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U(1) AND SU(2) QUANTUM DISSIPATIVE SYSTEMS: THE CALDEIRA–LEGGETT VERSUS AMBEGAOKAR–ECKERN–SCHÖN APPROACHES

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There are two paradigmatic frameworks for treating quantum systems coupled to a dissipative environment: the Caldeira–Leggett and Ambegaokar–Eckern–Schön approaches. Here, we recall the differences between them and explain the consequences of applying each to a zero-dimensional spin (having an SU(2) symmetry) in a dissipative environment (a dissipative quantum dot near or beyond the Stoner instability point).

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The diagrammatic technique for nonequilibrium systems developed in the pioneering works of Schwinger and Keldysh plays a predominant role in theoretical condensed matter physics [1, 2]. It is designed to tackle real-time evolution of systems at and away from equilibrium. Due to the developments of the last two decades [3–5], it now provides a nonperturbative tool to tackle interaction-induced strong correlations in quantum many-body systems. In this paper, we discuss an important prototypical problem, a quantum zero-dimensional degree of freedom in a dissipative environment, in which the Keldysh technique is of tremendous use, providing an insight into the physics involved.

1. GENERAL PERSPECTIVE

We consider the dynamics of a quantum system coupled to a dissipative environment. The resulting equation of motion is stochastic, which can be formulated on any of the following three levels: (i) a fully classical Langevin equation, where both the variables are classical (expectation values of observables) and the frequency range of interest is $\hbar\omega < k_B T$ (for Ohmic dis-
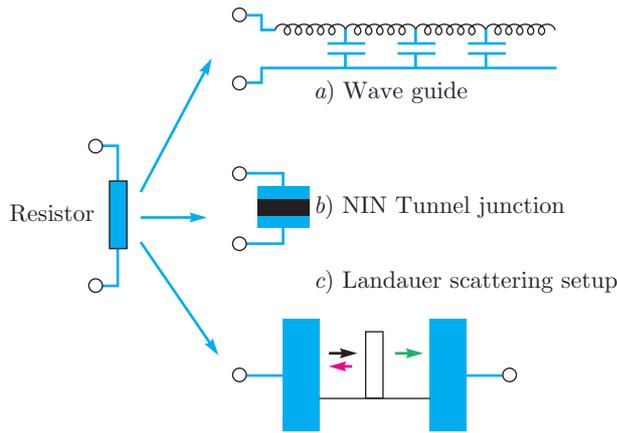
Three approaches to envisioning dissipation: *a)* an LC circuit (waveguide) extracting energy from the system; *b*) a dissipative tunnel junction suitable for the AES picture; *c*) the Landauer picture consisting of a tunnel barrier (or a tunnel juction) coupled to dissipative reservoirs

sipation, the noise spectrum is white); (ii) a semiclassical hybrid description, within which the variables are still classical coordinates, but the noise may be quantum, having a high-frequency component $\hbar\omega > k_B T$ (Ref. [6]); (iii) a full-fledged quantum mechanical description, according to which the noise may contain high-frequency quantum components, and the variables of the quantum Langevin equation are operators within the Heisenberg description. This approach is practiced, e. g., in the field of quantum optics [7].

A paradigmatic framework to represent a dissipative environment, in a way that connects to our preformed classical intuition, is to model an Ohmic resistor quantum mechanically. We mention three approaches.

1. The Caldeira–Leggett (CL) modeling [8]: One introduces an effective circuit consisting of an L–C transmission line (with an infinitesimal imaginary term), which can extract energy and current from the bare quantum system (cf. Figure *a*).

2. The Ambegaokar–Eckern–Schön (AES) modeling [9, 10]: Here, we model a tunnel junction (see Figure *b*) assuming explicitly that its transparency is low, and hence only lowest-order contributions to the tunneling should be accounted for. The resulting Hamiltonian represents reservoir degrees of freedom that give rise to dissipation. Traditional applications of the CL picture employed extended coordinates (which are not indispensable, however: the CL action in the case of a spin degree of freedom consists of compact coordinates). By contrast, the AES approach introduces compact (periodic) coordinates.

3. The Landauer picture [11–13]. Here, one models the resistor by a tunnel barrier (of arbitrary transparency) (see Figure *c* for the single-channel case). The contribution of this tunnel barrier to the resistance is given by $R/(1-R)$, where the reflection probability off the barrier is equal to the modulus square of the reflection amplitude, $R = |r|^2$. This elastic backscattering process yields the magnitude of the resistor; the actual inelastic dissipation takes place in the connected reservoirs. Such a model has been discussed, e. g., in Ref. [14]. We do not consider this picture here.

The outline of this paper is as follows. In Sec. 2, we briefly review earlier work, emphasizing the difference between the CL and AES approaches to dissipative dynamics and focusing on charge dynamics. The gauge symmetry underlying charge transport is U(1). In Sec. 3, we recall the physics of a quantum dot (QD) tuned to be near (but below) the Stoner instability. As such, the QD supports large magnetization. With the fluctuations in the magnitude of the spin ignored, the spin degree of freedom has an SU(2) symmetry. The coupling of such a QD to external leads gives rise to dissipation, which is formulated and studied in the framework of the AES approach (Sec. 4). In Sec. 4, we first compare our AES analysis for the spin case to our results obtained within the CL framework. We then note that this AES versus CL contrast differs from that in the standard charge U(1) case. We conclude in Sec. 6.

2. CALDEIRA–LEGGETT VERSUS AMBEGAOKAR–ECKERN–SCHÖN: THE CHARGE U(1) CASE

We consider the dynamics associated with a current running through a resistor, and compare the two paradigmatic representation thereof, the CL and AES ones.

2.1. CL action

The CL action of a current-biased linear resistor (modeled as a transmission line) is

$$i\mathcal{S}_{CL} = -\int dt_1 dt_2\, \alpha(t_1, t_2) \frac{[\varphi(t_1) - \varphi(t_2)]^2}{2} + {}$$
$$+ i\mathcal{S}^{source}, \quad (1)$$

where the dimensionless phase variable $\varphi(t)$ represents the effective flux variable $\Phi(t)$ via $\varphi = 2\pi(\Phi/\Phi_0)$, where $\Phi_0 = h/e$ is the flux quantum. The voltage across the resistor is given by $V = d\Phi/dt$, and Φ is the degree of freedom canonically conjugate to the charge

that has flown through the resistor $Q = \int dt\, I$. In Eq. (1), $\alpha(t_1, t_2)$ is the kernel of the Ohmic bath [8]. Dropping the time-local terms (important for avoiding renormalization of the nondissipative part of the action), we obtain

$$i\mathcal{S}_{CL} = \int dt_1 dt_2\, \alpha(t_1, t_2)\varphi(t_1)\varphi(t_2) + i\mathcal{S}^{source}. \quad (2)$$

We note that in the Keldysh notation, this action can be written as

$$i\mathcal{S}_{CL} = i\mathcal{S}^{source} + 2\int\limits_{-\infty}^{\infty} dt_1 \int\limits_{-\infty}^{\infty} dt_2 \left(\varphi_c(t_1)\, \frac{\varphi_q(t_1)}{2} \right) \times$$
$$\times \begin{pmatrix} 0 & \alpha^A \\ \alpha^R & \alpha^K \end{pmatrix}_{(t_1 - t_2)} \begin{pmatrix} \varphi_c(t_2) \\ \dfrac{\varphi_q(t_2)}{2} \end{pmatrix}, \quad (3)$$

where c and q refer to the classical and quantum components on the Keldysh contour [4][1]. The subscripts R, A, and K refer to the retarded, advanced, and Keldysh components of the matrix.

Employing the relation between the retarded and the advanced components of the kernel α, $\alpha^A(t_2, t_1) = -\left[\alpha^R(t_1, t_2)\right]^*$, we can write the action as

$$i\mathcal{S}_{CL} = i\mathcal{S}_{CL}^R + i\mathcal{S}_{CL}^K + i\mathcal{S}^{source} \quad (4)$$

with

$$i\mathcal{S}_{CL}^R = 2i\int dt_1 dt_2 \left[\operatorname{Im}\alpha^R(t_1 - t_2)\right] \varphi_q(t_1)\varphi_c(t_2), \quad (5)$$

$$i\mathcal{S}_{CL}^K = \frac{1}{2}\int dt_1 dt_2 \alpha^K(t_1 - t_2)\varphi_q(t_1)\varphi_q(t_2), \quad (6)$$

and

$$i\mathcal{S}^{source} = i\int dt\, I_{ex}(t)\frac{\Phi_0}{2\pi}\, \varphi_q(t). \quad (7)$$

One can [6] rewrite the Keldysh term of the action using the decoupling

$$\exp\left(\frac{i}{\hbar}\mathcal{S}_{CL}^K\right) = \int \mathcal{D}\xi \exp\left\{\frac{i}{\hbar}\int dt\, \hbar\xi(t)\varphi_q(t)\right\} \times$$
$$\times \exp\left\{\frac{1}{2}\int dt_1 dt_2 \hbar\, [\alpha_K]_{t_1, t_2}^{-1}\, \xi(t_1)\xi(t_2)\right\}. \quad (8)$$

It follows that

$$\langle \xi(t_1)\xi(t_2)\rangle = \frac{1}{\hbar}\,\alpha^K(t_1, t_2). \quad (9)$$

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[1] Unlike Ref. [4], we use the convention $\varphi_c = (\varphi_u + \varphi_d)/2$, $\varphi_q = \varphi_u - \varphi_d$, where u and d refer to the respective forward and backward parts of the Keldysh contour.

The resulting Langevin equation of motion is obtained by calculating the variation $i\delta\mathcal{S}_{CL}/\delta\varphi_q(t) = 0$. The equation obtained is

$$\frac{\dot{\Phi}_c(t)}{R} = I_{ex} + \delta I(t), \quad (10)$$

where $\delta I(t) \equiv e\xi(t)$ represents stochastic current noise. We note that the noise is additive, and is not affected by the bias current.

In deriving Eq. (10), we used the fact that the dissipative bath has an Ohmic spectrum [8], implying that

$$\operatorname{Im}\alpha^R(t) = \frac{1}{2}\frac{1}{R}\frac{\hbar^2}{e^2}\delta'(t) \quad \text{or}$$
$$\operatorname{Re}\alpha^R(\omega) = \frac{1}{2}\frac{1}{R}\frac{\hbar^2}{e^2}\omega, \quad (11)$$

where R is the resistance. The variation over the retarded part of the action leads to

$$\frac{i\delta S_{CL}^R}{\delta\Phi_q(t_1)} = \frac{2\pi}{\Phi_0}\, 2i\int dt_2 \left[\operatorname{Im}\alpha^R(t_1 - t_2)\right] \times$$
$$\times \left[2\pi\frac{\Phi_c(t_2)}{\Phi_0}\right] = \frac{i}{R}\,\dot{\Phi}_c(t_1). \quad (12)$$

The Fourier transform of the current noise correlator is given by

$$\langle \delta I(t_1)\delta I(t_2)\rangle_\omega = \frac{e^2}{\hbar}\,\alpha^K(\omega). \quad (13)$$

At equilibrium,

$$\alpha^K(\omega) = \left[\alpha^R(\omega) - \alpha^A(\omega)\right]\coth\frac{\hbar\omega}{2k_B T}. \quad (14)$$

The fluctuation–dissipation theorem follows from Eqs. (12) and (14):

$$\langle \delta I(t_1)\delta I(t_2)\rangle_\omega = \frac{\hbar\omega}{R}\,\coth\frac{\hbar\omega}{2k_B T}. \quad (15)$$

We note that the additivity of the noise and its independence from the bias current (Eq. (10)) imply that the noise is independent of I_{ex}, i.e., absence of shot noise.

2.2. AES action

The AES action is now given by

$$i\mathcal{S}_{AES} = -\int dt_1 dt_2\, \alpha(t_1, t_2) \times$$
$$\times (1 - \cos\left[\varphi(t_1) - \varphi(t_2)\right]) + i\mathcal{S}^{source}. \quad (16)$$

The source term is the same as in the preceding case. Similarly to the CL case, Eq. (4), we can write the action as

$$i\mathcal{S}_{AES} = i\mathcal{S}_{AES}^R + i\mathcal{S}_{AES}^K + i\mathcal{S}^{source}. \quad (17)$$

The retarded part is essentially identical to that in the CL case, having to do with the fact that t_1 and t_2 are very close to each other (cf. Eq. (11)), which allows us to expand the cos term in Eq. (16). But the Keldysh term is very different:

$$i\mathcal{S}_{AES}^K = \frac{1}{2} \int dt_1 dt_2 \alpha^K(t_1 - t_2) \times$$
$$\times \Big\{ [\cos\varphi(t_1)]_q [\cos\varphi(t_2)]_q +$$
$$+ [\sin\varphi(t_1)]_q [\sin\varphi(t_2)]_q \Big\}. \quad (18)$$

Decoupling the action via two auxiliary fields ξ_1 and ξ_2, we obtain [10]

$$\exp\left(\frac{i}{\hbar}\mathcal{S}_{AES}^K\right) = \int \mathcal{D}\xi_1 \mathcal{D}\xi_2 \times$$
$$\times \exp\left\{ \frac{i}{\hbar}\int dt\,\hbar\left(\xi_1(t)\,[\cos\varphi(t)]_q + \xi_2(t)\,[\sin\varphi(t)]_q\right)\right\} \times$$
$$\times \exp\left\{ \frac{1}{2}\int dt_1 dt_2\,\hbar\,[\alpha^K]_{t_1,t_2}^{-1}\,(\xi_1(t_1)\xi_1(t_2) +\right.$$
$$\left.+ \xi_2(t_1)\xi_2(t_2))\right\}. \quad (19)$$

The resulting equation of motion for the AES action is

$$\frac{\dot{\Phi}_c(t)}{R} = I_{ex} - e\xi_1 \sin\left(2\pi\frac{\Phi_c}{\Phi_0}\right) +$$
$$+ e\xi_2 \cos\left(2\pi\frac{\Phi_c}{\Phi_0}\right). \quad (20)$$

This equation can be cast into the form of Eq. (10) by writing $\delta I(t) = \delta I_1(t) + \delta I_2(t)$ with the two independent terms of current fluctuations defined as

$$\delta I_1 = -e\xi_1 \sin\left(2\pi\frac{\Phi_c}{\Phi_0}\right),$$
$$\delta I_2 = e\xi_2 \cos\left(2\pi\frac{\Phi_c}{\Phi_0}\right). \quad (21)$$

Equation of motion (20) implies that the noise is nonadditive, as can be shown explicitly from the following iterative procedure. The zeroth iteration gives $\Phi_c = Vt$, where $V = I_{ex}R$. Next, we introduce a correction $\Phi_c = Vt + \delta\Phi_c$ and obtain

$$\frac{\delta\dot{\Phi}_c(t)}{R} = -e\xi_1 \sin\left(2\pi\frac{Vt + \delta\Phi_c}{\Phi_0}\right) +$$
$$+ e\xi_2 \cos\left(2\pi\frac{Vt + \delta\Phi_c}{\Phi_0}\right). \quad (22)$$

The first iteration consists in dropping $\delta\Phi_c$ in the right-hand side of Eq. (22). The resulting stochastic

terms give rise to shot noise [10] (unlike the CL equation of motion). For $eV \gg k_B T$, we find

$$\langle\delta I_1(t_1)\delta I_1(t_2)\rangle_{\omega\to 0} = \langle\delta I_2(t_1)\delta I_2(t_2)\rangle_{\omega\to 0} =$$
$$= \frac{1}{2}\,e\,\frac{V}{R}. \quad (23)$$

3. A QUANTUM DOT NEAR THE STONER PHASE TRANSITION

Over the past few decades, the physics of quantum dots has become a focal point of research in nanoelectronics. The introduction of the Universal Hamiltonian [15–18] has made it possible to take the effects of electron–electron (e–e) interaction within a QD into account in a controlled way. This approach is applicable to a normal-metal QD when the Thouless energy E_{Th} and the mean single-particle level spacing δ satisfy $g_{QD} \equiv E_{Th}/\delta \gg 1$. Here, g_{QD} is the dimensionless conductance of the QD. The single-particle level spacing is given by $\delta \sim 1/V\nu_0$, where V is the volume of the QD and ν_0 is its density of states (DoS), and therefore $\delta \sim 1/L^d$ for a d-dimensional QD. The Thouless energy E_{Th} is the inverse time of flight (or diffusion time) of an electron across the quantum dot.

Within this scheme, the interactions are split into a sum of three spatially independent contributions in the charging, spin-exchange, and Cooper channels. Ignoring the latter (see below), the charging term leads to the phenomenon of Coulomb blockade, while the spin-exchange term can drive the system towards the Stoner instability [19]. In bulk systems, the exchange interaction competes with the kinetic energy, leading to Stoner instability. In finite-size systems, a mesoscopic Stoner regime may be a precursor of the bulk thermodynamic Stoner instability [15, 16]: a new phase, intermediate between paramagnetic and ferromagnetic ones, emerges, in which the total spin of the QD is finite but not extensive (i. e., not proportional to the volume of the dot). The mesoscopic Stoner regime can be realized in QDs made of materials close to the thermodynamic Stoner instability.

A QD in the metallic regime $g_{QD} \gg 1$ is described by the universal Hamiltonian [15]

$$H = H_0 + H_C + H_J + H_\lambda. \quad (24)$$

The noninteracting part of the universal Hamiltonian is

$$H_0 = \sum_{\alpha,\sigma} \epsilon_\alpha a_{\alpha,\sigma}^\dagger a_{\alpha,\sigma}, \quad (25)$$

where ϵ_α denotes the energy of a spin-degenerate (index σ) single-particle level α. The charging interaction term

$$H_C = E_C \left(\hat{N} - N_0 \right)^2 \qquad (26)$$

accounts for the Coulomb blockade. Here, $E_C \equiv e^2/2C$ denotes the charging energy of the QD with the self-capacitance C, N_0 represents the background charge, and

$$\hat{N} = \sum_{\alpha,\sigma} a^\dagger_{\alpha,\sigma} a_{\alpha,\sigma}$$

is the operator of the total number of electrons of the dot. For an isolated QD, the total number of electrons is fixed and, therefore, the charging interaction term can be omitted. The term

$$H_J = -J\hat{\mathbf{S}}^2 \qquad (27)$$

represents the ferromagnetic ($J > 0$) exchange interaction within the dot, where

$$\hat{\mathbf{S}} = \sum_{\alpha} a^\dagger_{\alpha,\sigma_1} \mathbf{S}_{\sigma_1\sigma_2} a_{\alpha,\sigma_2}$$

is the operator of the total spin of the dot. Here, $\mathbf{S}_{\sigma_1\sigma_2} \equiv (1/2)\boldsymbol{\sigma}_{\sigma_1\sigma_2}$, where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is a vector made of Pauli matrices. The interaction in the Cooper channel is described by

$$H_\lambda = \lambda T^\dagger T , \quad T = \sum_{\alpha} a_{\alpha,\uparrow} a_{\alpha,\downarrow}. \qquad (28)$$

In what follows, we do not take H_λ into account for the following reasons. For the dots defined in a 2D electron gas, the interaction in the Cooper channel is typically repulsive and, therefore, renormalizes to zero [16]. In the case of 3D QDs realized as small metallic grains, the interaction in the Cooper channel can be attractive, giving rise to interesting competition between superconductivity and ferromagnetism [20–22]. In that case, we assume that there is a weak magnetic field that suppresses the Cooper channel.

The starting point of our analysis of a dissipative Stoner QD (near the Stoner instability point) is the QD Hamiltonian

$$H_{dot} = \sum_{\alpha,\sigma} \epsilon_\alpha a^\dagger_{\alpha,\sigma} a_{\alpha,\sigma} - J\boldsymbol{S}^2, \qquad (29)$$

where we ignore possible correlations between the charging state and the spin configuration of the QD [23].

We note that for an isotropic spin exchange interaction (the Heisenberg model), the mesoscopic Stoner phase extends over $1/2 \le J/\delta \le 1$. In the anisotropic case [24, 25], the lower boundary of this inequality slides towards 1, with no mesoscopic Stoner phase for the

Ising spin [23, 26]. In the isotropic case, the ground-state spin S is the integer (for an even number of electrons in the QD) or half-integer (for an odd number) that is closest to $J/2(\delta - J)$. This value increases with increasing J and diverges as $J \to \delta$, which marks the onset of the macroscopic Stoner ferromagnetic phase. The problem is apparently easy to tackle theoretically. The interaction terms of the universal Hamiltonian consist only of zero-mode (zero wavenumber) contributions, which commute with each other. But the inclusion of the exchange term renders the problem nontrivial: the resulting action, which consists of Pauli matrices, is non-Abelian (more specifically, it is underlain by an SU(2) symmetry). Attempts to study the problem from different standpoints included the Ising limit [23] and the perturbation theory in the Ising anisotropy [24]. An exact solution that employs states classified by the total number of electrons and the total spin [27–29] requires the calculation of Clebsch–Gordan coefficients, which is not an easy task. In this way, Alhassid and Rupp have found an exact solution for the partition function in the absence of Zeeman splitting. Elements of their analysis were then incorporated into a master-equation analysis of electric [27, 28] and thermal [30] properties. Independently, electron transport through a QD for low temperatures ($T \ll \delta$) was studied in [31]. That analysis, accounting for the charging and exchange interactions, also employed a master-equation approach.

An exact solution based on the Wei–Norman–Kolokolov approach was presented in [32], and was then extended to include randomness-induced spectral fluctuations [33]. The tunneling density of states and the spin susceptibility were calculated; other thermodynamic and linear response correlations are calculable as well. The study of shot noise near the Stoner point was reported in [34].

We note that the exact solution approaches mentioned above, although elegant and powerful, are very difficult to generalize to more complex setups, in particular, to setups where external leads are added (a common means for the introduction of dissipation). An efficient approximation that can be generalized to such setups employs the adiabatic approximation of the spin stochastic dynamics [35].

4. AES APPROACH FOR SU(2) SPIN

Our approach [36] can be viewed as a generalization of the Landau–Lifschitz–Gilbert (LLG)–Langevin equation [37, 38], central to the field of spintronics [39],

to a regime where quantum dynamics dominates. Stochastic LLG equations have been derived in numerous publications for both a localized spin in an electronic environment (a situation of the Caldeira–Leggett type) [40, 41] and a magnetization formed by itinerant electrons [42, 43]. In all these works, the precession frequency was assumed to be lower than the temperature or the voltage, thus justifying the semiclassical treatment of the problem. In this regime, the geometric phase did not influence the Langevin terms.

Our derivation here is technically close to that in Ref. [42]. However, in contrast to Ref. [42], we do not limit ourselves to small deviations of the spin from the instantaneous direction, but consider the action on global trajectories covering the entire Bloch sphere.

To demonstrate the emergence of an AES-like effective action, we consider a QD with strong exchange interaction coupled to a normal lead. The Hamiltonian is $H = H_{dot} + H_{lead} + H_{tun}$. The QD is described by the magnetic part[2] of the universal Hamiltonian [15]

$$H_{dot} = \sum_{\alpha, \sigma} \epsilon_\alpha a_{\alpha, \sigma}^\dagger a_{\alpha, \sigma} - J\mathbf{S}^2 + \mathbf{B} \cdot \mathbf{S}, \qquad (30)$$

where $\mathbf{S} \equiv (1/2) \sum_{\alpha, \sigma_1, \sigma_2} a_{\alpha, \sigma_1}^\dagger \boldsymbol{\sigma}_{\sigma_1, \sigma_2} a_{\alpha, \sigma_2}$ is the operator of the total spin on the QD, $\mathbf{B}$ is the external magnetic field, and $J > 0$ is the corresponding "zero mode" ferromagnetic exchange constant. The Hamiltonian of the lead and that describing the tunneling between the dot and the lead are standard: $H_{lead} = \sum_{\gamma, \sigma} \epsilon_\gamma c_{\gamma, \sigma}^\dagger c_{\gamma, \sigma}$ and $H_{tun} = \sum_{\alpha, \gamma, \sigma} V_{\alpha, \gamma} a_{\alpha, \sigma}^\dagger c_{\gamma, \sigma} + \text{H.c.}$ We here assume a nonmagnetic lead; γ is the orbital quantum number describing eigenmodes of the lead.

We consider the Keldysh generating functional $\mathcal{Z} = \int D\bar{\Psi} D\Psi \exp[i \mathcal{S}_\Psi]$, where the Keldysh action is given by $\mathcal{S}_\Psi = \oint_K dt \, (i\bar{\Psi} \partial_t \Psi - H)$ (plus the necessary source terms, which are not written explicitly). For brevity, Ψ denotes all fermionic fields and the time t runs along the Keldysh contour. After standard Hubbard–Stratonovich manipulations [4, 32, 35] decoupling the interaction term $-J\mathbf{S}^2$, we obtain $\mathcal{Z} = \int D\mathcal{M} \exp[i \mathcal{S}_M]$, and the action for the bosonic vector $\mathcal{M}(t)$ becomes

$$i\mathcal{S}_M = \text{tr} \ln \begin{pmatrix} G_{dot}^{-1} & -\hat{V} \\ -\hat{V}^\dagger & G_{lead}^{-1} \end{pmatrix} - i \oint_K dt \frac{|\mathcal{M}|^2}{4J}, \quad (31)$$

where

$$G_{dot}^{-1} \equiv i\partial_t - \epsilon_\alpha - (\mathcal{M}(t) + \mathbf{B}) \cdot \boldsymbol{\sigma}/2$$

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[2] Here, we disregard the charging part of the "universal" Hamiltonian, having in mind, e. g., systems of the type considered in Refs. [42, 43]. Consequently, no Kondo physics is expected.

and $G_{lead}^{-1} \equiv i\partial_t - \epsilon_\gamma$. Both G_{dot}^{-1} and G_{lead}^{-1} are matrices with time, spin, and orbital indexes. We introduce $\mathbf{M}(t) \equiv \mathcal{M}(t) + \mathbf{B}$. Expanding (31) in powers of the tunneling matrix $\hat{V}$ and re-summing, we easily obtain

$$i\mathcal{S}_M = \text{tr} \ln G_{lead}^{-1} + \text{tr} \ln \left(G_{dot}^{-1} - \Sigma \right) - \\ - i \oint_K dt \frac{|\mathbf{M} - \mathbf{B}|^2}{4J}, \quad (32)$$

where the self energy is $\Sigma \equiv \hat{V} G_{lead} \hat{V}^\dagger$. The first term is trivial, i. e., it can never contain the source fields, and is therefore dropped in what follows.

Rotating frame. We introduce the unit-length vector

$$\mathbf{n}(t) = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta) \qquad (33)$$

through $\mathbf{M}(t) = M(t)\mathbf{n}(t)$ and transform to a coordinate system in which $\mathbf{n}$ coincides with the z axis: $\mathbf{n}(t) \cdot \boldsymbol{\sigma} = R(t)\sigma_z R^\dagger(t)$. This condition identifies the unitary rotation matrix R as an element of SU(2)/U(1). Indeed, if we use the Euler angle representation

$$R = \exp\left(-\frac{i\phi}{2}\sigma_z\right) \exp\left(-\frac{i\theta}{2}\sigma_y\right) \times \\ \times \exp\left(-\frac{i\psi}{2}\sigma_z\right), \quad (34)$$

then the angles $\phi(t)$ and $\theta(t)$ determine the direction of $\mathbf{n}(t)$, while $\psi(t)$ is arbitrary, i. e., the condition $\mathbf{n}(t) \cdot \boldsymbol{\sigma} = R\sigma_z R^\dagger$ is satisfied at any value of $\psi(t)$. Therefore, ψ represents the gauge freedom of the problem. We introduce a shifted gauge field $\chi(t) \equiv \phi(t) + \psi(t)$. This way, a periodic boundary condition, e.g., in the Matsubara representation $R(\tau) = R(\tau + \beta)$, is satisfied for $\chi(\tau + \beta) = \chi(\tau) + 4\pi m$. (The fact that m is integer is intimately related to the spin quantization [44].) We can always assume trivial boundary conditions for χ, i. e., $m = 0$. We keep this representation of the rotation matrix R also for the Keldysh technique.

We perform a transition to the rotating frame and obtain

$$i\mathcal{S}_M = \text{tr} \ln \left[R^\dagger \left(G_{dot}^{-1} - \Sigma \right) R \right] - \\ - i \oint_K dt \left[\frac{M^2}{4J} - \frac{\mathbf{B} \cdot \mathbf{M}}{2J} \right] \quad (35)$$

(we omit the constant term $\propto |\mathbf{B}|^2$). For the Green's function of the dot, this gives

$$R^\dagger G_{dot}^{-1} R = i\partial_t - \epsilon_\alpha - \frac{M(t)\sigma_z}{2} - Q,$$

where we define the gauge (Berry) term as $Q \equiv R^\dagger(-i\partial_t)R = Q_\parallel + Q_\perp$. Here,

$$Q_\parallel \equiv [\dot\phi(1 - \cos\theta) - \dot\chi]\frac{\sigma_z}{2}$$

and

$$Q_\perp \equiv -\frac{1}{2}\exp[i\chi\sigma_z]\left[\dot\theta\,\sigma_y - \dot\phi\sin\theta\,\sigma_x\right]\exp[i\phi\sigma_z].$$

We note that Q depends on the choice of the gauge field χ. Finally, we obtain

$$i\mathcal{S}_M = \mathrm{tr}\ln\left[G_{dot,z}^{-1} - Q - R^\dagger\Sigma R\right] -$$
$$- i\oint_K dt\left[\frac{M^2}{4J} - \frac{\mathbf{B}\cdot\mathbf{M}}{2J}\right], \quad (36)$$

where $G_{dot,z}^{-1} \equiv i\partial_t - \epsilon_\alpha - (1/2)\,M(t)\,\sigma_z$.

To find the semiclassical trajectories of the magnetization, we need to consider paths $M(t)$, $\theta(t)$, $\phi(t)$ on the Keldysh contour such that the quantum components are small. The quantum (q) and classical (c) components of the fields are expressed in terms of the forward (u) and backward (d) components [4], i.e., $\phi_q(t) = \phi_u(t) - \phi_d(t)$ and $\phi_c(t) = (\phi_u(t) + \phi_d(t))/2$. Performing the standard Keldysh rotation [4], we thus obtain

$$i\mathcal{S}_M = \mathrm{tr}\ln\left[\tilde{G}_{dot,z}^{-1} - \tilde{Q} - \tilde{R}^\dagger\tilde{\Sigma}\tilde{R}\right] +$$
$$+ i\int dt\,\frac{\mathbf{B}\cdot\mathbf{M}_q}{2J} - i\int dt\,\frac{M_c M_q}{2J}, \quad (37)$$

where $\tilde{G}_{dot,z}^{-1} \equiv \tau_x G_{dot,z}^{-1}$. The local-in-time matrix fields $Q(t)$ and $R(t)$ also acquire a 2×2 matrix structure in the Keldysh space, e.g., $\tilde{Q} = Q_c\tau_x + Q_q\tau_0/2$, where $\tau_{x,y,z,0}$ are the standard Pauli matrices.

The adiabatic limit. Thus far, we have made no approximations. The action in (37) governs the dynamics of both the magnetization amplitude $M(t)$ and the magnetization direction $\mathbf{n}(t)$. Here, we focus on the case of a large amplitude M (more precisely, M fluctuating around a large average value M_0). Such a situation arises either on the ferromagnetic side of the Stoner transition or on the paramagnetic side, but very close to the transition. In the latter case, as was shown in Refs. [32, 35], integrating out the fast angular motion of $\mathbf{n}$ creates an effective potential for M, forcing it to acquire a finite average value. More precisely the angular motion with frequencies $\omega \gg \max[T, B]$ (we adopt the units $\hbar = k_B = 1$) can be integrated out, renormalizing the effective potential for the slow part of $M(t)$. The very interesting question of the dissipative dynamics of slow longitudinal fluctuations of $M(t)$

in the mesoscopic Stoner regime will be addressed elsewhere. Here, we focus on the slow angular motion and substitute $M(t) = M_0$. Thus, the last term in (37) can be dropped. We note that in the adiabatic limit, we can neglect $\tilde{Q}_\perp$ because it contributes only in the second order in $d\mathbf{n}/dt$ [35].

The idea now is to expand action (37) in both $\tilde{Q}$ (which is small due to the slowness of $\mathbf{n}(t)$) and $\tilde{R}^\dagger\tilde{\Sigma}\tilde{R}$ (which is small due to the smallness of the tunneling amplitudes). A straightforward analysis reveals that a naive expansion to the lowest order in both violates the gauge invariance with respect to the choice of $\chi(t)$. It can be shown that the expansion in $\tilde{R}^\dagger\tilde{\Sigma}\tilde{R}$ is gauge invariant only if all orders of $\tilde{Q}$ are taken into account, i.e., if $(\tilde{G}_{dot,z}^{-1} - \tilde{Q})^{-1}$ is used as the zeroth-order Green's function in the expansion. This problem necessitates a clever choice of gauge, such that $(\tilde{G}_{dot,z}^{-1} - \tilde{Q})^{-1}$ is as close to $\tilde{G}_{dot,z}$ as possible, i.e., the effect of $\tilde{Q}$ is "minimized".

Choice of gauge. Because action (37) is gauge invariant, we are allowed to choose the most convenient form of $\chi(t)$. We make the choice

$$\dot\chi_c(t) = \dot\phi_c(t)\,(1 - \cos\theta_c(t)),$$
$$\chi_q(t) = \phi_q(t)\,(1 - \cos\theta_c(t)), \quad (38)$$

which satisfies the necessary boundary conditions $\chi_q(t = \pm\infty) = 0$.

Here, we present a detailed justification of the gauge in Eq. (38). Ideally, we should have chosen a gauge that would lead to $Q_\parallel = 0$. Seemingly, this might have been achieved with the choice $\dot\chi(t) = \dot\phi(t)\,(1 - \cos\theta(t))$ on both branches of the Keldysh contour. But this choice violates our desired boundary conditions because the integrals over $\dot\chi$ accumulated between $t = -\infty$ and $t = +\infty$ on the upper and lower Keldysh branches are different. Such a difference would show up as nontrivial boundary conditions on χ_q at either $t = -\infty$ or $t = +\infty$. In other words, had we selected $\dot\chi(t) = \dot\phi(t)\,(1 - \cos\theta(t))$, we whould have violated the requirement $\chi_q(t = \pm\infty) = 0$. We note, however, that to the linear order in the quantum components, the condition

$$\dot\chi(t) = \dot\phi(t)\,(1 - \cos\theta(t))$$

yields

$$\dot\chi_q = \dot\phi_q(1 - \cos\theta_c) + \theta_q\sin\theta_c\,\dot\phi_c,$$

leading to

$$\chi_q(t) = \int^t dt' \left[\dot{\phi}_q(t')(1 - \cos\theta_c(t')) + \right.$$
$$+ \theta_q(t') \sin\theta_c(t') \, \dot{\phi}_c(t') \Big] = \phi_q(t)(1 - \cos\theta_c(t)) +$$
$$+ \int^t dt' \, \sin\theta_c(t') \Big[\theta_q(t') \, \dot{\phi}_c(t') - \dot{\theta}_c(t') \, \phi_q(t') \Big].$$

The first term vanishes at $t = \pm\infty$, but the last term does not. We thus include only the first term in χ_q, leading to Eq. (38). The gauge in (38) satisfies the boundary conditions and leads to the desired cancelation $Q_\parallel^c = 0$, whereas the quantum component of $Q_\parallel$ remains nonzero:

$$Q_{\parallel,q} = \frac{1}{2} \, \sigma_z \, \sin\theta_c \left[\dot{\phi}_c \theta_q - \dot{\theta}_c \phi_q \right]. \qquad (39)$$

At the same time, this choice allows expanding the Keldysh action in the small ϕ_q and θ_q because there are no $\dot{\phi}_q$ terms remaining in (39).

Berry phase (Wess–Zumino–Novikov–Witten (WZNW) action). Expanding the term of the zeroth order in $\tilde{\Sigma}$ in action (37) to the first order in $\tilde{Q}$, we obtain the Berry phase (WZNW) action well known in spin physics (see, e. g., Refs. [44, 45])

$$i\mathcal{S}_{WZNW} = -\frac{1}{2} \int dt \, \mathrm{tr} \left[G^K_{dot,z}(t,t) Q_{\parallel,q}(t) \right], \qquad (40)$$

which after a straightforward calculation becomes

$$i\mathcal{S}_{WZNW} = iS \int dt \, \sin\theta_c \left[\dot{\phi}_c \theta_q - \dot{\theta}_c \phi_q \right], \qquad (41)$$

where $S \equiv N(M_0)/2$ is the (dimensionless) spin of the dot. Here, $N(M_0)$ is the number of orbital levels of the dot in the energy interval M_0 around the Fermi energy. Roughly, $S = M_0 \bar{\rho}_{dot}/2$, where $\bar{\rho}_{dot}$ is the density of states averaged over the energy interval M_0. The effects of mesoscopic fluctuations of the density of states were considered in Ref. [33].

AES action. The central result in this paper is the AES-like [9, 10] effective action, which we obtain by expanding (37) to the first order in $\tilde{R}^\dagger \tilde{\Sigma} \tilde{R}$: $i\mathcal{S}_{AES} = -\mathrm{tr}\left[\tilde{G}_{dot,z} \tilde{R}^\dagger \tilde{\Sigma} \tilde{R} \right]$. This gives

$$i\mathcal{S}_{AES} = -g \int dt_1 dt_2 \mathrm{tr} \left[\left(R^\dagger_c(t_1) \, \frac{R^\dagger_q(t_1)}{2} \right) \times \right.$$
$$\times \left(\begin{matrix} 0 & \alpha^A \\ \alpha^R & \alpha^K \end{matrix} \right)_{(t_1-t_2)} \left(\begin{matrix} R_c(t_2) \\ \dfrac{R_q(t_2)}{2} \end{matrix} \right) \right], \quad (42)$$

where $g \equiv (\hbar/e^2)(G_\uparrow + G_\downarrow)/2$. Here, $G_\sigma \equiv 2\pi \, (e^2/\hbar) \times \times |V|^2 \, \rho_{lead} \rho^\sigma_{dot}$ is the tunneling conduction of the spin projection σ, $\rho^{\uparrow/\downarrow}_{dot}$ are the densities of states at the respective $\uparrow$ and $\downarrow$ Fermi levels, and the density of states in the lead, ρ_{lead}, is spin independent. The standard [10] Ohmic kernel functions are given by $\alpha^R(\omega) - \alpha^A(\omega) = 2\omega$ and $\alpha^K(\omega) = 2\omega \coth(\omega/2T)$. Action (42) strongly resembles the AES action [10], with the U(1) exponents $\exp\left[i\varphi/2\right]$ replaced by the SU(2) matrices R. Fixing the gauge of R is an essential part of our procedure.

Semiclassical equations of motion. From effective action (42), we derive the semiclassical equation of motion. We follow the ideas proposed in Ref. [6]. Using the representation

$$R = A_0\sigma_0 + iA_x\sigma_x + iA_y\sigma_y + iA_z\sigma_z,$$

with

$$A_0 \equiv \cos\frac{\theta}{2} \cos\frac{\chi}{2}, \quad A_x \equiv \sin\frac{\theta}{2} \sin\left(\phi - \frac{\chi}{2}\right),$$
$$A_y \equiv -\sin\frac{\theta}{2} \cos\left(\phi - \frac{\chi}{2}\right), \quad A_z \equiv -\cos\frac{\theta}{2} \sin\frac{\chi}{2},$$

we rewrite the AES action (Eq. (42)) as

$$\mathcal{S}_{AES} = \mathcal{S}^R_{AES} + \mathcal{S}^K_{AES},$$

where

$$i\mathcal{S}^R_{AES} = -2ig \int dt_1 dt_2 \left[\mathrm{Im}\,\alpha^R(t_1 - t_2) \right] \times$$
$$\times \sum_j A_{j,q}(t_1) A_{j,c}(t_2), \quad (43)$$

$$i\mathcal{S}^K_{AES} = -\frac{g}{2} \int dt_1 dt_2 \, \alpha_K(t_1 - t_2) \times$$
$$\times \sum_j A_{j,q}(t_1) A_{j,q}(t_2). \quad (44)$$

Here, $j = 0, x, y, z$. The Keldysh part of action (44) leads to random Langevin forces. This can be shown [6] using the Hubbard–Stratonovich transformation

$$\exp\left(i\mathcal{S}^K_{AES} \right) = \int \left(\prod_{j=0,x,y,z} D\xi_j \right) \times$$
$$\times \exp\left[\int dt \left\{ i \sum_{j=0,x,y,z} \xi_j A_{j,q} \right\} + i\mathcal{S}_\xi \right], \quad (45)$$

where the action $\mathcal{S}_\xi$ is given by

$$i\mathcal{S}_\xi = -\frac{1}{2g} \sum_j \int dt_1 dt_2 \left[\alpha^K \right]^{-1}_{(t_1-t_2)} \xi_j(t_1)\xi_j(t_2). \quad (46)$$

In other words, $\langle \xi_j(t_1)\xi_k(t_2)\rangle = \delta_{jk}\, g\, \alpha^K(t_1 - t_2)$ and $\langle \xi_j \rangle = 0$. We obtain Langevin equations (47) from

$$\frac{\delta i \mathcal{S}_{total}}{\delta \phi_q(t)} = \frac{\delta i \mathcal{S}_{total}}{\delta \theta_q(t)} = 0,$$

where

$$i\mathcal{S}_{total} \equiv i\mathcal{S}_B + i\mathcal{S}_{WZNW} + i\mathcal{S}_{AES}^R + \int dt \sum_j i\xi_j A_{j,q}.$$

Here, $i\mathcal{S}_B = -iS\gamma B \int dt \sin\theta_c\, \theta_q$ is the action related to the magnetic field (in the z direction). Prior to performing the variation of the action, the field χ is replaced according to the gauge-fixing choice (Eq. (38)). Finally, we use the relation $\alpha_R''(t) = (\partial_t + C)\delta(t)$ (the constant C is important for causality but drops out in our calculation) and obtain the equations of motion

$$\dot{\theta}_c + \tilde{g}\,\sin\theta_c\dot{\phi}_c = \eta_\theta,$$
$$\sin\theta_c\left(\dot{\phi}_c - \gamma B\right) - \tilde{g}\,\dot{\theta}_c = \eta_\phi. \tag{47}$$

Here, $\tilde{g} \equiv g/2S$ and $\gamma = (J\bar{\rho}_{dot})^{-1}$ is the "gyromagnetic" constant of the order of unity. The Langevin forces (torques) are given by

$$\eta_\theta = \frac{1}{2S}\,\cos\frac{\theta_c}{2}\left[\xi_x\,\cos\left(\phi_c - \frac{\chi_c}{2}\right) + \right.$$
$$+ \xi_y\,\sin\left(\phi_c - \frac{\chi_c}{2}\right)\right] -$$
$$- \frac{1}{2S}\sin\frac{\theta_c}{2}\left[\xi_z\,\cos\frac{\chi_c}{2} + \xi_0\,\sin\frac{\chi_c}{2}\right],$$
$$\eta_\phi = -\frac{1}{2S}\,\cos\frac{\theta_c}{2}\left[\xi_x\,\sin\left(\phi_c - \frac{\chi_c}{2}\right) - \right.$$
$$- \xi_y\,\cos\left(\phi_c - \frac{\chi_c}{2}\right)\right] -$$
$$- \frac{1}{2S}\sin\frac{\theta_c}{2}\left[\xi_z\,\sin\frac{\chi_c}{2} - \xi_0\,\cos\frac{\chi_c}{2}\right]. \tag{48}$$

The left-hand sides of Eqs. (47) represent the standard Landau–Lifshitz–Gilbert (LLG) equations [37] (without a random torque). The right-hand sides represent the random Langevin torque. The latter is expressed in terms of four independent stochastic variables ξ_j $(j = 0, x, y, z)$, which satisfy $\langle \xi_j(t_1)\xi_k(t_2)\rangle = \delta_{jk}\, g\, \alpha^K(t_1 - t_2)$ and $\langle \xi_j \rangle = 0$. On the Gaussian level, i.e., if fluctuations of θ_c and ϕ_c are neglected in Eqs. (48), the Langevin forces η_θ and η_ϕ are independent of each other and have the same autocorrelation functions: $\langle \eta_\theta(t_1)\eta_\phi(t_2)\rangle = 0$ and $\langle \eta_\theta(t_1)\eta_\theta(t_2)\rangle = \langle \eta_\phi(t_1)\eta_\phi(t_2)\rangle$. We emphasize that, in general, the noise depends on the angles θ_c and ϕ_c, leading to complicated dynamics within Eqs. (47). In the classical domain, i.e., for frequencies much lower than T, we can approximate

$$\langle \xi_j(t_1)\xi_k(t_2)\rangle = 4gT\delta(t_1 - t_2)\,\delta_{jk}.$$

Then

$$\langle \eta_\phi(t_1)\eta_\phi(t_2)\rangle = \langle \eta_\theta(t_1)\eta_\theta(t_2)\rangle = (gT/S^2)\delta(t_1 - t_2).$$

Thus, the situation is simple and we reproduce the treatment in Ref. [38].

Effective temperature. In the quantum high-frequency domain, the situation is different. We cannot interpret the four independent fields ξ_n as representing the components of a fluctuating magnetic field. A close inspection of Eqs. (47) shows that in the regime of weak dissipation, $S \gg 1$ and $\tilde{g} \ll 1$, the spin can precess with the frequency $\tilde{B} \equiv \gamma B/(1 + \tilde{g}^2)$ at an almost constant θ for a long time of the order of (shorter than) $(\tilde{g}\tilde{B})^{-1}$. For such time scales, we can approximate $\phi_c = \tilde{B}t$ and $\chi_c = (1 - \cos\theta_c)\phi_c = (1 - \cos\theta_c)\tilde{B}t$. Thus, the Langevin fields ξ_n in (48) are multiplied by fast oscillating cosines and sines with the frequencies $\omega_{cos} \equiv \tilde{B}\cos^2(\theta_c/2)$ and $\omega_{sin} \equiv \tilde{B}\sin^2(\theta_c/2)$. Therefore[3],

$$\langle \eta_{\phi,\theta}(t_1)\eta_{\phi,\theta}(t_2)\rangle_{\omega=0} =$$
$$= \frac{g}{4S^2}\left[\cos^2\left(\frac{\theta_c}{2}\right)\alpha_K\left(\omega_{cos}\right) + \right.$$
$$\left. + \sin^2\left(\frac{\theta_c}{2}\right)\alpha_K\left(\omega_{sin}\right)\right]. \tag{49}$$

In the quantum regime $T \ll \tilde{B}$, these correlation functions differ substantially from the classical ones,

$$\langle \eta_\phi(t)\eta_\phi(t')\rangle_{\omega=0} = \langle \eta_\theta(t)\eta_\theta(t')\rangle_{\omega=0} = \frac{gT}{S^2}.$$

Hence, if the spin could be held for a long time on a constant $\theta_c = \theta_0$ trajectory (one possible way to do so was proposed in Ref. [36]), the diffusion would be determined by quantum noise at the frequencies ω_c and ω_s, which are governed by the geometric phase. More precisely, the spread of θ_c and ϕ_c (in the rotating frame) would then be given by $(\Delta\theta)^2 = \sin^2\theta_0\,(\Delta\phi)^2 = Dt$, where

$$D = (g/S^2)T_{eff}, \tag{50}$$

and the effective temperature is calculated from (49) to be

$$T_{eff} = \frac{\tilde{B}}{2}\,\cos^4\left(\frac{\theta_0}{2}\right)\coth\left[\frac{\tilde{B}}{2T}\cos^2\left(\frac{\theta_0}{2}\right)\right] +$$
$$+ \frac{\tilde{B}}{2}\,\sin^4\left(\frac{\theta_0}{2}\right)\coth\left[\frac{\tilde{B}}{2T}\sin^2\left(\frac{\theta_0}{2}\right)\right]. \tag{51}$$

---

[3] We drop nonstationary terms depending on $t_1 + t_2$ here.

We emphasize once again that this semiclassical analysis is valid for a highly nonequilibrium situation in which the spin is driven and is kept artificially at a trajectory with $\theta_c = \theta_0 \neq 0$.

Semiclassical approximation. We are now ready to discuss the physical meaning of the semiclassical approximation, i. e., the expansion of action (42) through the second order in θ_q and ϕ_q. The nonexpanded action is periodic in both θ_q and ϕ_q. The periodicity in ϕ_q corresponds to the quantization of the z spin component $S_z = S \cos \theta_c$. By expanding, we restrict ourselves to the long-time limit, in which S_z has already "jumped" many times by $\Delta S_z = 1$ in the course of spin diffusion. We therefore neglect cumulants of spin noise higher than the second (see, e. g., Ref. [46] for a similar discussion of charge noise). However, we obtain a correct second cumulant with down-converted quantum noise (similar to shot noise in the charge sector). This is due to the "multiplicative noise" character of our Keldysh action (42) similar to the original AES case [10] (see also [47]).

Equilibrium dynamics near $\boldsymbol{\theta_c = 0}$. In the absence of external driving at $T \ll \tilde{B}$, Eqs. (47) lead to fast relaxation of the spin towards the north pole of the Bloch sphere, i. e., $\theta_c = 0$. Here, we show that the effective temperature introduced above loses its meaning in this case. Near the north pole, the spherical coordinates are not applicable and we rewrite Langevin equations (47) in Cartesian coordinates. Namely, we define $x = \sin \theta_c \cos \phi_c \approx \theta_c \cos \phi_c$ and $y \approx \theta_c \sin \phi_c$. The new Langevin equations for x and y (valid for $x, y \ll 1$) become

$$
\begin{aligned}
\dot{x} &= -\tilde{B}y - g\tilde{B}x + \frac{1}{2S(1+\tilde{g}^2)}\left(\xi_x - g\xi_y\right), \\
\dot{y} &= \tilde{B}x - g\tilde{B}y + \frac{1}{2S(1+\tilde{g}^2)}\left(\xi_y + g\xi_x\right).
\end{aligned}
\tag{52}
$$

A straightforward analysis of these linear equations leads to the stationary widths (standard deviations) of the order of $\Delta x = \Delta y \sim 1/\sqrt{S}$. Taking the standard relation $\langle \mathbf{S}^2 \rangle = \langle S_x^2 \rangle + \langle S_y^2 \rangle + \langle S_z^2 \rangle = S(S+1)$ into account, we observe that in the pure state $S_z = S$, the relation $\langle S_x^2 \rangle + \langle S_y^2 \rangle = S^2(\Delta x^2 + \Delta y^2) = S$ holds. Thus, fluctuations of the order of $\Delta x = \Delta y \sim 1/\sqrt{S}$ are purely quantum (they would be of this order also for $\Delta S_z \sim 1$) and the semiclassical analysis is inapplicable in this case.

5. CL VERSUS AES

In this section, we compare the SU(2) AES model described in Sec. 4 with the straightforward generalisa-

tion of the Caldeira–Legget model for the spin SU(2) case. We further analyze the similarities and the differences between AES and CL approaches in the U(1) and SU(2) cases.

5.1. CL in the spin SU(2) case

The CL action arises from the interaction of the type $H_{int} = \mathbf{h} \cdot \mathbf{n}$. Here, $\mathbf{n} \equiv \mathbf{S}/S$ and the vector field $\mathbf{h}$ represents isotropic fluctuations of an effective magnetic field with the Keldysh correlation function $\langle T_K h_n(t_1) h_m(t_2) \rangle = g\alpha(t_1, t_2) \delta_{n,m}$, where the times t_1 and t_2 are on the Keldysh contour. In reality, the field $\mathbf{h}$ can be, e. g., due to the Kondo coupling of the localized spin $\mathbf{S}$ to the electron–hole continuum. The coupling constant g is chosen such that the equations of motion are exactly the same as in the AES case, where g was proportional to the tunneling conductance. Assuming that the fluctuations are Gaussian, we obtain the effective action

$$
\mathcal{S}_{CL} = \frac{g}{2} \oint_K dt_1 \oint_K dt_2 \, \alpha(t_1, t_2) \, (1 - \mathbf{n}(t_1)\mathbf{n}(t_2)). \tag{53}
$$

The Keldysh analysis similar to that presented above again produces Eqs. (47), but the Langevin terms look different:

$$
\begin{aligned}
\eta_\theta &= \frac{1}{2S}\left(-\xi_x \sin \phi + \xi_y \cos \phi\right), \\
\eta_\phi &= \frac{\sin \theta}{2S}\xi_z - \frac{\cos \theta}{2S}\left(\xi_x \cos \phi + \xi_y \sin \phi\right).
\end{aligned}
\tag{54}
$$

Only three random fields ξ_n ($n = x, y, z$) are needed. Their fluctuations are given by

$$
\langle \xi_n(t_1)\xi_m(t_2) \rangle = \delta_{nm} \, g \, \alpha^K(t_1 - t_2).
$$

Exactly these equations are derived in Ref. [38] before making the high-temperature approximation, which makes the $\cos \phi$ and $\sin \phi$ factors in the right hand side unimportant. In contrast to the AES case, we obtain

$$
\begin{aligned}
\langle \eta_\theta(t_1)\eta_\theta(t_2) \rangle_{\omega=0} &= \frac{g}{4S^2} \, \alpha^K\left(\omega = \tilde{B}\right), \\
\langle \eta_\phi(t_1)\eta_\phi(t_2) \rangle_{\omega=0} &= \frac{g}{4S^2} \times \\
\times \Big[\cos^2 \theta_c \, \alpha^K\left(\omega = \tilde{B}\right) &+ \sin^2 \theta_c \, \alpha^K\left(\omega = 0\right)\Big].
\end{aligned}
\tag{55}
$$

We observe that the diffusion is not isotropic is this case. That is, in the θ-direction, the diffusion is characterized by $D_\theta = (g/S^2)T_\theta$, where $T_\theta = (\tilde{B}/2) \times \coth\left(\tilde{B}/2T\right)$. For the ϕ-direction, we obtain $D_\phi = (g/S^2)T_\phi$ with

$$
T_\phi = \frac{1}{2} \cos^2 \theta \, \tilde{B} \, \coth\frac{\tilde{B}}{2T} + \sin^2 \theta \, T = \cos^2 \theta \, T_\theta + \sin^2 \theta \, T.
$$

We observe that $T_\theta > T_\phi$. This anisotropy is most pronounced for $\theta = \pi/2$ and $T \ll \tilde{B}$. We emphasise once again that the above-mentioned diffusion occurs in a highly nonequilibrium case of a spin being artificially held on a trajectory with constant $\theta \neq 0$. At equilibrium, as in the AES case, the semiclassical analysis is not applicable.

5.2. Comparisons: CL and AES for U(1) and SU(2) cases

Here, we compare the similarities and differences between the CL and the AES pictures in the U(1) charge case with those in the SU(2) spin case. In the U(1) case, the semiclassical equation of motion can be cast in the form of Eq. (10) for both the CL and AES models. But the Langevin term, i. e., the fluctuating current δI, is entirely different in the two models at low temperatures $k_B T \ll eV$. In the CL case, $\delta I = e\xi$ is produced by one stochastic variable ξ, whose noise spectrum is Ohmic at equilibrium. In the AES case, two independent stochastic variables ξ_1 and ξ_2 are needed (see Eq. (21)). Both these variables have equilibrium Ohmic noise, but, due to the multiplicative oscillating factors in Eq. (21), the noise of δI at zero frequency is determined by the noise of $\xi_{1,2}$ at the frequency V. This leads to the appearance of shot noise.

Analogously, in the SU(2) spin case, both CL and AES models lead to the semiclassical stochastic LLG equations of form (47). The two Langevin terms (spin torques) η_θ and η_ϕ are different in the two models, however. In the CL case, η_θ and η_ϕ can be expressed (see Eq. (54)) via three independent stochastic variables ξ_x, ξ_y, ξ_z (all having Ohmic equilibrium noise spectra). But in the AES case, we need four independent stochastic variables ξ_x, ξ_y, ξ_z, ξ_0 with an Ohmic equilibrium spectrum (see Eq. (48)).

In both SU(2) CL and SU(2) AES models, the noise is multiplicative. That is, in both Eq. (54) and Eq. (48), the independent Langevin variables are multiplied by trigonometric functions of the Euler angles θ and ϕ. Hence, in both models, the frequency shifts are similar to those leading to shot noise in the U(1) case. But these frequency shifts are very different in the CL and the AES cases. We again consider the example of the spin being held artificially on a trajectory with $\theta = \theta_0 \neq 0$ and precessing with a frequency $\tilde{B}$. In the CL model, the spectrum of ξ_z is not shifted, whereas the spectra of ξ_x and ξ_y are shifted by $\tilde{B}$. By contrast, in the AES case, the spectra of ξ_x and ξ_y are shifted by $\omega_{cos} \equiv \tilde{B} \cos^2(\theta_0/2)$ and the spectra of ξ_z and ξ_0 are

shifted by $\omega_{sin} \equiv \tilde{B} \sin^2(\theta_0/2)$. Both these factors are geometrical and are determined by the Berry phase of the spin's trajectory.

6. CONCLUSIONS

In this paper, we review the Caldera–Leggett and Ambegaokar–Eckern–Schön approaches to dissipation. We first remind the reader about the well-known physics of dissipative charge dynamics underlain by the U(1) symmetry. Then we provide an analogous treatment for the dissipative SU(2) spin dynamics. In both cases, the Keldysh technique allows deriving the semiclassical Langevin equations of motion. Except in the CL U(1) case, the noise is multiplicative, which leads to the admixture of high-frequency (quantum) noise components to the low-frequency dynamics. This gives rise to shot noise in the charge dynamics as well as to the novel phenomenon of geometric dephasing in the dynamics of large spins.

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