

VISCOUS COUPLED FLUIDS IN INFLATIONARY COSMOLOGY

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Received September 3, 2015

We consider the inflation produced by two coupled fluids in a flat Friedmann–Robertson–Walker universe. Different cosmological models for describing inflation with the use of an inhomogeneous equation of state for the fluid are investigated. The gravitational equations for energy and matter are solved, and analytic representations for the Hubble parameter and the energy density are obtained. Corrections to the energy density for matter inducing the inflation and the coupling to energy are discussed. We analyze the description of inflation induced by nonconstant equation-of-state parameters from fluid viscosity. The correspondence between the spectral index and the tensor-to-scalar ratio recently observed by the Planck satellite is considered.

DOI: 10.7868/S0044451016040064

1. INTRODUCTION

According to recent observational data, the current universe is accelerating [1]. There exists an early-time accelerated epoch after the Big Bang, called the "hot scenario" or inflationary period (for reviews, see Refs. [2, 3]). During inflation, the total energy and the scale factor both increase exponentially [4]. This phenomenon suggests that the cosmic fluid has properties quite different from those of standard matter and radiation. The simplest description of the accelerated expansion of the universe is given by the dark fluid model (the Friedmann–Robertson–Walker (FRW) model), in which the cosmic fluid is taken to satisfy a homogeneous or alternatively an inhomogeneous equation of state.

Various problems associated with viscous cosmology have been discussed extensively in Refs. [5–21]. Papers [22–25] were devoted to inhomogeneous fluids, of which a time-dependent equation of state in the presence of viscosity represents a subclass. The behavior of a nonclassical inflation was considered in Ref. [26]. Inhomogeneous fluid cosmology may also be understood as modified gravity (see Refs. [27, 28] for reviews) because it can be modeled as a gravitational fluid with an inhomogeneous equation of state [29]. It is then to be expected that following the unification of inflation with dark energy in modified gravity [30], one can also achieve this unification in viscous cosmology, including the presence of dark matter.

Some examples of inhomogeneous viscous coupled fluids were considered in Refs. [31–33]. In [34, 35], a bounce cosmology induced by an inhomogeneous viscous fluid in an FRW space–time was investigated. In [36], different kinds of inhomogeneous viscous fluids were analyzed with regard to the possibility of reproducing the current cosmic acceleration in a flat FRW space–time and the presence of finite-future time singularities; in Ref. [37], the noticeable fact was proven that a viscous fluid with an inhomogeneous equation of state is able to produce a Little Rip cosmology as a purely viscosity effect. Various cosmological models of the coupling between energy and matter were considered in review [38].

In this paper, we investigate cosmological models in the presence of inflation. Some variants of inhomo-

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geneous viscous coupled fluids, of interest in the inflationary regime, are considered. The influence of the interaction between energy and matter on the description of inflationary cosmology is analyzed. The slow roll parameters, the spectral index, and the tensor-to-scalar ratio are calculated. Restrictions on the thermodynamic parameters, needed to satisfy the Planck data, are obtained. The agreement between theoretical inflationary models and observational data is discussed.

2. VISCOUS FLUID MODELS FOR INFLATION

In this section, we consider the early-time universe, applying the formalism of an inhomogeneous viscous fluid in a flat FRW space–time. We describe the inflation in terms of the equation-of-state parameters and the bulk viscosity. We pay attention to the behavior of the Hubble parameter and the energy density at the beginning and at the end of inflation.

We start from a universe filled with two coupled fluids: energy and matter in a spatially flat metric with a scale factor a. The background equations are (see, e. g., Ref. [39])

$$\begin{aligned} &\dot{\rho} + 3H(p+\rho) = -Q, \\ &\dot{\rho}_1 + 3H(p_1+\rho_1) = Q, \\ &\dot{H} = -\frac{k^2}{2}(p+\rho+p_1+\rho_1), \end{aligned} \qquad (1)$$

where $H = \dot{a}/a$ is the Hubble parameter and $k^2 = 8\pi G$, with G being Newton's gravitational constant. Further, p, ρ and p_1, ρ_1 are the respective pressure and energy density of energy and matter, and Q is a function that accounts for the energy exchange between the fluids. A dot denotes the derivative with respect to the cosmic time t. The cosmological constant Λ is set equal to zero.

As is known, there are a number of studies on cosmology with multicomponent viscous fluids, especially in order to explain the late-time cosmic acceleration (see, e. g., Refs. [18–20, 33, 40–43].) It is then most natural to allow for interaction terms between the components. This means physically that each component is to be regarded as a nonclosed physical system, corresponding to an energy–momentum tensor that is not divergence-free in general. The requirement of a vanishing four-divergence can be imposed only on the total energy–momentum tensor: $T^{\mu\nu}_{Total;\nu} = 0$. A new element in our analysis is to adopt a two-component model also in the inflationary region. One may ask: is there a physical reason for adopting this model also in the inflationary epoch? To our knowledge, there is currently no such basic reason. Rather, the model is introduced on phenomenological grounds, to make the theory in the inflationary epoch similar to that commonly accepted in other epochs. We emphasize, however, that the state of acceleration is a common feature for inflation and for the late universe, in the latter case especially when the Big Rip is approached. In a future work, we plan to extend the present analysis so as to unify inflation with dark energy, such that one component describes inflation whereas the other component effectively describes the dark energy epoch. In this way, we hope to present a unified universe evolution with the use of a two-component fluid.

We now write Friedmann's equation for the Hubble rate:

$$H^2 = \frac{k^2}{3}(\rho + \rho_1). \qquad (2)$$

We assume that there is no relativistic matter ($\tilde{w} = 0$), and we assume, as usual, that the dark matter is dust. Hence, $p_1 = 0$. We also set $c = 1$, as usual.

The gravitational equation for matter reduces to

$$\dot{\rho}_1 + 3H\rho_1 = Q. \qquad (3)$$

We now study some cases of inhomogeneous viscous coupled fluids in the inflationary period.

2.1. Fluid model with $\omega = -\rho/(\rho+\rho_*)$, and viscosity proportional to H

We choose the viscosity to have the form [44]

$$\zeta(H) = \exp(-H/H_*)\, f(H), \qquad (4)$$

where the star subscript refers to the end of the inflationary period. The function $f(H)$ is to be adjusted in what follows to have a physically reasonable form. Correspondingly, $H_* = H(t_*)$ refers to the end of inflation. Function (4) is assumed to vary slowly with time, because inflation can be realized when the viscosity is slight.

We take the equation of state to have the inhomogeneous form

$$p = \omega(\rho)\rho + \zeta(H), \qquad (5)$$

where the thermodynamic parameter is $\omega(\rho) = -\rho/(\rho+\rho_*)$ [44], and ρ_* is the energy density at the end of inflation ($\rho_* \equiv H_*^2/k^2$).

First of all, we find the solution of the gravitational equation for the energy component [29]:

$$\dot{\rho} + 3H\rho[1+\omega(\rho)] - 3H^2\zeta(H) = -Q. \qquad (6)$$

We suppose that the ratio $r = \rho_1/\rho$ is constant, and rewrite Friedmann's equation (2) in the form

$$\rho = \frac{3H^2}{k^2(1+r)}. \qquad (7)$$

We consider the coupling between the fluids given by

$$Q = 9\delta H^3, \qquad (8)$$

where δ is a positive dimensional constant. In geometric units, its dimension is $[\delta] = \mathrm{cm}^{-2}$.

Since there exists no fundamental theory specifying the functional form of the coupling between the fluids, the present coupling model has necessarily to be phenomenological [38], as mentioned above.

We analyze the model in the asymptotic limit corresponding to the initial stage of inflation,

$$\frac{H}{H_*} \ll 1. \qquad (9)$$

Then $\zeta(H/H_* \ll 1) = f(H)$, and the function $f(H)$ can be taken to be proportional to H,

$$f(H) = \theta H, \qquad (10)$$

with $\theta = 3/k^2(1+r)$.

With Eqs. (7), (8), and (10) taken into account, the continuity equation (6) for energy is simplified to

$$\frac{2}{3}\dot{H} + H^2 \frac{\rho_*}{\theta H^2 + \rho_*} = 0. \qquad (11)$$

Because we are considering the initial stage of inflation, we can choose the density ρ_* in the form $\rho_* = \theta H_{in}^2$ with $H_{in} = H(t_{in})$, where t_{in} is the initial time. In this approximation, the solution of Eq. (11) becomes

$$H(t) = \left(\sqrt{\tau^2+1} - \tau\right) H_{in}, \qquad (12)$$

where $\tau = (3/4)H_{in}(t - t_{in})$. The Hubble parameter is positive, and we hence have an expanding universe.

Now, we solve the gravitational equation of motion (3) for matter with coupling terms (8) and Hubble parameter (12):

$$\rho_1(t) = H^2 \left[6\delta(e^\tau - 1) + \frac{(\rho_1)_m}{H_{in}^2}\right] \times \exp\left(-2\tau \frac{H}{H_{in}}\right). \qquad (13)$$

Here, $(\rho_m)_m = \rho_m(t_{in})$, and the time lies in the interval $t \in [t_{in}, t_{in} + 4/(3H_{in})]$. Finally, from Friedmann's equation (2), we find the energy density

$$\rho(t) = H^2 \left\{\frac{3}{k^2} - \left[6\delta(e^\tau - 1) + \frac{(\rho_1)_{in}}{H_{in}^2}\right] \times \exp\left(-2\tau \frac{H}{H_{in}}\right)\right\}. \qquad (14)$$

Thus we have shown how the inflation is realized, by using an inhomogeneous equation of the state parameter and viscous coupled fluids.

Further, we can investigate how this inflationary model conforms with recent results from the Planck satellite. We first calculate the slow-roll parameter ε,

$$\varepsilon = -\frac{\dot{H}}{H^2} = \frac{3}{4}\left(1 + \frac{\tau}{\sqrt{\tau^2+1}}\right). \qquad (15)$$

Acceleration occurs at $\varepsilon < 1$. This requirement is fulfilled when $t \in [t_{in}, t_{in} + \sqrt{2}/(3H_{in})]$. But the spectral parameters n_s and r become negative. Consequently, the results of the recent BICEP2 experiments are not reproduced in the model.

2.2. Fluid model with constant $\omega(\rho) = \omega_0$ and the viscosity proportional to H^2

In this section, we assume that the thermodynamic parameter $\omega(\rho) = \omega_0$ is constant in Eq. (5). We study the initial stage of inflation and take the function $f(H)$ to be proportional to the square of H,

$$f(H) = \tilde{\theta} H^2, \qquad (16)$$

where $\tilde{\theta} = \theta/3$. As in the preceding section, the interacting term Q has the same form as in Eq. (8).

Gravitational equation of motion (6) takes the form:

$$\frac{2}{3}\dot{H} + \left(\omega_0 + \frac{\delta}{\tilde{\theta}}\right) H^2 = 0. \qquad (17)$$

The Hubble parameter is

$$H = \frac{2}{3\left(\omega_0 + \delta/\tilde{\theta}\right)(t - t_{in}) - 2/H_{in}}. \qquad (18)$$

We find the solution of the gravitational equation of motion for matter as

$$\rho_1(t) = \left(\frac{H}{H_{in}}\right)^{2/(\omega_0+\delta/\tilde{\theta})} \left(\tilde{\rho}_{in} + \frac{3\delta}{\omega_0 - 1 + \delta/\tilde{\theta}} H_{in}^2\right) - \frac{3\delta}{\omega_0 - 1 + \delta/\tilde{\theta}} H^2, \qquad (19)$$

where $\tilde{\rho}_{in} = \rho_1(t_{in})$ is the energy density of matter at the beginning of inflation.

The energy density is given by the expression

$$\rho(t) = 3\left(\frac{1}{k^2} + \frac{\delta}{\omega_0 + 1 + \delta/\tilde{\theta}}\right) H^2 - \left(\frac{H}{H_{in}}\right)^{2/(\omega_0+\delta/\tilde{\theta})} \times \left(\tilde{\rho}_{in} + \frac{3\delta}{\omega_0 - 1 + \delta/\tilde{\theta}} H_{in}^2\right). \qquad (20)$$

Thus, we have constructed an example of an inhomogeneous viscous fluid cosmology coupled to matter, applied to inflation.

We next use solution (18) to calculate the slow-roll parameter in the inflation,

$$\varepsilon = \frac{3}{2}\left(\omega_0 + \frac{\delta}{\tilde{\theta}}\right), \tag{21}$$

where $\omega_0 + \delta/\tilde{\theta} > 0$. As above, in order to have a regime of acceleration, we must require that $\varepsilon < 1$, whence $\omega_0 < 2/3 - \delta/\tilde{\theta}$.

Another important slow-roll parameter is [44]

$$\eta = \varepsilon - \frac{1}{2\varepsilon H}\dot{\varepsilon}. \tag{22}$$

In our case, $\varepsilon = \eta$. The power spectrum is [44]

$$\Delta_R^2 = \frac{k^2 H^2}{8\pi^2\varepsilon}. \tag{23}$$

With Eq. (18) for the Hubble parameter, the power spectrum takes the form

$$\Delta_R^2 = k^2 \left\{ 3\pi^2 \left(\omega_0 + \frac{\delta}{\tilde{\theta}}\right) \times \right. \\ \left. \times \left[3\left(\omega_0 + \frac{\delta}{\tilde{\theta}}\right)(t - t_{in}) - \frac{2}{H_{in}}\right]^2 \right\}^{-1}. \tag{24}$$

From the slow-roll parameters, we can calculate the spectral index n_s and the tensor-to-scalar ratio r:

$$n_s = 1 - 6\varepsilon + 2\eta, \quad r = 16\varepsilon. \tag{25}$$

We obtain

$$n_s = 1 - 6\left(\omega_0 + \frac{\delta}{\tilde{\theta}}\right), \quad r = 24\left(\omega_0 + \frac{\delta}{\tilde{\theta}}\right). \tag{26}$$

In the particular case $\delta = \tilde{\theta}$, we obtain the same values for parameters (21), (22), and (26) as in Ref. [44].

From the observations by the Planck satellite, it is known that $n_s = 0.9603 \pm 0.0073$. In order to satisfy this result, we must require that $\omega_0 + \delta/\tilde{\theta} \approx 0.00(6)$. Consequently, the present model can produce inflation.

3. QUASI-DE SITTER EXPANSION FOR INFLATION PRODUCED BY TWO COUPLED FLUIDS

In this section, we investigate a nonviscous model for the cosmic fluid. The equation of state is taken to have the inhomogeneous form [44]

$$\omega(\rho) = -1 + a_1\rho^{1/2} - a_2\rho^{-1/2}, \tag{27}$$

where a_1 and a_2 are positive dimensional constants. This model describes a quasi-de Sitter inflationary expansion. We find an exact solution of this model taking the coupling of two different fluids into account.

We start from the gravitational equation for energy,

$$\dot{\rho} + 3H\rho^{1/2}(a_1\rho - a_2) = -Q. \tag{28}$$

We take the energy exchange between energy and matter in the form

$$Q = 3\tilde{\delta}H^2, \tag{29}$$

where $\tilde{\delta}$ is a positive dimensional constant, and we put the constants a_1 and a_2 in Eq. (27) equal to

$$a_1 = \frac{1}{\sqrt{3\theta}}, \quad a_2 = \frac{\tilde{\delta}}{\sqrt{3\theta}}. \tag{30}$$

Taking Eqs. (7), (20), and (29) into account, we obtain the continuity equation for energy (28) in the simple form

$$2\dot{H} + \frac{3}{\sqrt{\theta}}H^3 = 0. \tag{31}$$

The solution gives the Hubble parameter in the form

$$H = \frac{1}{\sqrt{\left(3/\sqrt{\theta}\right)(t - t_{in}) + 1/H_{in}^2}}. \tag{32}$$

Further, we find the solution for the gravitational equation of matter,

$$\rho_1(t) = 3\tilde{\rho}\sqrt{\theta}\exp\left(-\frac{2\sqrt{\theta}}{H}\right) \times \\ \times \left[C + 2\,\mathrm{Ei}\left(\frac{2\sqrt{\theta}}{H}\right)\right], \tag{33}$$

where $\mathrm{Ei}(2\sqrt{\theta}/H)$ is the integral exponential function and C is an arbitrary constant.

Then the expression for the energy density becomes

$$\rho(t) = 3\left\{\frac{H^2}{k^2} - \tilde{\rho}\sqrt{\theta}\exp\left(-\frac{2\sqrt{\theta}}{H}\right) \times \right. \\ \left. \times \left[C + 2\,\mathrm{Ei}\left(\frac{2\sqrt{\theta}}{H}\right)\right]\right\}. \tag{34}$$

Thus, we have obtained a situation in which the equation-of-state parameter changes slowly during the inflationary period, taking the coupling of two fluids into account.

As in the preceding section, we consider how this inflationary model conforms with the Planck observational data. First, we calculate the slow-roll parameters ε and η. From solution (32), we obtain

$$\varepsilon = \frac{3}{2\sqrt{\theta}}H, \quad \eta = \frac{3}{2}\varepsilon, \tag{35}$$

and the power spectrum is given by the expression

$$\Delta_R^2 = \frac{k^2\sqrt{\theta}}{12\pi^2\sqrt{\left(3/\sqrt{\theta}\right)(t-t_{in})+1/H_{in}^2}}. \qquad (36)$$

The spectral index n_s and the tensor-to-scalar ratio r are given by

$$n_s = 1-3\varepsilon, \quad r = 16\varepsilon. \qquad (37)$$

For a de Sitter expansion, we can introduce the number of N-folds,

$$N = \int\limits_{t_{in}}^{t_*} H(t)\,dt. \qquad (38)$$

In our case,

$$N = \frac{2\sqrt{\theta}}{3}\left(\frac{1}{H_*}-\frac{1}{H_{in}}\right). \qquad (39)$$

The inflation is viable if $N > 76$. Therefore, the slow-roll parameters change during inflation,

$$\varepsilon = \frac{1}{N} \ll 1, \quad \eta = \frac{3}{2N} \ll 1, \qquad (40)$$

and the slow-roll conditions are satisfied.

The spectral indices are given by

$$n_s = 1-\frac{3}{N}, \quad r = \frac{16}{N}. \qquad (41)$$

For $N = 76$, the indices are $n_s = 0.9605$ and $r = 0.221$. Consequently, the results from the Planck satellite data can be realized in this model. According to the recent Planck data [45], the tensor-to-scalar ration is constrained to be $r < 0.11$ (95 % CL).

4. CONCLUSION

In the present paper, we investigated coupled-fluid cosmological models that take into account viscosity properties of the fluid in an FRW flat space–time, in a hot universe. We considered the influence of the interaction between energy and matter during the inflationary very early stages of the evolution. We paid attention especially to the initial stage of the inflation. Studying the agreement between the theoretical inflationary models and the last results of the Planck satellite data, we showed that some restrictions on the thermodynamic parameters allowed a satisfactory correspondence with the observations.

Inhomogeneous fluid cosmology may be understood as some kind of modified gravity (for a review, see Ref. [37]), owing to the fact that it may be presented as a gravitational fluid with an inhomogeneous equation of state [39]. It then follows that the unification of inflation with dark energy in modified gravity [37] can also be achieved in viscous cosmology, with the inclusion of dark matter.

Our theory may be extended to the case of multiple coupled viscous fluids. The calculation can be done in the same way as above.

This work was supported by a grant from the Russian Ministry of Education and Science, project TSPU-139 (A. V. T.). We also thank S. D. Odintsov for valuable information.

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