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ON A CHIRAL ANALOG OF THE EINSTEIN–de HAAS EFFECT

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The Einstein–de Haas effect reveals a transfer of anguar momentum from microscopic constituents (electrons) to a macroscopic body, but in the case of massless fermions, one could expect the transfer of the chirality of constituents to macroscopic helical motion. For such a picture to be consistent, the macroscopic helicity is to be conserved classically, to echo the conservation of the angular momentum of a rotating body. The helicity conservation would in turn impose constraints on hydrodynamics of chiral liquids (whose constituents are massless fermions). Essentially, the chiral liquids are dissipation-free, on the classical level. Reservations and alternatives to this scenario are discussed.

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1. INTRODUCTION. CHIRAL LIQUIDS

Theory of liquids with massless fermionic constituents has been greatly highlighted recently (for a review, see, e. g., lecture volume [1]). The interest in such chiral liquids was triggered by the discovery of QCD plasma, with its nearly massless quarks (see, e. g., [2]). The quark–gluon plasma exhibits remarkable properties. In particular, it is characterized by a low ratio of the viscosity η to the entropy density s, close to its conjectured quantum lower bound [3]. However, this property of the quark–gluon plasma has not yet been related to the (nearly) chiral nature of the plasma, and we return to this point later.

Vector and axial-vector currents are natural probes of the chiral nature of the underlying field theory. Moreover, from the theoretical standpoint, the consideration of chiral media with an asymmetric right–left composition or a nonvanishing chiral chemical potential $\mu_5 \neq 0$ represents an especially clean case. In particular, one predicts the existence of the chiral magnetic effect [4–6], or a flow of electric current along the magnetic field in equilibrium,

$$j^{el}_\mu = \sigma_M B_\mu, \tag{1}$$

where $B_\mu \equiv (1/2)\epsilon_{\mu\nu\alpha\beta}u^\nu F^{\alpha\beta}$, u^μ is the 4-velocity of an element of the liquid, and $F^{\alpha\beta}$ is the standard electromagnetic field tensor. In the rest frame, B_μ reduces to the magnetic field. We mostly focus on the vortical chiral effect [7–9], according to which helical macroscopic motion of the liquid contributes to the axial current j^5_μ:

$$j^5_\mu = (1/2)\sigma_\omega \epsilon_{\mu\nu\rho\sigma} u_\nu \partial_\rho u_\sigma. \tag{2}$$

We note that σ_ω is actually a function of both the chemical potential and temperature. For simplicity, we mostly suppress the temperature dependences. This does not affect our conclusions.

Currents (1) and (2) are predicted to exhibit remarkable properties. First, the coefficients σ_M and σ_ω are uniquely determined in terms of the chiral anomaly. Thus, for a single massless Dirac fermion with an electric charge e,

$$\sigma_M = \frac{e\mu_5}{2\pi^2}, \tag{3}$$

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where $\mu_5 = \mu_L - \mu_R$ is the chiral chemical potential. For the vortical conductivity σ_ω, we obtain

$$\sigma_\omega = \frac{\mu^2}{2\pi^2}, \qquad (4)$$

where $\mu = \mu_L + \mu_R$. Amusingly, Eqs. (1) and (3) imply that the laws of classical electrodynamics are modified for chiral liquids.

Another intriguing feature of chiral liquids is that currents (1) and (2) are nondissipative. This conclusion already follows from the observation that the currents exist in equilibrium. Another way of reasoning [10] is that both the r.h.s. and the l.h.s. of (1) are odd under time reversal. This is a strong indication that the dynamics behind Eqs. (1) and (2) is Hamiltonian and there is no dissipation. For a discussion of the analogy with supeconductivity, we refer the reader to [11].

As mentioned above, the numerical values of σ_M and σ_ω can be traced back to the coefficients in front of the product of electric and magnetic fields in the expression for the famous chiral anomaly [12]:

$$\partial_\mu j^5_\mu = \frac{e^2}{8\pi^2} B_\alpha E^\alpha, \qquad (5)$$

where the definition of the magnetic field adjusted to the consideration of hydrodynamics is given above, while the electric field in the medium is defined as $E_\alpha = u^\beta F_{\beta\alpha}$. In the hydrodynamic approximation, relations (3) and (4) were originally obtained in Ref. [9]. In their approach, the authors of [9] start with both electric and magnetic external fields present and then let $E^\alpha \to 0$. Remarkably enough, currents (1) and (2) survive in the limit of chiral anomaly (5) being switched off by taking the limit $E_\alpha \to 0$. This implies that already in the limit of the electromagnetic coupling $\alpha_{el} \to 0$ the conserved axial charge is modified in the hydrodynamic setup.

The reason for such a modification can be explained in a number of ways (see in particular, [13–17]). What is specific for hydrodynamics, is the change of the original Hamiltonian H_0 of the system to a modified one:

$$H_0 \to H_0 - \mu Q, \qquad (6)$$

where μ is the chemical potential associated with a conserved charge Q. As a result, there is a change already in the *conserved* axial current (i. e., in the limit of vanishing electromagnetic coupling). In a somewhat simplified form, the axial charge within the hydrodynamic approach is given by

$$\mathcal{Q}^A_{hydro} = \mathcal{Q}^A_{naive} + \frac{1}{2\pi^2}\mathcal{H}_{fluid} + O(e), \qquad (7)$$

where $\mathcal{Q}^A_{naive}$ counts the number of elementary chiral constitutents and the fluid helicity is $\mathcal{H}_{fluid} = \int d^3x\, \mu^2 \omega_0$, where $\omega_\alpha = (1/2)\epsilon_{\alpha\beta\gamma\delta} u^\beta \partial^\gamma u^\delta$ and we reserve for the possibility of the chemical potential varying in space[1)].

The conservation of hydrodynamic axial charge (7) suggests a possibility of transition of the chirality of the constituents into helical macroscopic motion of the liquid. As is mentioned in the abstract, this is an analog of the Einstein–de Haas effect. A new point is what can be called the *clash of symmetries*: on the microscopic level, chirality is conserved, but on the macroscopic level, we are using the standard hydrodynamic description, which does not incorporate the conservation of chirality in general and was originally developed for nonrelativistic motion of the constituents.

One way to resolve this contradiction is to impose extra constraints on the hydrodynamic description [18]. Generically, the solution of these constraints is that classically chiral liquids are dissipation-free. In particular,

$$\eta_{classical} = 0. \qquad (8)$$

We note that phenomenological consequences from the (hypothesized) conservation of fluid helicity were studied in great detail in magnetohydrodynamics[2)] (see, e. g., [19] and the references therein).

The outline of this paper is as follows. In Sec. 2, we discuss the issue of the conservation of macroscopic helical motion in hydrodynamics in more detail. The main conclusion is that the conservation of the axial charge implies dissipation-free hydrodynamics of chiral liquids in the classical approximation. In Sec. 3, we discuss reservations and problems.

2. AXIAL CHARGE IN HYDRODYNAMICS

2.1. Hydrodynamics as an effective field theory

Hydrodynamics is a unversal framework to describe motions in the infrared limit, when the wave lengths of perturbations are much larger than the mean free path of constituents. The beauty of this approach is that hydrodynamic equations of motion reduce to ge-

[1)] For simplicity, we quote the expression for the fluid helicity in flat space. In curved space, there is an extra geometric factor of $\sqrt{-g}$ in the integrand.

[2)] Note, however, that in magnetohydrodynamics, the electromagnetic field is considered to be dynamical, while many results we are quoting refer to the case of global symmetries, or external magnetic and electric fields.

neral conservation laws. In particular, in the absence of external fields, these equations are

$$\partial^\mu T_{\mu\nu} = 0, \quad \partial^\mu j_\mu^{(i)} = 0,$$

where $T_{\mu\nu}$ is the energy–momentum tensor and $j_\mu^{(i))}$ is a set of conserved currents[3)].

Since explicit expressions for $T_{\mu\nu}$ and $j_\mu^{(i)}$ involve phenomenological expansions in derivatives, hydrodynamics is usually considered as a "typical" effective field theory. However, apart from integrating out hard, or ultraviolet degrees of freedom, the hydrodynamic approximation also assumes a change of language. Indeed, the μQ term in hydrodynamic Hamiltonian (6) does not correspond literally to any integration over fundamental interactions and the very notion of the chemical potential can be introduced only on average, or thermodynamically (see, e. g., [20]). Also, the problem we are considering here is somewhat specific since we need a closed expression for the axial charge, with no further contributions [13] from the gradient expansion.

The simplest way to argue that the hydrodynamic axial charge contains extra pieces, see (7), is as follows [13]. We first assume the chemical potential to be small, such that the μQ term in hydrodynamic Hamiltonian (6) can be treated as a perturbation. Using the relation $\delta L = -\delta H$ for a small variation of the Lagrangian, we find for small μ:

$$(\delta L)_{hydro} = \mu u^\alpha j_\alpha, \tag{9}$$

where the charge Q above is related to the current j_α in the standard way, $Q = \int d^3x j_0$. Finally, using the analogy with the electromagnetic interaction, $\delta L_{el} = e \int d^4x\, A^\alpha j_\alpha$, we come to the substitution

$$eA_\mu \;\to\; eA_\mu + \mu u_\mu. \tag{10}$$

Extra pieces in the axial charge are generated via this substitution.

In more detail, we recall that chiral anomaly (5) can be reformuated [21] as the statement that the actually conserved axial charge contains a term with external electromagnetic potentials:

$$Q^A_{conserved} = \mathcal{Q}^A_{naive} + \frac{e^2}{4\pi^2}\mathcal{H}_{magn}, \tag{11}$$

where we introduce the notation $\mathcal{H}_{magn}$, common in papers on magnetohydrodynamics, which stands for the magnetic helicity,

$$\mathcal{H}_{magn} = \int d^3x\, \epsilon^{ijk} A_i F_{jk},$$

where (i, j, k) range over 1, 2, 3, A_i and F_{jk} are the electromagnetic potential and field strength tensor, and e is the electric charge of the massless fermions.

Now, by substitution (10), we generate further terms in the hydrodynamic expression for the conserved axial charge [16, 18]:

$$\mathcal{Q}^A_{hydro} = \mathcal{Q}^A_{naive} + \frac{\mu^2}{2\pi^2}\mathcal{H}_{fluid} + \\ + \frac{e}{2\pi^2}\mathcal{H}_{mixed} + \frac{e^2}{4\pi^2}\mathcal{H}_{magn}, \tag{12}$$

where the so-called naive axial charge is expressed in terms of the density ρ^A of the fermionic constituents, $\mathcal{Q}^A_{naive} = \int d^3x \rho^A$, the so-called mixed helicity is given by

$$\mathcal{H}_{mixed} = e\int d^3x\, \mu \epsilon_{ijk} u^i F^{jk},$$

and $\mathcal{H}_{fluid}$ and $\mathcal{H}_{magn}$ are defined in Eqs (7) and (11). We emphasize again that only the last term in the r.h.s. of Eq. (12), that is, H_{magn}, corresponds to chiral anomaly (5) on the fundamental level of the underlying field theory, while the sum of the first three terms is to be conserved classically.

So far, we treated the μQ piece in Hamiltonian (6) as a perturbation, and the whole construction seems to be straightforward. In the spirit of the hydrodynamic approximation, one could introduce further terms in the derivative expression for the currents and look for the solution of the hydrodynamic equations order by order in terms of such expansions [9]. However, from the experience with evaluating the axial charge in perturbative vacuum (11), we learn that the anomalous term $\mathcal{H}_{magn}$ is uniquely fixed and has no extention in terms of perturbative expansions either in the electric charge or in derivatives. We expect a similar phenomenon to occur in the hydrodynamic approximation, such that the densities of $\mathcal{H}_{fluid}$ and $\mathcal{H}_{mixed}$ receive no further contributions [13]. To achieve this, we need to formulate the hydrodynamic approximation with an explicitly chiral invariant infrared regularization (which fails, exceptionally, in the one-loop calculation in the effective theory).

In the case of an ideal chiral liquid, such a formalism is worked out in [17, 22] and the references therein. The infrared degrees of freedom in the field theoretic language are provided by real scalar fields φ^i and ψ, where

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[3)] For a moment, we ignore possible quantum anomalies. Moreover, we consider only $U(1)$ anomalies, and then we can redefine the anomaly as a new conserved charge, such that the external electromagnetic field has a nonvanishing axial charge if $E^\alpha B_\alpha \neq 0$.

the number of φs is equal to the number of spatial coordinates and the φ^i can be thought of as comoving coordinates of an element of liquid. This identification introduces symmetries that have a geometric origin, like the invariance of the volume under reparametrization of the coordinates. Another real field ψ is needed to realize a flavor symmetry, or a conserved charge. We can develop intuition on symmetries obeyed by the interactions of the field ψ and its relation to hydrodynamics if we think of ψ as of a relativistic generalization of the phase of the wave function in the case of superfluidity.

The interaction of the fields φ, ψ is highly nonlinear. The main advantage, however, is that symmetries of the theory can now be realized in field-theoretic terms. Thus, we can expect that the modified axial current, like (7), arises as a Noether current, which is conserved, as usual, on the mass shell, or with the account of hydrodynamic equations of motion. These expectations are indeed realized. For details, we refer the reader to Ref. [17] and quote here only the final result, relevant to our purposes:

$$\partial_\alpha\left(\mu^2\omega^\alpha + \mu B^\alpha\right) = B_\alpha E^\alpha - \hat{B}_\alpha \hat{E}^\alpha, \qquad (13)$$

where $\omega^\alpha = (1/2)\epsilon^{\alpha\beta\gamma\delta}u_\beta\partial_\gamma u_\delta$, B_α and E_α are magnetic and electric fields in the medium defined above (with the constant e included into the definition of the electromagnetic potential), and $\hat{B}_\alpha$ and $\hat{E}_\alpha$ are constructed on the hydrodynamic "shadow" potential $\mu u_\mu + eA_\mu$ in a similar way. The chemical potential μ satisfies the standard thermodynamic relation $dP = T\,ds + q\,d\mu$, while field-theoretically it is expressed in terms of the covariant derivative of the field ψ mentioned above, $\mu = u^\alpha D_\alpha\psi$.

A crucial point is that the so-called transverse electric field $\hat{E}_\alpha$ entering Eq. (13) vanishes on the hydrodynamic equations of motion for the ideal liquid, or in equlibrium [17]. Moreover, the hydrodynamic axial charge is defined to all orders in the derivative expansion, as desired (see the discussion above). A reservation is that these properties hold if the liquid velocity u_α is defined in a specific frame. Namely, in this frame, also called the "entropy frame" [22], the entropy current is simply $s_\alpha = su_\alpha$, where s is the entropy density. Moreover, the entropy current is defined geometrically in such a way that it is conserved automatically, i. e., off the mass shell:

$$s^\alpha = \epsilon^{\alpha\beta\gamma\delta}\epsilon_{ijk}(\partial_\beta\varphi^i)(\partial_\gamma\varphi^j)(\partial_\delta\varphi^k).$$

There are no further corrections to the entropy current due to the expansion in derivatives. We also note that we are using general curvilinear coordinates and the expression for the axial charge in terms of the current density contains a geometric factor due to the invariant volume element.

2.2. Clash of symmetries?

We now consider a nonideal liquid. Then there seemingly arises a problem with the axial current conservation. In its generality, it can be formulated as the lack of matching between symmetries at microscopic and macroscopic scales. Microscopically, we consider a chiral invariant theory of massless constituents (barring the chiral anomaly (5) for the moment, which is of the second order in electromagnetic interactions). The classification of particles according to their chiral charges is specific for massless fermions. In general, there is no macroscopic conservation law matching the chirality conservation in the underlying field theory.

To reiterate the point, we compare the conservation of the angular momentum and of chirality. We invoke the conservation of the total angular momentum when interpreting the Einstein–de Haas experiment. In this case, we have the conserved total angular momentum that incorporates both the spin angular momenta of the constutuents (electrons) and the angular momentum of a rotating body:

$$(M^z)_{total} = \sum_i s_i^z + M^z_{rotation}, \qquad (14)$$

where the summation is over all the constituents and the axis of rotation is directed along the z coordinate. As a resut of absorbing spinning elementary electrons, there arises macroscopic rotation of a rigid body, in accordance with conservation law (14).

Now, in the case of chiral liquids, we have an extra condition of the axial charge conservation. In the limit of vanishing electromagnetic coupling, $\alpha_{el} \to 0$, the conserved axial charge is given by (7):

$$\mathcal{Q}^A_{total} = \sum_i \chi^i + \frac{1}{2\pi^2}\mathcal{H}_{fluid}, \qquad \frac{d\mathcal{Q}^A_{total}}{dt} = 0, \qquad (15)$$

where χ^i are chiralities of the constituents and $H_{fluid} = \int d^3x\,\mu^2\omega_0$ is the helical charge associated with the axial current $j^5_\alpha = (\mu^2)\omega_\alpha$ (see the discussion above). The conservation of $\mathcal{Q}^A_{total}$ suggests the possibility of transition of the chirality of the constituents into helical macroscopic motion. Such a transition could be called a chiral analog of the Einstein–de Haas effect.

As we discussed in the preceding subsection, in the case of the ideal liquid, both $\sum_i \chi^i$ and $\mathcal{H}_{fluid}$ are separately conserved in equilibrium. The transition, say, from $\sum_i \chi^i \neq 0$, $\mathcal{H}_{fluid} = 0$ to $\mathcal{H}_{fluid} \neq 0$, with the conservation of $\mathcal{Q}^A_{total}$, is still possible if the configuration with $\mathcal{H}_{fluid} = 0$ does not correspond to the minimum of energy and is in fact unstable. (For related discussions, see Refs. [23–27] and Sec. 3 below.)

We now try to include dissipation. Then the macroscopic helicity $\mathcal{H}_{fluid}$ is not conserved. Indeed, relation (13) reduces to the standard anomaly relation (5) only upon the use of the equations of motion of the ideal liquid, which correspond to the vanishing viscosity $\eta = 0$. Actually, the observation that the fluid helicity conservation assumes the vanishing viscosity was made a long time ago, in the context of mgnetohydrodynamics (see, in particular, [19][4)]).

We can readily understand why the conservation of axial charge (7) requires the vanishing viscosity. Because of the viscosity, or friction, the helical motion slows down and recedes. Thus, $\mathcal{H}_{fluid}$ diminishes as time progresses. On the other hand, we assume that $\eta \neq 0$ arises due to some chiral invariant interaction and neglect the chiral anomaly. This implies that the total chirality of the constituents, $\sum_i \chi^i$, is conserved. For consistency with the axial charge conservation, we therefore need the validity of Eq. (8).

It is interesting to note that the limit of vanishing electric resistivity, or infinite conductivity σ_E is also intimitely related to the conservation of extended axial charge (12). Namely, in the limit as $\sigma \to \infty$, the electric field in the rest frame of an element of liquid vanishes. Therefore, the Lorentz invariant $E_\alpha B^\alpha$ vanishes in any frame as well:

$$\partial_\alpha \epsilon^{\alpha\beta\gamma\delta} A_\beta \partial_\gamma A_\delta \sim E_\alpha B^\alpha \to 0 \quad \text{with} \quad \sigma_E \to \infty, \tag{16}$$

as is emphasized in many papers on magnetohydrodynamics (see, e. g., [19]). Moreover, for a large finite σ_E, we obtain the estimate

$$\frac{d}{dt}\int d^3x\, \mathbf{A}\cdot\mathbf{B} \approx -\frac{\alpha}{2\pi\sigma_E}\int d^3x\, \mathbf{B}\cdot(\operatorname{curl}\mathbf{B}). \tag{17}$$

This relation has been used in many applications; for recent examples and further references, we refer the reader to [25].

4) On the detailed level, there are important differences between the formalism in Ref. [17], which we quoted in Sec. 2, and that in papers [19] on traditional magnetohydrodynamics.

3. RESERVATIONS AND CONCLUSIONS

3.1. Instabilities

In the preceding section, we argued that the axial current conservation in the hydrodynamic approach imposes constraints on the hydrodynamics itself (in the classical approximation). Essentially, dissipation-free hydrodynamics is favored in the classical approximation (see, in particular, Eq. (8)). Such a scenario looks very attractive since it allows appreciating the most striking effects — the low ratio η/s, chiral magnetic effect (1), chiral vortical effect (2) — in a unified way, as consequences of the chiral nature of the underlying field theories. It is therefore important to analyze reservations and possible alternative scenarios.

We begin with a discussion of the expansion in electromagnetic interaction. To derive (8), we neglected electromagnetic interactions and used a "shortened" version of the conserved axial charge (see Eq. (7)) instead of the full expression (see Eq. (12)). At first sight, solving hydrodynamic equations order by order in the expansion in electromagnetic interactions is a legitimate procedure.

There is a possibility, however, that in fact there are instabilities, and the true equilibrium state corresponds to the (approximate) equality of all four terms contributing to the total conserved axial charge (12):

$$\mathcal{Q}^A_{naive} \approx \frac{1}{4\pi^2}\mathcal{H}_{fluid} \approx \frac{e}{2\pi^2}\mathcal{H}_{mixed} \approx \approx \frac{e^2}{4\pi^2}\mathcal{H}_{magn}, \tag{18}$$

in apparent defiance of the expansion in electromagnetic interactions. We note that in this regime, the smallness of the extra powers of the electromagnetic coupling is compensated by large amplitudes of electromagnetic potentials in the components with small momenta, $k \sim 1/\alpha_{el}$.

The instability implies, for example, that if we start with a state with $\mathcal{Q}^A_{naive} \neq 0$ and $\mathcal{H}_{magn} = 0$, then there is spontaneous production of domains with $\mathcal{H}_{magn} \neq 0$ [27]. This scenario is supported, in particular, by an explicit identification of an unstable mode (see [26]). The time needed for developing the instability is of the order of

$$\tau_{instability} \sim 1/\mu_5 \alpha_{el}^2 \tag{19}$$

[26] and becomes infinite in the limit of vanishing electromagnetic coupling, $\alpha_{el} \to 0$.

3.2. Chiral dynamics, dependence on infrared physics

To avoid confusion, it is worth emphasizing that various possible scenarios for the dynamics of chiral liquids can be considered. First, one can assume that there exists a chiral invariant interaction, much stronger than the electromagnetic interaction, which is responsible for the gross features of the liquid. Then the condition of the conservation of the total conserved axial charge (12) can apparently be imposed in each order in electromagnetic interactions. This scenario essentially implies dissipation-free hydrodynamics, or $\eta_{classical} \to 0$, as discussed in detail in Sec. 2. Unexpectedly, the total charge is split into at least two pieces that are conserved separately according to the equations of motion. Therefore, the transitions between the states with the same total charge and different "subcharges" can occur only through formation of bubbles of a new vacuum.

Closer analysis reveals, however, that the back-reaction of the medium, or higher orders in electromagnetic interaction can crucially change the properties of chiral liquids because of the instabilities [23, 25–27]. To compensate for the smallness of α_{el}, one has to include into consideration distances and time intervals inversely proportional to α_{el} (see, e. g., (19)). One can visualize this instability in the following way. Because of the chiral magnetic effect, there is flow of electric currents. These currents induce electric fields, which in turn change the axial charge of the constituents because of the anomaly. This example demonstrates that the properties of plasma might depend on the details of the infrared regularization, as is emphasized in Refs. [27, 28]. Indeed, for the instability to be realized, the total volume is to be large in units $(\mu_5\alpha_{el})^{-3}$.

Moreover, if we make one step further and account for the back-reaction of the medium to the electric field arising as a result of the instability, then the dynamical scenario can change again because of the possibility of screening of the electric field in the medium. In particular, if we consider magnetohydrodynamics, or the case where electrodynamics entirely determines the properties of the plasma, the dissipation-free limit implies complete screening of the electric field [19]:

$$\sigma_E \to \infty, \quad (E_\alpha B^\alpha)_{medium} \to 0.$$

As a result, the instability would be curbed. Spontaneous production of domains with a nonzero $\mathcal{H}_{magn}$ is still possible, but the lifetime of the false vacuum with $\mathcal{H}_{magn} = 0$ would be exponentially large,

$$\tau_{bubbe\ formation} \sim \mu_5^{-1} \exp\left(\mathrm{const}/\alpha_{el}\right). \tag{20}$$

To our knowledge, no explicit calculations of this lifetime were attempted in the literature.

3.3. Instability in a Euclidean mirror

Instabilities discussed so far refer to the Minkowskian picture. Many papers on the subject, however, use the finite temperature $T \neq 0$ to fix the theory in the infrared (see, e. g., [15, 17]) and start with the Euclidean picture, with its cyclic time coordinate $0 \le \tau \le 1/T$. One might suspect that since the lifetime of false vacuum (19) is large, the instability does not develop at temperatures $T \gg 1/\tau_{instability}$.

The question can be phrased in another way. The magnetic conductivity is related to a static correlator of two spatial components of the electromagnetic current. In the momentum space,

$$\sigma_M = \lim_{\omega\equiv 0, q_i \to 0} \frac{\epsilon^{ijk}\langle j_i^{el},\ j_j^{el}\rangle_q}{iq_k}, \tag{21}$$

where ω is the frequency, $(i,j,k) = (1,2,3)$, and there is no summation over repeated indices. The chiral anomaly is encoded in the $3D$ action

$$S_{3D} = \frac{e^2}{2\pi^2}\int d^3x\, \mu_5 \epsilon^{ijk} A_i\, \partial_j A_k, \tag{22}$$

which can be reconstructed, e. g., from Eqs. (6) and (11) and in many other ways. The magnetic conductivity is uniquely fixed by action (22), which is linear in derivatives.

We note that the standard criterion for superfluidity also refers to a static correlator, this time of the spatial components T_{0i} and T_{0k} of the energy–momentum tensor. If we start from the Minkowskian definition, the superfluidity is signaled by the following form of the correlator:

$$\lim_{\omega\to 0, q_i\to 0}\langle T_{0i},\ T_{0k}\rangle_q \sim \frac{\delta_{ik}}{q^2}\delta(\omega). \tag{23}$$

In this case, however, the continuation to the Euclidean space is much more subtle because of $\delta(\omega)$ in the r.h.s. of Eq. (23). Also, a pole in q^2, or a long-range force is required for the superfluidity, while correlator (21) is saturated by a polynomial. In the cases of both superfluidity and the chiral magnetic effect, the currents are evaluated in equilibrium, and hence the striking differences between correlators (21) and (23) might look puzzling.

The chiral-plasma instability mentioned above arises if the electromagnetic field is treated as dynamical. To clarify the Euclidean counterpart of the phenomenon, we add the standard $3D$ kinetic term

$$L_{kin} = -\frac{1}{4}\int d^3x\, F_{ij}^2 \quad (i,j = 1,2,3)$$

to action (23) and evaluate the static photon propagator with the anomalous piece (22) taken into account [27]. The result is

$$D_{ij}(q) = \frac{1}{q^2-\sigma^2}\left(\delta_{ij} - \frac{q_i q_j}{q^2}\right) - \frac{i\sigma\epsilon_{ijl}q^l}{q^2(q^2-\sigma^2)}, \quad (24)$$

where $\sigma = \mu_5 e^2/2\pi^2$.

If we now use propagator (24) to evaluate correlator (21), then there is a pole at $q^2 = 0$, similar to the case of the superfluidity-related correlator (23). Thus, the apparent simplicity of the evaluation of correlator (23) starting from action (22) is due to the fact that we have not found the $3D$ spectrum of excitations. On the other hand, checking the criterion of superfluidity (23) does require the knowledge of the spectrum.

Moreover, and more importantly, there is an unphysical pole at $q^2 = \sigma^2$, which reveals the nonunitary nature of the theory we are considering. This loss of unitarity can be specified in the following way. The static limit of the $4D$ theory that we are considering can be compared with the Euclidean version of the $(2+1)$ theory with a nonvanishing topological photon mass. Then the anomalous action (22) corresponds to an imaginary topological photon mass $m_\gamma^{topological} = i\sigma$ [23, 24, 27], or

$$(m_\gamma^{topological})^2 = -\sigma^2. \quad (25)$$

This is the signature of the plasma instability in Euclidean disguise. The problem of the unphysical pole is not removed by introducing finite temperature.

3.4. Double counting?

Finally, we mention another reservation concerning our conclusions in Sec. 2. Namely, in the case of a perfect liquid, there seem to be *two conserved currents*. Indeed, the total current is represented as

$$J_\mu = \rho u_\mu + (\mu^2/2\pi^2)\omega_\mu + \mu B_\mu, \quad (26)$$

where ρ is the corresponding charge density and the rest of the notation is the same as in Eq. (13). According to (13), the second term in (26) by itself satisfies the (anomalous) conservation law in equilibrium. Hence, the first term, ρu_μ, is to be conserved separately[5)]. This seems uncomfortable, especially in view of the fact that on the fundamental level, in terms of massless fermions, there exists a single current. The sum of the two terms in the r.h.s. of (26) refers to the hydrodynamic matrix element of this fundamental current. The splitting of the total current into two terms seems not well defined in general.

The problem could also be formulated in the following way. One can derive the chiral vortical effect by considering field theory in a rotating frame [7]. Another purely geometric derivation can be given in terms of the Fermi sphere in the momentum space [16]. Thus, we could speculate that the origin of the chiral vortical effect is similar to the origin of, say, the Unruh effect and is rooted in a (hidden) use of a noninertial frame. Then the (anomalous) conservation of the total current could have a kinematical origin. We note that there is an explicit construction [17] of an off-shell (anomalous) conservation of the total current within the Schwinger–Keldysh formalism. This construction has not been derived from first principles. The mechanism behind it could be simiar to what we are describing here as the use of a noninertial frame.

At this moment, we cannot provide an educated appreciation of the physics behind the possible off-shell conservation of the current.

3.5. Conclusions

The main problem that we addressed in this note is how to reconcile the chiral symmetry of underlying theories with the general hydrodynamic framework. The point is that chiral symmetry is a property of (some of) theories of massless fermion fields. The classification of massless spin particles is different from the classification of massive particles. The standard hydrodynamics, on the other hand, uses the symmetries that are rooted only in symmetries of space-time and, as a result, apply to both relativistic and nonrelativistic motions, with or without dissipation.

One way to avoid this "clash of symmetries" is to impose constraints on hydrodynamics, by requiring the conservation of macroscopic helical motion. Essentially, the constraints require the liquid to be ideal, and therefore describable in terms of the field theory. The axial current is then a Noether current, (anomalously) conserved on the mass shell, i. e., with the equation of motion of the ideal liquid taken into account. The derivation of such currents can be found, in particular, in Refs. [17, 22]. The construction turns in fact highly nontrivial and the expression for the conserved current contains a finite number of terms in the derivative expansion if a specific choice of the frame is made. An unexpected problem emerges: there seem to arise two independently conserved currents.

Another possibilty is that we should reserve for an off-shell conservation of the axial current in hydrodynamics. We have not found any precise mechanism for the off-shell conservation. However, the analogy with the Unruh effect, where radiation of particles

5) For a related discussion, see [29] and the references therein.

arises because of the use of a noninertial, accelerated frame, might serve as a guide. Indeed, the appearence of specifically hydrodynamic terms in the axial charge seems to be related to the use of noninertial frames, like a rotating frame. This possibility might correspond to the construction of an automatically (anomalously) conserved current within the Schwinger–Keldysh formalism presented in Ref. [17].

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