta\omega/\omega_0} = 1. \qquad (57)$$

However, it was shown quite recently that freak waves can also occur in systems where the MI is absent and BFI = 0 [28].

This apparent contradiction is easy to explain if we note that (57) is equivalent to the condition for

the maximal increment (8) and consequently to chain equation (9), which in turn can be described without invoking modulation instability, as was demonstrated in (III).

5. DISCUSSION

In this paper, we have demonstrated how to apply the increment chain equation method for computing the energy spectrum of an ensemble of weakly nonlinear gravity–capillary waves with the dispersion function $\omega^2 = gk + \sigma k^3$ and a small parameter $\varepsilon \sim 0.1$–0.25. The energy spectrum is computed analytically as a function of g/σ, see (32); the D-spectra in the two limit cases — pure gravity, $\omega^2 = gk$, and pure capillary, $\omega^2 = \sigma k^3$, — coincide with the known results first presented in [15].

The D-cascade among gravity–capillary water waves is formed at the time scale of the order of dozens of seconds and can easily be observed in laboratory experiment. Various characteristics of the D-cascade in this case (its direction, possible scenarios of cascade termination, etc.) can be studied analytically, similarly to the case of pure gravity waves presented in [21]. This work is in progress.

We have also demonstrated that the D-cascade, though being a novel model, is directly connected with other important topics widely studied in fluid mechanics, e. g., resonance clustering of modes with nonlinear Stokes corrected frequencies or the criterion for the freak wave appearance. This allows transferring the ideas, concepts, and approaches from one scientific area to another and studying them in a new setting. Knowledge of the connections between different models is crucial in the understanding and forecasting specifics of the energy transport in a variety of multicomponent nonlinear wave systems occurring virtually everywhere from oceanography to optics, from plasma physics to acoustics.

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