ЖЭТФ, 2014, том 146, вып. 3 (9), стр. 471–482

ROBUST STATIONARY DISTRIBUTED DISCORD IN THE JORDAN–WIGNER FERMION SYSTEM UNDER PERTURBATIONS OF THE INITIAL STATE

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Received April 22, 2014

We investigate the Jordan–Wigner fermion clusters with a stationary distributed quantum pairwise discord. Such clusters appear after the Jordan–Wigner transformation of a spin chain governed by the nearest-neighbor XY Hamiltonian with the particular initial state having one polarized node. We show that the quantum discord stationarity in such systems is not destroyed by the "parasitic" polarization of at least two types. The first type appears because the initial state with a single polarized node is hardly realizable experimentally, and therefore the low polarization of neighboring nodes must be taken into account. The second is the unavoidable additional noise polarization of all nodes. Although the stationarity may not be destroyed by perturbations of the above two types, the parasitic polarizations deform the pairwise discord distribution and may destroy clusters of correlated fermions with equal pairwise discords. Such deformations are studied in this paper.

DOI: 10.7868/S0044451014090077

1. INTRODUCTION

Quantum correlations are responsible for the effective operation of quantum information devices having the essential advantages in comparison with their classical counterparts [1–14]. According to the current standpoint, the total correlations in a multi-particle system are described by mutual information, and quantum correlations for both pure and mixed states are characterized by the quantum discord [2–5, 10–14].

In studing quantum correlations, it is important to choose a proper quantum system possessing the desirable properties and realizable in practice. In this regard, we note the chains of nuclear spins, which are suitable for realization of quantum registers and quantum devices transferring and manipulating quantum information. It is challenging that the multiple quantum (MQ) NMR methods [15, 16] allow constructing the XY interacting spin chains experimentally. Moreover, using the NMR method, it is possible to create conditions providing the concentration of polarization at a single node of the chain (up to the unavoidable experimental errors) [17]. The dynamics of quantum correlations in this model was first studied in Ref. [18]. Moreover, it was shown recently [19] that such chains are convenient for studying the dependence of the discord on the representation basis of the density matrix describing the quantum system state. The quantum discord calculated for interacting nuclear spins differs from that between the fermions arising after the Jordan–Wigner transformation [20] of the density matrix operator [19, 21]. It turned out that the quantum discord between fermions may exhibit very interesting properties [19], which have not been observed in the discord between nuclear spins. The most important property is the stationarity of the pairwise discord in a fermion cluster with the above initial state of a spin-1/2 chain. Besides, if we polarize the proper initial node, then the quantum discord is the same for any fermion pair in the selected fermion cluster. Apparently, this fact is important for the implementation of fermion registers in quantum devices because all fermion nodes are equivalent from the quantum correlation standpoint.

The existence of such clusters motivates the study of their stability with respect to both experimental errors in creating single-node polarization and noise effects. We note that the stability of spin dynamics in the presence of different types of noise is a relevant problem because noise is unavoidable in any quantum process. In particular, the fidelity of the perfect state transfer

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(in the absence of noises) under noise perturbations of the coupling constants in the Hamiltonian was considered in Refs. [22–26] for two chains: a completely engineered chain and a chain with remote endnodes. In both cases, the important result is that the noise reduces the fidelity without changing the state transfer time.

In this paper, we study the stability of the discord distribution relative to perturbations of the initial state in a homogeneous spin chain (i. e., the coupling constants in the Hamiltonian are assumed to be stable). We show that the stationarity of the quantum discord in the system with a single initially polarized node may not be destroyed by the additional low polarizations of the neighboring nodes, which unavoidably appear in the experiment. This perturbation just leads to the deformation of the pairwise quantum discord and may eventually destroy the clusters of fermions with the equal pairwise discord. The threshold value of the low polarization is found. We also consider the deformation of the stationary discord distribution caused by the noise polarization appearing in all nodes of the spin chain. It is remarkable that the discord stationarity is not disturbed in both cases.

The paper is organized as follows. The Jordan–Wigner transformation of the XY Hamiltonian with nearest-neighbor interactions is briefly discussed in Sec. 2. The stability of the pairwise discord stationarity in the Jordan–Wigner fermion system of a spin-$1/2$ chain with single initially polarized nodes under perturbations of the initial state is demonstrated in Secs. 3 and 4 with numerical simulations of the spin dynamics of a 17-node chain. First, in Sec. 3, the parasitic polarization of two neighboring nodes (with respect to the selected inner polarized node) is considered. Then, in Sec. 4, the noise polarization of all nodes is taken into account using the perturbation method. Deformations of the fermion clusters with equal pairwise discord under the above perturbations are also considered in Secs. 3 and 4. The basic results are discussed in Sec. 5. A formula for calculating the discord in the X-type density matrix [27] is represented in the Appendix.

2. JORDAN–WIGNER TRANSFORMATION OF THE XY HAMILTONIAN WITH THE NEAREST-NEIGHBOR INTERACTION

We study quantum correlations in the one-dimensional open spin-$1/2$ chain of N nodes governed by the XY Hamiltonian with the nearest-neighbor interactions,

$$H = \omega_0 \sum_{i=1}^{N} I_{iz} + D \sum_{i=1}^{N-1} (I_{ix} I_{(i+1)x} + I_{iy} I_{(i+1)y}), \quad (1)$$

where ω_0 is the Larmor frequency in the external magnetic field, D is the spin–spin coupling constant between the nearest neighbors, and $I_{i\alpha}$ ($i = 1, \ldots, N$, $\alpha = x, y, z$) is the ith spin projection on the α axis.

Following Refs. [18, 19, 21], we diagonalize Hamiltonian (1) using the Jordan–Wigner transformation method [20],

$$H = \sum_k \varepsilon_k \beta_k^\dagger \beta_k - \frac{1}{2} N \omega_0, \quad \varepsilon_k = D\cos(k) + \omega_0, \quad (2)$$

where the fermion operators β_j are expressed in terms of other fermion operators c_j by means of the Fourier transformation

$$\beta_k = \sum_{j=1}^{N} g_k(j) c_j, \quad (3)$$

and the fermion operators c_j are defined as [20]

$$c_j = (-2)^{j-1} I_{1z} I_{2z} \ldots I_{(j-1)z} I_j^- . \quad (4)$$

Here,

$$g_k(j) = \left(\frac{2}{N+1}\right)^{1/2} \sin kj, \quad k = \frac{\pi n}{N+1}, \quad n = 1, 2, \ldots, N. \quad (5)$$

We can readily express the projection operators I_{jz} in terms of the fermion operators c_j as

$$I_{jz} = c_j^\dagger c_j - \frac{1}{2}, \ \forall j. \quad (6)$$

Hereafter, diagonal representation (2) of the XY Hamiltonian is used to describe the dynamics of the density matrix associated with the spin-$1/2$ chain.

3. INITIAL STATE WITH THREE POLARIZED NODES

The dynamics of the Jordan–Wigner fermions associated with the spin-$1/2$ chain with a single initially polarized node j_0 has been studied in Refs. [18, 19, 21]. There, the stationarity of the pairwise discord in such systems is demonstrated and fermion clusters with equal pairwise discord are revealed.

We now consider the initial state with an inner initially polarized node j_0 (i. e., $1 < j_0 < N$) and assume the parasitic low polarization of two neighboring nodes; the initial density matrix is therefore given by

$$\rho_0 = \frac{1}{Z}\exp\left(\sum_{k=j_0-1}^{j_0+1} b_k I_{k,z}\right) = \frac{1}{2^N}\prod_{k=j_0-1}^{j_0+1}\left(1+2I_{kz}\,\mathrm{th}\,\frac{b_k}{2}\right), \quad (7)$$

$$Z = \mathrm{Tr}\left(\prod_{k=j_0-1}^{j_0+1} e^{b_k I_{kz}}\right) = 2^N\prod_{k=j_0-1}^{j_0+1}\mathrm{ch}\,\frac{b_k}{2},$$

where $b_j = \hbar\omega_{j0}/kT$, $\hbar$ is the Planck constant, k is the Boltzmann constant, and T is the temperature of the system.

The motivation for considering this initial state is discussed in the introduction. Namely, an experimental scheme may not provide the ideal single-node polarization. Hence, two neighboring nodes $j_0 \pm 1$ also acquire some polarization whenever j_0 is an inner node, i. e., $1 < j_0 < N$. This polarization might be called parasitic. As was shown in Refs. [19, 21], a fermion cluster with equal pairwise discords (which is our subject in this paper) may be obtained if the polarized node j_0 is an inner one. This case is related to the density matrix in Eq. (7) and is discussed below.

The evolution of the initial density matrix (7) in the fermion representation of Hamiltonian (2) is given by

$$\rho(t) = \exp\left\{-it\sum_k \varepsilon_k\beta_k^\dagger\beta_k\right\}\rho_0 \times \exp\left\{it\sum_k\varepsilon_k\beta_k^\dagger\beta_k\right\}. \quad (8)$$

Using the identity

$$\exp\left(-i\varphi\beta_k^\dagger\beta_k\right)\beta_k^\dagger\exp\left(i\varphi\beta_k^\dagger\beta_k\right) = \exp(-i\varphi)\beta_k^\dagger, \quad \forall\,\varphi, \quad (9)$$

we rewrite density matrix (8) as [18]

$$\begin{aligned}\rho(t) &= \frac{1}{2^N}\prod_{j=j_0-1}^{j_0+1}\left(1-\mathrm{th}\,\frac{b_j}{2}+2\,\mathrm{th}\,\frac{b_j}{2}\times\sum_{k,k'}\exp\{-it(\varepsilon_k-\varepsilon_{k'})\}\,g_k(j)g_{k'}(j)\beta_k^\dagger\beta_{k'}\right) = \\ &= A_0^{j_0} + \sum_{k,k'}A_{kk'}^{j_0}\exp\{-it(\varepsilon_k-\varepsilon_{k'})\}\,\beta_k^\dagger\beta_{k'} + \\ &+ \sum_{k,k',q,q'}A_{kqk'q'}^{j_0}\exp\{-it(\varepsilon_k+\varepsilon_q-\varepsilon_{k'}-\varepsilon_{q'})\}\times\beta_k^\dagger\beta_{k'}\beta_q^\dagger\beta_{q'} + \sum_{k,k',q,q',l,l'}A_{kqlk'q'l'}^{j_0}\times \\ &\times\exp\{-it(\varepsilon_k+\varepsilon_q+\varepsilon_l-\varepsilon_{k'}-\varepsilon_{q'}-\varepsilon_{l'})\}\times\beta_k^\dagger\beta_{k'}\beta_q^\dagger\beta_{q'}\beta_l^\dagger\beta_{l'}, \end{aligned}\quad (10)$$

where

$$A_0^{j_0} = \frac{1}{2^N}\prod_{j=j_0-1}^{j_0+1}\left(1-\mathrm{th}\,\frac{b_j}{2}\right), \quad (11)$$

$$\begin{aligned}A_{kk'}^{j_0} &= \frac{1}{2^{N-1}}\left(\left(1-\mathrm{th}\,\frac{b_{j_0}}{2}\right)\times\left(1-\mathrm{th}\,\frac{b_{j_0+1}}{2}\right)\mathrm{th}\,\frac{b_{j_0-1}}{2}g_k(j_0-1)g_{k'}(j_0-1) + \right.\\ &+\left(1-\mathrm{th}\,\frac{b_{j_0-1}}{2}\right)\left(1-\mathrm{th}\,\frac{b_{j_0+1}}{2}\right)\mathrm{th}\,\frac{b_{j_0}}{2}g_k(j_0)g_{k'}(j_0) + \\ &+\left(1-\mathrm{th}\,\frac{b_{j_0-1}}{2}\right)\left(1-\mathrm{th}\,\frac{b_{j_0}}{2}\right)\times\left.\mathrm{th}\,\frac{b_{j_0+1}}{2}g_k(j_0+1)g_{k'}(j_0+1)\right),\end{aligned}$$

$$\begin{aligned}A_{kqk'q'}^{j_0} &= \frac{1}{2^{N-2}}\left(\left(1-\mathrm{th}\,\frac{b_{j_0-1}}{2}\right)\mathrm{th}\,\frac{b_{j_0}}{2}\times\mathrm{th}\,\frac{b_{j_0+1}}{2}g_k(j_0)g_{k'}(j_0)g_q(j_0+1)g_{q'}(j_0+1) + \right.\\ &+\left(1-\mathrm{th}\,\frac{b_{j_0}}{2}\right)\mathrm{th}\,\frac{b_{j_0-1}}{2}\,\mathrm{th}\,\frac{b_{j_0+1}}{2}\times g_k(j_0-1)g_{k'}(j_0-1)g_q(j_0+1)g_{q'}(j_0+1) + \\ &+\left(1-\mathrm{th}\,\frac{b_{j_0+1}}{2}\right)\mathrm{th}\,\frac{b_{j_0-1}}{2}\,\mathrm{th}\,\frac{b_{j_0}}{2}g_k(j_0-1)\times\left.g_{k'}(j_0-1)g_q(j_0)g_{q'}(j_0)\right),\end{aligned}$$

$$A_{kqlk'q'l'}^{j_0} = \frac{1}{2^{N-3}}\mathrm{th}\,\frac{b_{j_0-1}}{2}\,\mathrm{th}\,\frac{b_{j_0}}{2}\,\mathrm{th}\,\frac{b_{j_0+1}}{2}\times g_k(j_0-1)g_{k'}(j_0-1)g_q(j_0)g_{q'}(j_0)g_l(j_0+1)g_{l'}(j_0+1).$$

Finally, using the fermion anticommutation relations [28] and relations among the coefficients $A_{kqk'q'}^{j_0}$ and $A_{kqlk'q'l'}^{j_0}$,

$$\sum_{q=1}^N A_{kqqk'}^{j0} = \sum_{q=1}^N A_{klqk'ql'}^{j0} = \sum_{q=1}^N A_{kqlqk'l'}^{j0} = \sum_{q=1}^N A_{klqqk'l'}^{j0} = 0, \quad (12)$$

which follow from the orthogonality relation

$$\sum_{k=1}^{N} g_k(j) g_k(l) = \delta_{jl}, \tag{13}$$

we easily transform Eq. (10) to the canonic form

$$\begin{aligned}
\rho(t) = {} & \frac{1}{2^N} \prod_{j=j_0-1}^{j_0+1} \left(1 - \operatorname{th}\frac{b_j}{2}\right) + \\
& + \sum_{k,k'} A^{j_0}_{kk'} \exp\{-it(\varepsilon_k - \varepsilon_{k'})\} \beta^\dagger_k \beta_{k'} - \\
& - \sum_{\substack{k,k',q,q' \\ k\neq q,\, k'\neq q'}} A^{j_0}_{kqk'q'} \exp\{-it(\varepsilon_k + \varepsilon_q - \varepsilon_{k'} - \varepsilon_{q'})\} \times \\
& \times \beta^\dagger_k \beta^\dagger_q \beta_{k'} \beta_{q'} - \sum_{\substack{k,k',q,q',l,l' \\ k\neq q\neq l,\, k'\neq q'\neq l'}} A^{j_0}_{kqlk'q'l'} \times \\
& \times \exp\{-it(\varepsilon_k + \varepsilon_q + \varepsilon_l - \varepsilon_{k'} - \varepsilon_{q'} - \varepsilon_{l'})\} \times \\
& \times \beta^\dagger_k \beta^\dagger_q \beta^\dagger_l \beta_{k'} \beta_{q'} \beta_{l'}.
\end{aligned} \tag{14}$$

3.1. Reduced density matrix

Studying quantum correlations, we consider only the pairwise discord. First, in calculating the discord between the nth and mth fermions, we have to reduce density matrix (14) with respect to all nodes except the nth and mth, which leads to the marginal density matrix of the form

$$\begin{aligned}
\rho^{j_0}_{nm}(t) = {} & B^{j_0}_{nm} + \sum_{k,k'=n,m} B^{j_0}_{nmkk'} \times \\
& \times \exp\{-it(\varepsilon_k - \varepsilon_{k'})\} \beta^\dagger_k \beta_{k'} + C^{j_0}_{nm} \beta^\dagger_n \beta^\dagger_m \beta_m \beta_n,
\end{aligned} \tag{15}$$

where all coefficients are independent of the time t:

$$\begin{aligned}
B^{j_0}_{nm} = {} & \frac{1}{4} \prod_{j=j_0-1}^{j_0+1} \left(1 - \operatorname{th}\frac{\beta_j}{2}\right) + \\
& + 2^{N-3} \sum_{k\neq n,m} A^{j_0}_{kk} + \\
& + 2^{N-4} \sum_{\substack{k,q \\ k\neq q;\, k,q\neq n,m}} (-A^{j_0}_{kqkq} + A^{j_0}_{kqqk}) - 2^{N-5} \times \\
& \times \sum_{\substack{k,q,l\neq n,m \\ k\neq q\neq l}} (-A^{j_0}_{kqlkql} + A^{j_0}_{kqlklq} + A^{j_0}_{kqlqkl} - \\
& - A^{j_0}_{kqllkq} - A^{j_0}_{kqlqlk} + A^{j_0}_{kqllqk}), \\
B^{j_0}_{nmkk'} = {} & 2^{N-2} A^{j_0}_{kk'} + 2^{N-3} \times \\
& \times \sum_{q,q\neq k,k'} (-A^{j_0}_{kqk'q} + A^{j_0}_{kqqk'} + A^{j_0}_{qkk'q} - A^{j_0}_{qkqk'}) - \\
& - 2^{N-4} \sum_{\substack{q,l \\ q\neq l\neq k\neq k'}} (-A^{j_0}_{kqlk'ql} + A^{j_0}_{kqlk'lq} + \\
& + A^{j_0}_{kqlqk'l} - A^{j_0}_{kqllk'q} - A^{j_0}_{kqlqlk'} + \\
& + A^{j_0}_{kqllqk'} + A^{j_0}_{qklk'ql} - \\
& - A^{j_0}_{qklk'lq} - A^{j_0}_{qklqk'l} + A^{j_0}_{qkllk'q} + A^{j_0}_{qklqlk'} - \\
& - A^{j_0}_{qkllqk'} - A^{j_0}_{qlkk'ql} + A^{j_0}_{qlkk'lq} + \\
& + A^{j_0}_{qlkqk'l} - A^{j_0}_{qlklk'q} - A^{j_0}_{qlkqlk'} + A^{j_0}_{qlklqk'}), \\
C^{j_0}_{nm} = {} & \\
= {} & 2^{N-2}(A^{j_0}_{nmmn} - A^{j_0}_{nmnm} - A^{j_0}_{mnmn} + A^{j_0}_{mnnm}) - \\
& - 2^{N-3} \sum_{k\neq n,m} (A^{j_0}_{knmkmn} - A^{j_0}_{knmknm} - \\
& - A^{j_0}_{kmnkmn} + A^{j_0}_{kmnknm} - A^{j_0}_{knmmkn} + \\
& + A^{j_0}_{knmnkm} + A^{j_0}_{kmnmkn} - A^{j_0}_{kmnnkm} + \\
& + A^{j_0}_{knmmnk} - A^{j_0}_{knmnmk} - A^{j_0}_{kmnmnk} + \\
& + A^{j_0}_{kmnnmk} - A^{j_0}_{nkmkmn} + A^{j_0}_{nkmknm} + \\
& + A^{j_0}_{mknkmn} - A^{j_0}_{mknknm} + A^{j_0}_{nkmmkn} - \\
& - A^{j_0}_{nkmnkm} - A^{j_0}_{mknmkn} + A^{j_0}_{mknnkm} - \\
& - A^{j_0}_{nkmmnk} + A^{j_0}_{nkmnmk} + A^{j_0}_{mknmnk} - \\
& - A^{j_0}_{mknnmk} + A^{j_0}_{nmkkmn} - A^{j_0}_{nmkknm} - \\
& - A^{j_0}_{mnkkmn} + A^{j_0}_{mnkknm} - A^{j_0}_{nmkmkn} + \\
& + A^{j_0}_{nmknkm} + A^{j_0}_{mnkmkn} - A^{j_0}_{mnknkm} + \\
& + A^{j_0}_{nmkmnk} - A^{j_0}_{nmknmk} - A^{j_0}_{mnkmnk} + A^{j_0}_{mnknmk}).
\end{aligned} \tag{16}$$

Next, using the basis

$$|00\rangle,\ |01\rangle, |10\rangle,\ |11\rangle, \tag{17}$$

we represent the marginal density matrix operator (15) in the matrix form

$$\rho^{j_0}_{nm}(t) = \begin{pmatrix}
B^{j_0}_{nm} & 0 & 0 & 0 \\
0 & B^{j_0}_{nm} + B^{j_0}_{nmnn} & B^{j_0}_{nmnm} e^{-it(\varepsilon_n - \varepsilon_m)} & 0 \\
0 & B^{j_0}_{nmmn} e^{it(\varepsilon_n - \varepsilon_m)} & B^{j_0}_{nm} + B^{j_0}_{nmmm} & 0 \\
0 & 0 & 0 & B^{j_0}_{nm} + B^{j_0}_{nmmm} + B^{j_0}_{nmnn} + C^{j_0}_{nm}
\end{pmatrix}. \tag{18}$$

Formula (18) shows that the diagonal elements of marginal matrix (18) are independent of the time t. The t-dependence appears only in nondiagonal elements. However, the pairwise discord for the X-matrix depends on the absolute value $|B^{j_0}_{nmnm}|$ of nondiagonal elements (see the Appendix, Sec. 6) and, consequently, does not depend on t. This means that the perturbations considered in this section do not destroy the stationarity of the discord. However, the distribution of the discord becomes deformed, which may eventually destroy the fermion clusters with equal pairwise discords. Deformations caused by the parasitic polarizations of $(j_0 \pm 1)$th nodes are studied numerically in the next subsection.

3.2. Numerical simulations

We present numerical simulations for a particular case of an $N = 17$ node spin chain and assume that the polarization is initially concentrated at such a node j_0 that the fermion clusters with equal pairwise discord may be selected from the whole system of 17 fermions [21]. The interest in this case appears because such clusters might be promising in QIP devices as candidates for large quantum registers.

In accordance with Refs. [19, 21], such a cluster Cl appears in an odd-node spin chain in two cases. First, if j_0 is the middle node ($j_0 = 9$ in our case), then the cluster Cl is formed by the odd fermions. Second, if $N = 5+6i$ ($i = 1, 2, \ldots$) and $j_0 = 2(i+1)$ (in our case, with $N = 17$, $i = 2$ and hence $j_0 = 6$), then the cluster Cl is formed by all fermions except each third one. In both cases, the pairwise discord Q_{nm} between the nth and mth node is

$$Q_{nm} = \\ = \begin{cases} Q_0 = \text{const}, & n \in Cl,\ m \in Cl \\ 0, & n \notin Cl \ \text{and/or} \ m \notin Cl. \end{cases} \tag{19}$$

In other words, the discord Q_{nm} is zero if at least one of the subscripts n or m is not in the set Cl.

We therefore consider two clusters corresponding to two cases of the initially polarized node j_0:

1) $j_0 = 6$, the cluster of fermions with equal pairwise discords is formed by all fermions except each third one:

$$Cl = \{1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17\}, \tag{20}$$

2) $j_0 = 9$, the odd fermions form the cluster with equal pairwise discords:

$$Cl = \{1, 3, 5, \ldots, 17\}. \tag{21}$$

We characterize the polarization by the inverse temperatures

$$\begin{aligned} &b_{j_0+1} = b_{j_0-1} = b, \quad 0 \le b \le b_{j_0}, \\ &b_j = 0, \quad j \ne j_0, j_0 \pm 1. \end{aligned} \tag{22}$$

In practice, b must be such that $\mathrm{th}(b_{j_0\pm1}/2)$ is several times less than $\mathrm{th}(b_{j_0}/2)$. Below, we take $b_{j_0} = 10$ (the low-temperature limit).

The presence of parasitic polarization leads to a deformation of the ideal ($b_{j_0\pm1} = 0$) discord distribution shown in Fig. 2*a* (below). As a result, some spread of the discord appears in the cluster Cl. Besides, the zero-valued discords at $b = 0$ become nonzero for $b > 0$.

To characterize both these effects of parasitic polarization, we introduce the functions

$$\begin{aligned} Cl_{max}(b) &= \max_{(n,m)\in Cl} Q_{nm}(b), \\ Cl_{min}(b) &= \min_{(n,m)\in Cl} Q_{nm}(b), \end{aligned} \tag{23}$$

$$\begin{aligned} Z_{max}(b) &= \max_{(n,m)\notin Cl} Q_{nm}(b), \\ Z_{min}(b) &= \min_{(n,m)\notin Cl} Q_{nm}(b). \end{aligned} \tag{24}$$

Here, the notation $(n,m) \notin Cl$ means that $n \notin Cl$ and/or $m \notin Cl$. Functions Cl_{max} and Cl_{min} characterize the spread of the pairwise discord in clusters (20) and (21), while Z_{max} and Z_{min} characterize the spread of the "parasitic" discord that was zero in the unperturbed case:

$$\begin{aligned} &Cl_{min}(b) \le Q_{nm} \le Cl_{max}(b), \\ &\qquad\qquad n \in Cl, \quad m \in Cl, \\ &Z_{min}(b) \le Q_{nm} \le Z_{max}(b), \\ &\qquad\qquad n \notin Cl \ \text{and/or} \ m \notin Cl. \end{aligned} \tag{25}$$

The functions $Cl_{max}(b)$, $Cl_{min}(b)$, and $Z_{max}(b)$ for $j_0 = 6$ and $j_0 = 9$ are plotted in Figs. 1*a* and 1*b*. The function $Z_{min}(b)$ (although it is nonzero for $b > 0$) is not shown because it is not important in this section (but it is used in Sec. 4.1 to characterize the noise effects).

A natural question arises: what is the critical value of the parameter b (characterizing the value of the "parasitic" polarization) that still does not completely destroy the cluster Cl? We consider the value of b corresponding to the intersection point of $Cl_{min}(b)$ and $Z_{max}(b)$ as the critical value b_{cl} such that the cluster Cl does not exist if $b > b_{cl}$. The critical value is $b_{cl} = 0.480$ for $j_0 = 6$ and $b_{cl} = 0.533$ for $j_0 = 9$, as shown in Figs. 1*a*,*b*.

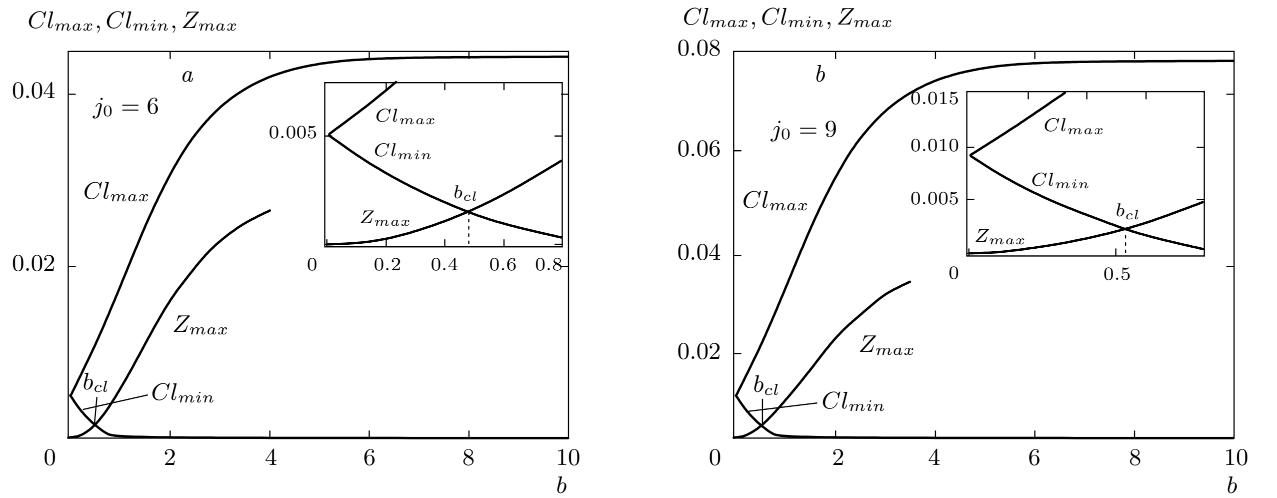

Fig. 1. Deformations of a fermion cluster in the system of $N = 17$ fermions. Graphs of the functions $Cl_{max}(b)$, $Cl_{min}(b)$, and $Z_{max}(b)$ are shown. The intersection point b_{cl} of the graphs $Cl_{min}(b)$ and $Z_{max}(b)$ can be considered the critical value of b such that the cluster Cl of the correlated fermions does not exist if $b > b_{cl}$. The insets show the graphs of $Cl_{max}(b)$, $Cl_{min}(b)$, and $Z_{max}(b)$ for small values of b. (a) The initially polarized node $j_0 = 6$, the critical value of the parameter b is $b_{cl} = 0.480$. (b) The initially polarized node $j_0 = 9$, the critical value $b_{cl} = 0.533$

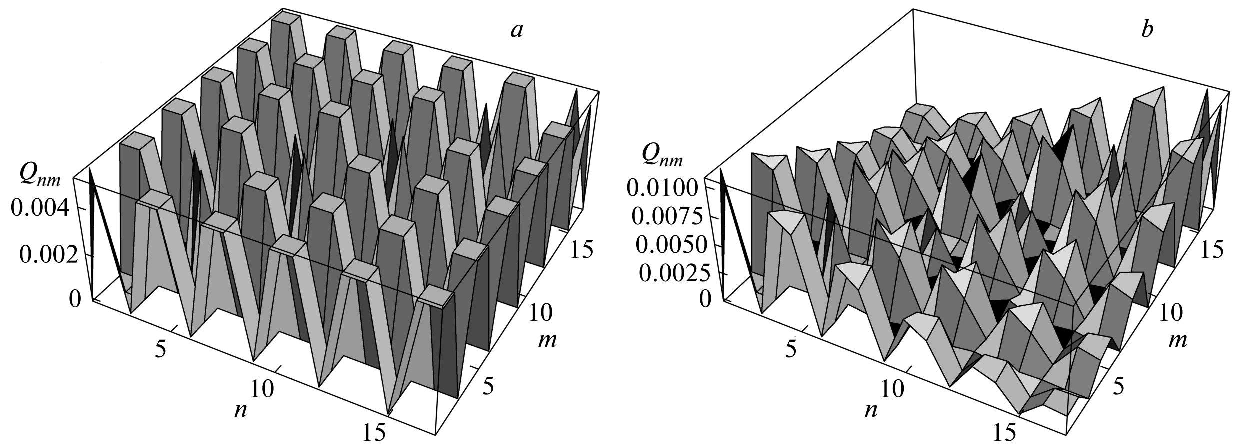

Fig. 2. Distribution of the discord Q_{nm} in the system of $N = 17$ fermions with the initially polarized node $j_0 = 6$. Here and in Fig. 4, we put $Q_{nn} \equiv 0$ following Ref. [21]. (a) Discord Q_{nm} in the fermion system without parasitic polarization, $b = 0$. (b) Discord Q_{nm} in the fermion system with the critical parasitic polarization of the 5th and 7th nodes, $b = b_{cl} = 0.480$

An example of the discord distribution for $j_0 = 6$ and $b = b_{cl} = 0.48$ (at the threshold value) is shown in Fig. 2b. We see that this distribution significantly differs from the unperturbed case $b = 0$ shown in Fig. 2a. However, we emphasize once again that the parasitic polarization does not lead to evolution of the discord in the considered fermion system, i. e., the stationarity of the pairwise discord is not destroyed.

4. NOISE EFFECT ON THE PAIRWISE DISCORD DISTRIBUTION

In this section, we assume that there is no parasitic polarization considered in Sec. 3 (i. e., $b_{j_0 \pm 1} = 0$), but there is noise polarization of all nodes. The initial density matrix (7) must be replaced with the following one:

$$\rho_0 = \frac{1}{2^N}\left(1 + I_{j_0 z}\left(2\,\mathrm{th}\,\frac{b_{j_0}}{2} + \epsilon\tilde{b}_{j_0}\right)\right) \times \prod_{\substack{k=1\\k\neq j_0}}^{N}\left(1 + I_{kz}\epsilon\tilde{b}_k\right), \quad Z = 2^N\,\mathrm{ch}\,\frac{b_{j_0}}{2}, \tag{26}$$

where ϵ is the double noise-polarization amplitude in a spin-1/2 chain, $\epsilon \ll 1$, and $\tilde{b}_j$ are the random numbers in the range $-\frac{1}{2} \le \tilde{b}_j \le \frac{1}{2}$. Using Eqs. (3) and (6), we transform the initial density matrix to [18]

$$\rho_0 = \frac{1}{2^N}\left(a_0^{j_0} + \sum_{k,k'} a_{kk'}^{j_0}\beta_k^\dagger\beta_{k'}\right) \times \prod_{\substack{j=1\\j\neq j_0}}^{N}\left(a_0^j + \sum_{k,k'} a_{kk'}^{j}\beta_k^\dagger\beta_{k'}\right), \tag{27}$$

where

$$a_0^j = \begin{cases} 1 - \mathrm{th}\,\frac{b_{j_0}}{2} - \frac{\epsilon\tilde{b}_{j_0}}{2}, & j = j_0, \\ 1 - \frac{\epsilon\tilde{b}_j}{2}, & j \neq j_0, \end{cases}$$
$$a_{kk'}^j = \begin{cases} \left(2\,\mathrm{th}\,\frac{b_{j_0}}{2} + \epsilon\tilde{b}_{j_0}\right) g_k(j_0)g_{k'}(j_0), & j = j_0, \\ \epsilon\tilde{b}_j g_k(j)g_{k'}(j), & j \neq j_0. \end{cases} \tag{28}$$

The quantities $a_{kk'}^j$, $j \neq j_0$, are proportional to ϵ and are therefore considered small parameters hereafter.

We now assume that the noise effect can be studied by the perturbation method for small ϵ. We then expand the initial density matrix in the $a_{kk'}^j$, $j \neq j_0$. We consider two density matrices corresponding to truncating the series and keeping the terms through the respective order $a_{kk'}^j$ and $(a_{kk'}^j)^2$. Taking the normalization condition (unit trace of the density matrix) into account, we write these matrices as follows:

$$\rho_{0i} = \frac{\tilde{\rho}_{0i}}{Z_i}, \quad Z_i = \mathrm{Tr}\,\tilde{\rho}_{0i}, \quad i = 1, 2,$$

$$\tilde{\rho}_{01} = \frac{1}{2^N}\left(a_0^{j_0} + \sum_{k,k'} a_{kk'}^{j_0}\beta_k^\dagger\beta_{k'}\right) \times \left(\prod_{\substack{l=1\\l\neq j_0}}^{N} a_0^l + \sum_{\substack{n=1\\n\neq j_0}}^{N}\left(\prod_{\substack{m=1\\m\neq j_0\neq n}}^{N} a_0^m\right) \times \sum_{k,k'} a_{kk'}^n\beta_k^\dagger\beta_{k'}\right),$$

$$\tilde{\rho}_{02} = \frac{1}{2^N}\left(a_0^{j_0} + \sum_{k,k'} a_{kk'}^{j_0}\beta_k^\dagger\beta_{k'}\right) \times \left(\prod_{\substack{i=1\\i\neq j_0}}^{N} a_0^l + \sum_{\substack{n=1\\n\neq j_0}}^{N}\left(\prod_{\substack{m=1\\m\neq j_0\neq n}}^{N} a_0^m\right)\sum_{k,k'} a_{kk'}^n\beta_k^\dagger\beta_{k'} + \sum_{\substack{n,n'=1\\n'\neq n\neq j_0}}^{N}\left(\prod_{\substack{m=1\\m\neq j_0\neq n\neq n'}}^{N} a_0^m\right) \times \sum_{k,k',q,q'} a_{kk'}^n a_{qq'}^{n'}\beta_k^\dagger\beta_{k'}\beta_q^\dagger\beta_{q'}\right). \tag{29}$$

The evolution of these matrices $e^{-itH}\rho_{0i}e^{itH}$, taking the fermion representation of the Hamiltonian (given by formula (2)) and relation (9) into account, is given by

$$\rho_i(t) = \frac{\tilde{\rho}_i(t)}{Z_i},$$

$$\tilde{\rho}_1(t) = \frac{1}{2^N}\left(a_0^{j_0} + \sum_{k,k'} a_{kk'}^{j_0}e^{-it(\varepsilon_k-\varepsilon_{k'})}\beta_k^\dagger\beta_{k'}\right) \times \left(\prod_{\substack{l=1\\l\neq j_0}}^{N} a_0^l + \sum_{\substack{n=1\\n\neq j_0}}^{N}\left(\prod_{\substack{m=1\\m\neq j_0\neq n}}^{N} a_0^m\right) \times \sum_{k,k'} a_{kk'}^n e^{-it(\varepsilon_k-\varepsilon_{k'})}\beta_k^\dagger\beta_{k'}\right),$$

$$\tilde{\rho}_2(t) = \frac{1}{2^N}\left(a_0^{j_0} + \sum_{k,k'} a_{kk'}^{j_0}e^{-it(\varepsilon_k-\varepsilon_{k'})}\beta_k^\dagger\beta_{k'}\right) \times \left(\prod_{\substack{l=1\\l\neq j_0}}^{N} a_0^l + \sum_{\substack{n=1\\n\neq j_0}}^{N}\left(\prod_{\substack{m=1\\m\neq j_0\neq n}}^{N} a_0^m\right) \times \sum_{k,k'} a_{kk'}^n e^{-it(\varepsilon_k-\varepsilon_{k'})}\beta_k^\dagger\beta_{k'} + \sum_{\substack{n,n'=1\\n'\neq n\neq j_0}}^{N}\left(\prod_{\substack{m=1\\m\neq j_0\neq n\neq n'}}^{N} a_0^m\right) \times \sum_{k,k',q,q'} a_{kk'}^n a_{qq'}^{n'} e^{-it(\varepsilon_k+\varepsilon_q-\varepsilon_{k'}-\varepsilon_{q'})} \times \beta_k^\dagger\beta_{k'}\beta_q^\dagger\beta_{q'}\right). \tag{30}$$

Formulas (30) can be transformed to form (10) with

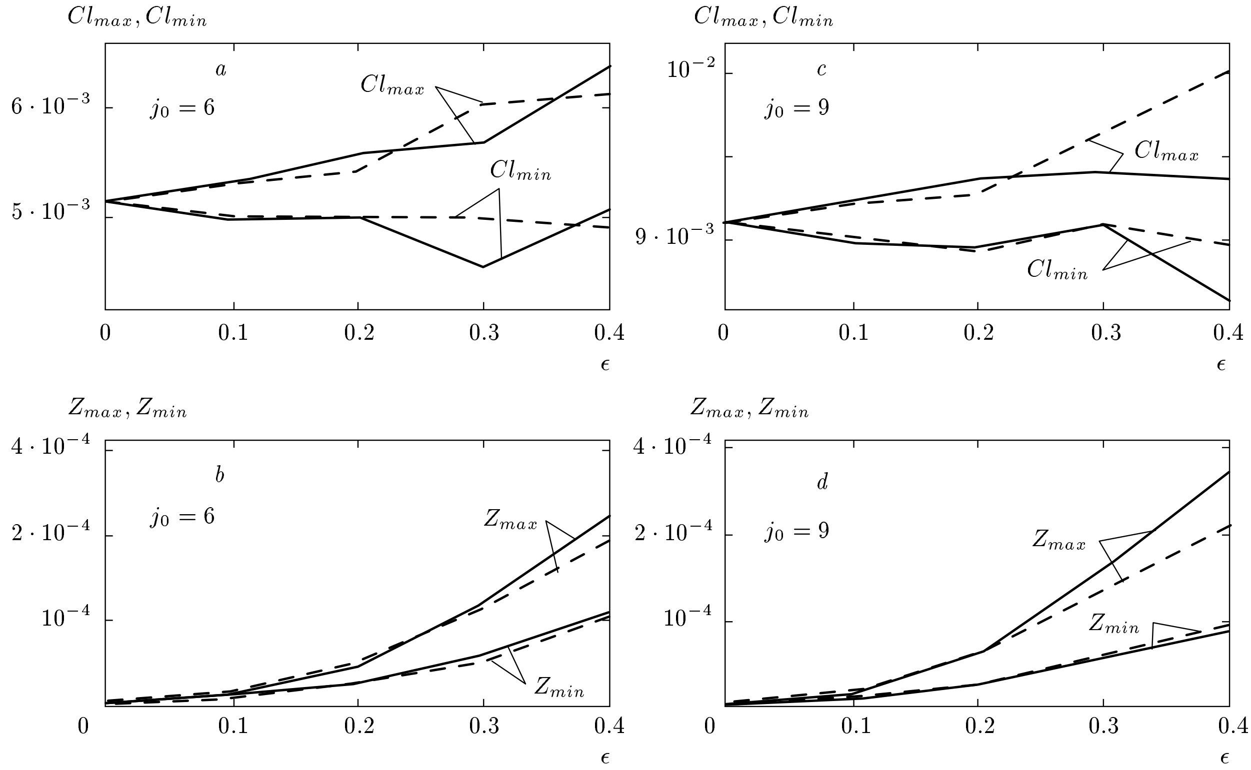

Fig. 3. Deformations of the fermion cluster in the system of $N = 17$ fermions with the noise polarization of the initial state. The pairwise discord is averaged over 10^2 realizations of random choices of the parameters $\tilde{b}_j$, $j = 1, \ldots, 17$, for each fixed value of the small parameter ϵ, $\epsilon = 0$, 0.1, 0.2, 0.3, 0.4. The cluster deformation by noise effects for the density matrix ρ_1 (solid lines) and ρ_2 (dashed lines) is characterized by the functions $Cl_{max}(\epsilon)$, $Cl_{min}(\epsilon)$, $Z_{max}(\epsilon)$, and $Z_{min}(\epsilon)$. (*a*) The initially polarized node $j_0 = 6$, the functions $Cl_{max}(\epsilon)$ and $Cl_{min}(\epsilon)$. (*b*) The initially polarized node $j_0 = 6$, the functions $Z_{max}(\epsilon)$ and $Z_{min}(\epsilon)$. (*c*) The initially polarized node $j_0 = 9$, the functions $Cl_{max}(\epsilon)$ and $Cl_{min}(\epsilon)$. (*d*) The initially polarized node $j_0 = 9$, the functions $Z_{max}(\epsilon)$ and $Z_{min}(\epsilon)$

$$
\begin{aligned}
A_0^{j_0} &= \frac{1}{2^N} \prod_{l=1}^{N} a_0^l, \\
A_{kk'}^{j_0} &= \frac{1}{2^N} \left(a_0^{j_0} \sum_{\substack{n=1 \\ n \neq j_0}}^{N} \left(\prod_{\substack{m=1 \\ m \neq j_0 \neq n}}^{N} a_0^m \right) a_{kk'}^n + \right. \\
&\qquad \left. + a_{kk'}^{j_0} \prod_{\substack{l=1 \\ l \neq j_0}}^{N} a_0^l \right), \\
A_{kqk'q'}^{j_0} &= \frac{a_{kk'}^{j_0}}{2^N} \sum_{\substack{n=1 \\ n \neq j_0}}^{N} \left(\prod_{\substack{m=1 \\ m \neq j_0 \neq n}}^{N} a_0^m \right) a_{qq'}^n, \\
A_{kqlk'q'l'}^{j_0} &= 0
\end{aligned}
\tag{31}
$$

for the density matrix ρ_1, or

$$
\begin{aligned}
A_0^{j_0} &= \frac{1}{2^N} \prod_{l=1}^{N} a_0^l, \\
A_{kk'}^{j_0} &= \frac{1}{2^N} \left(a_0^{j_0} \sum_{\substack{n=1 \\ n \neq j_0}}^{N} \left(\prod_{\substack{m=1 \\ m \neq j_0 \neq n}}^{N} a_0^m \right) a_{kk'}^n + \right. \\
&\qquad \left. + a_{kk'}^{j_0} \left(\prod_{\substack{l=1 \\ l \neq j_0}}^{N} a_0^l \right) \right), \\
A_{kqk'q'}^{j_0} &= \frac{1}{2^N} \left(a_{kk'}^{j_0} \sum_{\substack{n=1 \\ n \neq j_0}}^{N} \left(\prod_{\substack{m=1 \\ m \neq j_0 \neq n}}^{N} a_0^m \right) a_{qq'}^n + \right. \\
&\left. + a_0^{j_0} \sum_{\substack{n,n'=1 \\ n' \neq n \neq j_0}}^{N} \left(\prod_{\substack{m=1 \\ m \neq j_0 \neq n \neq n'}}^{N} a_0^m \right) a_{kk'}^n a_{qq'}^{n'} \right),
\end{aligned}
$$

$$A^{j_0}_{kqlk'q'l'} = \frac{a^{j_0}_{kk'}}{2^N} \sum_{\substack{n,n'=1 \\ n'\neq n\neq j_0}}^{N} \times \left(\prod_{\substack{m=1 \\ m\neq j_0\neq n\neq n'}}^{N} a^m_0 \right) a^n_{qq'} a^{n'}_{ll'} \quad (32)$$

for the density matrix ρ_2. In this section, formulas (14)–(18) hold as well. The stationarity of the pairwise quantum discord between fermions follows from the structure of the marginal matrix $\rho^{j_0}_{nm}$ in Eq. (18) and can be shown in a way similar to that proposed in Sec. 3.

4.1. Numerical simulations

Similarly to the numerical simulations in Sec. 3.2, we perform numerical simulations in the particular case of an $N = 17$ node spin chain with the initially polarized spins $j_0 = 6$ and $j_0 = 9$. For each fixed value of the small parameter ϵ in the interval $0 \le \epsilon \le 0.4$ ($\epsilon = 0, 0.1, 0.2, 0.3, 0.4$), we average the discord over 10^2 realizations of the random set of the parameters $\tilde{b}_j$, $j = 1, \ldots, 17$, characterizing the noise polarization of the jth node of the spin-1/2 chain. For the averaged discord, we use the same notation Q_{nm} in this section.

Again, to characterize the deformation of the discord distribution caused by the noise polarization, we use the functions $Cl_{max}(\epsilon)$, $Cl_{min}(\epsilon)$ and $Z_{max}(\epsilon)$, $Z_{min}(\epsilon)$ defined by formulas (23) and (24) in which we replace b with ϵ to characterize the spread of the discord in the cluster Cl and the spread of the parasitic discord, which is zero in the absence of noise. Pairs of functions $Cl_{max}(\epsilon)$, $Cl_{min}(\epsilon)$ and $Z_{max}(\epsilon)$, $Z_{min}(\epsilon)$ are respectively shown in Figs. 3a and 3b for $j_0 = 6$ and in Figs. 3c and 3d for $j_0 = 9$. We see that the difference between the discord distribution corresponding to the density matrices ρ_1 and ρ_2 is not significant inside the interval $0 \le \epsilon \le 0.4$, as is shown in Fig. 3. More exactly, the curves Cl_{max} corresponding to the maximal discord for the density matrices ρ_1 and ρ_2 are close to each other (see the upper solid and dashed lines in Figs. 3a and 3c), as well as the appropriate curves Cl_{min} (see the lower solid and dashed lines in Figs. 3a and 3c). The same statement holds for the curves Z_{max} and Z_{min} in Figs. 3b and 3d. We consider this a justification of using the perturbation theory.

To demonstrate the magnitude of deformation of the discord distribution under the small-amplitude noise polarization, we represent the discord distribution in the cluster of correlated fermions for the initially polarized node $j_0 = 6$, the density matrix $\rho_2 = \tilde{\rho}_2/Z_2$, and the noise amplitude $\epsilon = 0.4$ in Fig. 4. The comparison of Figs. 4 and 2a shows that the deformation of the discord distribution is approximately negligible in the cluster of the correlated fermions Cl. It is important that noise does not destroy the stationarity of the discord distribution, similarly to the case of parasitic polarization considered in Sec. 3.

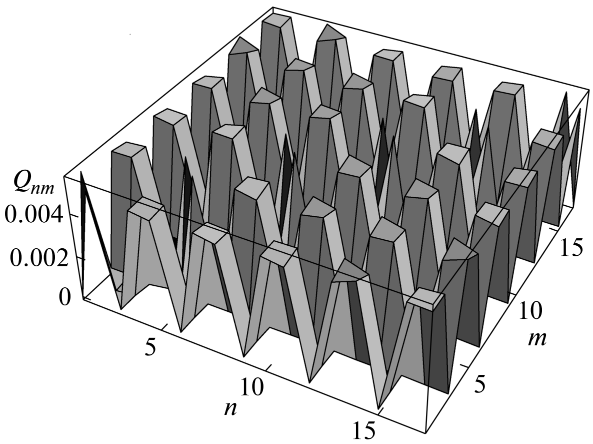

Fig. 4. The averaged discord distribution Q_{nm} in the system of $N = 17$ fermions with the initially polarized node $j_0 = 6$, density matrix ρ_2, and noise amplitude $\epsilon = 0.4$. The pairwise discord is averaged over 10^2 realizations of random choices of the parameters $\tilde{b}_j$, $j = 1, \ldots, 17$. This distribution is slightly deformed in comparison with that in Fig. 2a for $\epsilon = 0$. The difference is visible in the peaks

4.2. Stationarity of the pairwise discord in a fermion system with noise

In the preceding section, we demonstrated that the pairwise discord in the Jordan–Wigner fermion system with a single initially polarized node remains stationary under perturbations of two types, parasitic polarization of two neighboring nodes and noise polarization, considered by the perturbation method. In both cases, the density matrix operator involves at most three-fermion terms (see Eqs. (10) and (29)). However, it can be readily shown that the stationarity may not be destroyed by noise polarization even if we take all terms of the perturbed density matrix into account. Or, even more generally, the pairwise discord distribution is stationary for the initial density matrix of the form

$$\rho_0 = \frac{\tilde{\rho}}{Z}, \quad Z = \mathrm{Tr}\,\tilde{\rho},$$

$$\tilde{\rho} = 1 + \sum_i \gamma_i I_{zi} + \sum_{i_1 \neq i_2} \gamma_{i_1 i_2} I_{zi_1} I_{zi_2} + \\ + \sum_{i_1 \neq i_2 \neq i_3} \gamma_{i_1 i_2 i_3} I_{zi_1} I_{zi_2} I_{zi_3} + \ldots \\ \ldots + \gamma_{1\ldots N} I_{z1} \ldots I_{zN}, \tag{33}$$

where the γ are scalar constants. The evolution of the density matrix described by the Liouville equation $d\rho/dt = -i[H, \rho]$ is given by

$$\rho(t) = e^{-itH} \rho_0 e^{itH}. \tag{34}$$

After some transformations using Eqs. (2), (3), (6), and (9), we obtain the density matrix in the form (we write the t-dependence explicitly)

$$\rho(t) = \frac{1}{Z} \left(1 + \sum_{i=1}^{N} \alpha^i_{kk'} e^{-it(\varepsilon_k - \varepsilon_{k'})} \beta^\dagger_k \beta_{k'} + \right. \\ + \sum_{i_1, i_2 = 1}^{N} \alpha^{i_1 i_2}_{k_1 k_2 k'_1 k'_2} e^{-it(\varepsilon_{k_1} + \varepsilon_{k_2} - \varepsilon_{k'_1} - \varepsilon_{k'_2})} \times \\ \left. \times \beta^\dagger_{k_1} \beta_{k'_1} \beta^\dagger_{k_2} \beta_{k'_2} + \ldots \right), \tag{35}$$

where the α are expressed in terms of the γ. Equation (35) is an infinite series. An important fact regarding its structure is that the product of the operators $\beta^\dagger_k \beta_{k'}$ appears together with the exponential $e^{-it(\varepsilon_k - \varepsilon_{k'})}$.

Considering the reduced density matrix operator with respect to all fermions except the nth and mth, we obtain the density matrix in the form

$$\rho_{nm}(t) = \tilde{\alpha}^{nm}_0 + \sum_{k,k'=n,m} \tilde{\alpha}^{nm}_{kk'} e^{-it(\varepsilon_k - \varepsilon'_k)} \beta^\dagger_k \beta_{k'} + \\ + \tilde{\alpha}^{nm}_{nmnm} \beta^\dagger_n \beta^\dagger_m \beta_m \beta_n, \tag{36}$$

where all the coefficients $\tilde{\alpha}$ are expressed in terms of the coefficients α in Eq. (35) and do not depend on t. We do not give explicit expressions for the $\tilde{\alpha}$. Terms of higher degrees in the β operators do not appear in the two-particle density matrix operator (36) because of the fermion operator property $\beta^2_k = (\beta^\dagger_k)^2 = 0$. Using the basis in Eq. (17), we can represent density operator (36) in the matrix form

$$\rho_{nm}(t) = \begin{pmatrix} \tilde{\alpha}^{nm}_0 & 0 & 0 & 0 \\ 0 & \tilde{\alpha}^{nm}_0 + \tilde{\alpha}^{nm}_{nn} & e^{-it(\varepsilon_n - \varepsilon_m)} \tilde{\alpha}^{nm}_{nm} & 0 \\ 0 & e^{it(\varepsilon_n - \varepsilon_m)} (\tilde{\alpha}^{nm}_{nm})^* & \tilde{\alpha}^{nm}_0 + \tilde{\alpha}^{nm}_{mm} & 0 \\ 0 & 0 & 0 & \tilde{\alpha}^{nm}_0 + \tilde{\alpha}^{nm}_{nn} + \tilde{\alpha}^{nm}_{mm} + \tilde{\alpha}^{nm}_{nmnm} \end{pmatrix}. \tag{37}$$

Thus, the t-dependence appears only in the exponents in the nondiagonal elements.

We now repeat the arguments used in the demonstration of the discord stationarity in Secs. 3.1 and 4. Namely, it is shown in the Appendix that the pairwise discord in X-matrix (37) depends on the absolute value $|\tilde{\alpha}^{nm}_{nm}|$ of the nondiagonal element of this matrix. Consequently the discord does not depend on the time t.

5. CONCLUSIONS

We have shown that the property of stationarity for the pairwise discord in the system of Jordan–Wigner fermions is stable with respect to polarization-like perturbations of the initial state. Two types of such parasitic polarizations are considered in detail. The first is associated with the experimental error in the creation of the single-node polarization initial state, resulting in low polarizations of the neighboring nodes. The second type is related to the noise polarization of all nodes. The only effect of both such perturbations is deformation of the pairwise discord distribution in the cluster Cl of correlated fermions. In particular, such perturbations can destroy the cluster, which is explicitly demonstrated in Sec. 3 for two neighboring-node parasitic polarization. Hence, the discord stationarity in the Jordan–Wigner fermion system can be taken as a reliable and stable advantage of the considered fermion system in comparison with the original spin system. This encourages us to consider the possibility of a quantum gate realization on the basis of such systems of virtual particles.

This work is supported by the Program of the RAS Presidium No. 8 “Development of methods of obtaining chemical compounds and creation of new materials” and by the RFBR (grant No. 13-03-00017).

APPENDIX

Quantum discord in the X-type state (37)

The two-particle density matrix considered in this paper is a particular case of the so-called X-matrix [27]:

$$\rho^{red} = \begin{pmatrix} \rho_{11} & 0 & 0 & 0 \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{23}^* & \rho_{33} & 0 \\ 0 & 0 & 0 & \rho_{44} \end{pmatrix}, \quad \sum_{i=1}^{4} \rho_{ii} = 1. \quad (38)$$

The discord for the X-matrix was studied in [27]. We recall that the discord between particles n and m of a biparticle quantum system can be calculated as

$$Q_m = \mathcal{I}(\rho) - \mathcal{C}^m(\rho), \quad (39)$$

if the von Neumann type measurements are performed over the particle m. Here, $\mathcal{I}(\rho)$ is the total mutual information [4], which can be written as

$$\mathcal{I}(\rho) = S(\rho^{(n)}) + S(\rho^{(m)}) + \sum_{j=0}^{3} \lambda_j \log_2 \lambda_j, \quad (40)$$

where λ_j $(j = 0, 1, 2, 3)$ are the nonzero eigenvalues of the density matrix $\rho^{(nm)}$,

$$\begin{aligned} &\lambda_0 = \rho_{11}, \quad \lambda_1 = \rho_{44}, \\ &\lambda_{2,3} = \frac{1}{2}\left(\rho_{22}+\rho_{33} \pm \sqrt{(\rho_{22}-\rho_{33})^2+4|\rho_{23}|^2}\right), \end{aligned} \quad (41)$$

and $\rho^{(n)} = \mathrm{Tr}_m \rho^{(nm)}$ and $\rho^{(m)} = \mathrm{Tr}_n \rho^{(nm)}$ are the marginal density matrices. The appropriate entropies $S(\rho^{(n)})$ and $S(\rho^{(m)})$ are given by the formulas

$$\begin{aligned} S(\rho^{(n)}) &= -(\rho_{11} + \rho_{22}) \log_2(\rho_{11} + \rho_{22}) - \\ &\quad - (\rho_{33} + \rho_{44}) \log_2(\rho_{33} + \rho_{44}), \\ S(\rho^{(m)}) &= -(\rho_{11} + \rho_{33}) \log_2(\rho_{11} + \rho_{33}) - \\ &\quad - (\rho_{22} + \rho_{44}) \log_2(\rho_{22} + \rho_{44}). \end{aligned} \quad (42)$$

The so-called classical counterpart $\mathcal{C}^B(\rho^{(nm)})$ of the mutual information can be found by considering the minimization over projective measurements performed on the particle m as follows [27]:

$$\mathcal{C}^{(m)}(\rho) = S(\rho^{(n)}) - \min_{k\in[0,1]} (p_0 S_0 + p_1 S_1), \quad (43)$$

where

$$S(\theta_i) \equiv S_i = -\frac{1-\theta_i}{2} \log_2 \frac{1-\theta_i}{2} - \\ - \frac{1+\theta_i}{2} \log_2 \frac{1+\theta_i}{2}, \quad (44)$$

$$\begin{aligned} p_0 &= (\rho_{11} + \rho_{33})k + (\rho_{22} + \rho_{44})l, \\ p_1 &= (\rho_{11} + \rho_{33})l + (\rho_{22} + \rho_{44})k, \end{aligned} \quad (45)$$

$$\begin{aligned} &\theta_0 = \frac{1}{p_0} \times \\ &\times \sqrt{((\rho_{11} - \rho_{33})k + (\rho_{22} - \rho_{44})l)^2 + 4kl|\rho_{23}|^2}, \\ &\theta_1 = \frac{1}{p_1} \times \\ &\times \sqrt{((\rho_{11} - \rho_{33})l + (\rho_{22} - \rho_{44})k)^2 + 4kl|\rho_{23}|^2}. \end{aligned} \quad (46)$$

Here, the parameters k and l are related by the equation [27]

$$k + l = 1. \quad (47)$$

It is easy to show that the quantum discord Q_n obtained by performing the von Neumann type measurements on the particle n is related to Q_m as

$$Q_n = Q_m|_{\rho^{(nn)} \leftrightarrow \rho^{(mm)}} \quad (48)$$

for the system with the density matrix ρ^{red} given by Eq. (38). We then define the discord Q_{nm} as the minimum of Q_n and Q_m [29],

$$Q_{nm} = \min(Q_n, Q_m), \quad n \neq m, \quad (49)$$

with the obvious property $Q_{nm} = Q_{mn}$. We see that if the ρ_{nn}, $n = 1, 2, 3, 4$, and $|\rho_{23}|$ do not depend on the time t, then the discord does not evolve with time as well.

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