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INSTABILITY ANALYSIS OF A CYLINDRICAL STELLAR OBJECT IN BRANS–DICKE GRAVITY

M. Sharif [a*], ***R. Manzoor*** [b**]

[a] *Department of Mathematics, University of the Punjab*
54590, Lahore, Pakistan

[b] *Department of Mathematics, University of Management and Technology*
54782, Lahore, Pakistan

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This paper investigates instability ranges of a cylindrically symmetric collapsing cosmic filamentary structure in the Brans–Dicke theory of gravity. For this purpose, we use a perturbating approach to the modified field equations as well as dynamical equations and construct a collapse equation. The collapse equation with an adiabatic index (Γ) is used to explore the instability ranges of both isotropic and anisotropic fluid in Newtonian and post-Newtonian approximations. It turns out that the instability ranges depend on the dynamical variables of collapsing filaments. We conclude that the system always remains unstable for $0 < \Gamma < 1$, while $\Gamma > 1$ provides instability only in a special case.

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1. INTRODUCTION

Dark energy and gravitational collapse are the most fascinated and interesting phenomena of cosmology and gravitational physics. A number of astronomical observations such as supernova type I, Sloan Digital Sky Survey, large-scale structure, Wilkinson Microwave Anisotropy Probe, galactic cluster emission of X-rays and weak lensing, describe accelerated behavior of the expanding universe [1]. It is suggested that a mysterious type of energy known as dark energy is responsible for this accelerated expansion of the universe. This induces the problem of the correct theory of gravity, and hence several modified theories of gravity have been constructed using modified Einstein–Hilbert actions. The Brans–Dicke (BD) theory is one of the most explored examples among various modified theories, which provides convenient evidence of various cosmic problems like inflation, early and late behavior of the universe, the coincidence problem, and cosmic acceleration [2]. This is a generalized form of general relativity (GR), constructed by coupling a scalar field ϕ and the tensor field R. It contains a constant coupling parameter ω_{BD} (tuneable parameter), which can be adjusted according to suitable observations. This theory is compatible with Mach's principle, the weak equivalence principle, and Dirac's large number hypothesis [3]. It is also consistent with solar system observations and experiments (weak field test) for $|\omega| \geq 40000$ [4].

Gravitational collapse is a process in which stable stellar objects turn into unstable ones under the effects of their own gravity. The formation and dynamics of large-scale structures such as stars, celestial clusters, and galaxies are investigated through this phenomenon. It is believed that different instability ranges for astronomical bodies lead to different structure formation of collapsing models. Chandrasekhar [5] was the first to explore stability ranges of a spherically symmetric isotropic fluid in GR. He used the equation of state involving an adiabatic index (Γ) and concluded that the fluid remains unstable for $\Gamma < 4/3$. Subsequently, many researchers [6, 7] investigated dynamical instability of different types of fluids (anisotropic fluid, adiabatic, nonadiabatic, and shearing viscous fluid) in spherical and cylindrical configurations and found that stability ranges depend on physical properties of the respective fluid.

It is believed that the study of the collapse phenomenon in modified theories may reveal modification hidden in the formation of astronomical structures [8]. In 1969, Nutku [9] explored instability ranges of a spherically symmetric isotropic fluid in the BD theory

* E-mail: msharif.math@pu.edu.pk
** E-mail: rubab.manzoor@umt.edu.pk

and concluded that the BD fluid remains unstable for $\Gamma > 4/3$. An instability analysis of the Schwarzschild black hole in the BD gravity is presented in [10]. Stability ranges for spherical and cylindrical collapsing models in the $f(R)$ theory were investigated in [11], where the instability ranges were found to depend upon characteristics of fluids and dark energy components. The effects of the electromagnetic field on instability ranges for various $f(R)$ gravity models were studied in [12]. Dynamical instability of a spherically symmetric fluid in the $f(T)$ theory was explored in [13], with the conclusion that modified terms control instability ranges. In a recent paper [14], we have discussed the collapse of a spherically symmetric anisotropic BD fluid in terms of instability analysis and found that $0 < \Gamma < 1$ always leads to an unstable configuration while $\Gamma > 1$ provides instability only in one particular case.

The behavior of filamentary structures has important implications for the formation of structure in the universe. Galaxy filaments are the largest known cosmic structures in the universe. The filamentary structure is always present in the interstellar medium and instabilities within these filaments create a dense medium (dense core) where stars form [15]. N-body simulations of the formation of the large scale structure describe a wide range of filaments (with a cluster of galaxies forming at the intersection of filaments) [16]. Filamentary structures are associated with the cosmic web on large scales and are used to describe tidal tails (thrown off by merging galaxies) on the small scales [17]. To understand fragmentation of filament structures, cylindrically symmetric filament models are widely studied [18].

In this paper, we investigate dynamical instability of a cylindrically symmetric filaments collapsing structure in the BD gravity. The paper is organized as follows. In Sec. 2, we discuss BD equations, Darmois junction conditions, and dynamical equations. In Sec. 3, we use a perturbative technique to construct hydrostatic equilibrium (collapse equation) and describe the instability ranges (in the Newtonian and post-Newtonian (pN) limits) for isotropic and anisotropic fluid distributions. Finally, the last section summarizes the results.

2. BRANS–DICKE THEORY AND DYNAMICAL EQUATIONS

The BD theory (with a self-interacting potential $V(\phi)$) has the action [3]

$$S = \int d^4x\sqrt{-g}[\phi R - \frac{\omega_{BD}}{\phi}\nabla^\mu\phi\nabla_\mu\phi - V(\phi) + L_m], \quad (1)$$

where $8\pi G_0 = c = 1$ and L_m represents matter distribution. Varying Eq. (1) with respect to $g_{\alpha\beta}$ and ϕ, we obtain the BD equations

$$G_{\alpha\beta} = \frac{1}{\phi}(T^m_{\alpha\beta} + T^\phi_{\alpha\beta}), \quad (2)$$

$$\Box\phi = \frac{T^m}{3+2\omega_{BD}} + \frac{1}{3+2\omega_{BD}}\left[\phi\frac{dV(\phi)}{d\phi} - 2V(\phi)\right]. \quad (3)$$

Here, $G_{\alpha\beta}$ is the Einstein tensor, $T^m_{\alpha\beta}$ is the energy–momentum tensor for the matter distribution with T^m as its trace, and $\Box$ represents the d'Alembertian operator. The energy distribution due to the scalar field is given by

$$T^\phi_{\alpha\beta} = \phi_{,\alpha;\beta} - g_{\alpha\beta}\Box\phi + \frac{\omega_{BD}}{\phi}\left[\phi_{,\alpha}\phi_{,\beta} - \frac{1}{2}g_{\alpha\beta}\phi_{,\mu}\phi^{,\mu}\right] - \frac{V(\phi)}{2}g_{\alpha\beta}. \quad (4)$$

Equation (2) gives the BD field equations and (3) is a wave equation for the evolution of the scalar field.

We split the 4D geometry into interior and exterior regions by considering a timelike 3D hypersurface $\Sigma^{(e)}$ as an external boundary of the corresponding cylindrical body [7]. The interior region of a collapsing cylindrical filamentary structure is represented by

$$ds_-^2 = A^2(t,r)dt^2 - B^2(t,r)dr^2 - C^2(t,r)\,d\phi^2 - dz^2, \quad (5)$$

where we consider comoving coordinates inside the hypersurface. To preserve cylindrical symmetry, the coordinates satisfy the constraints

$$-\infty \le t \le \infty, \quad 0 \le r < \infty, \quad -\infty < z < \infty,$$

$$0 \le \phi \le 2\pi.$$

In the stationary or static region, a scalar field becomes constant and all stationary black holes in BD gravity are identical with GR solutions [19]. Therefore, for the region exterior to $\Sigma^{(e)}$, we take the line element of a static cylindrical black hole given by

$$ds_+^2 = -\frac{2M}{r}d\nu^2 + 2dr\,d\nu - r^2(d\phi^2 + \gamma^2 dz^2), \quad (6)$$

where M, ν, and γ describe the total gravitating mass, the retarded time, and an arbitrary constant [20]. The interior region is filled with the anisotropic matter distribution represented by

$$T^m_{\alpha\beta} = (\rho + p_r)u_\alpha u_\beta - p_r g_{\alpha\beta} + (p_z - p_r)S_\alpha S_\beta + (p_\phi - p_r)K_\alpha K_\beta, \quad (7)$$

where ρ, p_r, p_ϕ, and p_z denote the energy density and principal pressure stresses. The four-velocity u_α, and unit four-vectors S_α and K_α are calculated as $u_\alpha = A\delta^0_\alpha$, $S_\alpha = \delta^3_\alpha$, and $K_\alpha = C\delta^2_\alpha$, satisfying $u^\alpha u_\alpha = 1$, $S^\alpha S_\alpha = K^\alpha K_\alpha = -1$, and $S^\alpha u_\alpha = K^\alpha u_\alpha = S^\alpha K_\alpha = 0$. For the interior region, the BD equations are given in Appendix.

Junction conditions provide a smooth connection between the interior and exterior regions over $\Sigma^{(e)}$. We consider the Darmois junction conditions to discuss the connection between two regions [7]. For this, we take the C-energy (mass function) [21] given by

$$\tilde{E}(t,r) = m(t,r) = \frac{1}{8}(1 - l^{-2}\nabla^\beta\tilde{r}\nabla_\beta\tilde{r}), \tag{8}$$

where $\tilde{E}(t,r)$ is the gravitational energy per unit specific length of the cylinder, $\tilde{r}$ represents the areal radius, μ is the circumference radius, and l indicates the specific length. These are given by

$$\tilde{r} = \mu l, \quad \mu^2 = \xi_{(1)\beta}\xi^\beta_{(1)}, \quad l^2 = \xi_{(2)\beta}\xi^\beta_{(2)},$$

where $\xi_{(1)} = \partial/\partial\theta$ and $\xi_{(2)} = \partial/\partial z$ are the Killing vectors. For the interior spacetime, Eq. (8) takes the form

$$m(t,r) = \frac{l}{8}\left(1 + \frac{\dot{C}^2}{A^2} - \frac{C'^2}{B^2}\right), \tag{9}$$

where the dot and prime denote the respective derivatives with respect to t and r. In BD gravity, the scalar field and the metric tensor are considered gravitational variables, and therefore $\phi = \phi_{\Sigma^{(e)}} = \text{const}$ on the hypersurface $\Sigma^{(e)}$. The continuity of the first and second fundamental forms (Darmois conditions) yields the relations

$$\begin{gathered} r = r_{\Sigma^{(e)}} = \text{const}, \quad m(t,r) - M \overset{\Sigma^{(e)}}{=} \frac{l}{8}, \\ l \overset{\Sigma^{(e)}}{=} 4C, \quad \frac{p_r}{\phi} \overset{\Sigma^{(e)}}{=} \frac{-T^\phi_{11}}{B^2} - \frac{T^\phi_{01}}{AB} = -\frac{V(\phi)}{2\phi}. \end{gathered} \tag{10}$$

Dynamical equations obtained from the contracted Bianchi identities describe the conservation of total energy of the system given by

$$\begin{gathered} \left(\frac{T^{\alpha\beta}_m}{\phi} + \frac{T^{\alpha\beta}_\phi}{\phi}\right)_{;\alpha} u_\beta = 0, \\ \left(\frac{T^{\alpha\beta}_m}{\phi} + \frac{T^{\alpha\beta}_\phi}{\phi}\right)_{;\alpha} \chi_\beta = 0, \end{gathered} \tag{11}$$

where $\chi_\beta = -B\delta^1_\beta$ (unit four-vector), which yields

$$\left[\frac{\dot{\rho}}{A} - \frac{\rho\dot{\phi}}{\phi^2 A} + (\rho+p_r)\frac{\dot{B}}{AB} + (\rho+p_\phi)\frac{\dot{C}}{AC}\right] + K_1 = 0, \tag{12}$$

$$\left[\frac{p'_r}{B} + \frac{\phi' p_r}{\phi^2 B} + (\rho + p_r)\frac{A'}{AB} + (p_r - p_\phi)\frac{C'}{BC}\right] + K_2 = 0, \tag{13}$$

with K_1 and K_2 mentioned in Appendix.

3. INSTABILITY ANALYSIS

Here, we use the perturbative approach to construct a collapse equation that is to be used for instability analysis. We assume that initially, the system is in static equilibrium (metric and matter parts have a radial dependence only) and then all the dynamical variables along with metric functions are perturbed and time dependence appears [7]. The scalar field, the scalar potential, and the metric tensors have the same time dependence, while the density and pressure bear the same time dependence as follows:

$$A(t,r) = A_0(r) + \epsilon T(t)a(r), \tag{14}$$

$$B(t,r) = B_0(r) + \epsilon T(t)b(r), \tag{15}$$

$$C(t,r) = C_0(r) + \epsilon T(t)c(r), \tag{16}$$

$$\phi(r,t) = \phi_0(r) + \epsilon T(t)\Phi(r), \tag{17}$$

$$p_r(t,r) = p_{r0}(r) + \epsilon\bar{p}_r(t,r), \tag{18}$$

$$p_\phi(t,r) = p_{\phi0}(r) + \epsilon\bar{p}_\phi(t,r), \tag{19}$$

$$\rho(t,r) = \rho_0(r) + \epsilon\bar{\rho}(t,r), \tag{20}$$

$$V(\phi) = V_0(r) + \epsilon T(t)\bar{V}(r), \tag{21}$$

where $0 < \epsilon \ll 1$ and the static distribution is expressed by a zero subscript. For static and perturbed configurations of the field and dynamical equations, we take $C_0 = r$. The static configuration of the BD formalism, the perturbed form of BD equations, and junction condition (10) are given in Appendix.

The perturbed distribution of the first Bianchi identity gives

$$\bar{\rho} = -\left[\frac{(\rho_0 + p_{r0})b}{B_0} + \frac{(\rho_0 + p_{\phi0})c}{r} + \frac{\Phi\rho_0}{\phi_0} + A_0\phi_0\bar{K}_1\right]T. \tag{22}$$

The perturbed form of Eq. (13) yields

$$\bar{p}'_r + (\bar{\rho} + \bar{p}_r)\frac{A'_0}{A_0} + (\bar{p}_r - \bar{p}_\phi)\frac{1}{r} + \frac{\bar{p}_r\phi'_0}{\phi_0 B_0} + \bar{K}_2\phi_0 B_0 = 0, \tag{23}$$

where $\bar{K}_1$ and $\bar{K}_2$ are given in Appendix. Equation (55) along with the junction conditions leads to

$$T(t) = c_1 \exp(\gamma_{\Sigma^{(e)}} t) + c_2 \exp(\lambda_{\Sigma^{(e)}} t), \tag{24}$$

where

$$\gamma_{\Sigma^{(e)}} = \sqrt{\frac{v}{u}}, \quad \lambda_{\Sigma^{(e)}} = -\sqrt{\frac{v}{u}}$$

with

$$u \overset{\Sigma^{(e)}}{=} \frac{\Phi}{\phi_0 A_0^2} - \frac{2c}{rA_0^2}, \quad v \overset{\Sigma^{(e)}}{=} \frac{\Phi}{\phi_0}\left[\frac{\omega_{BD}}{\phi_0} - \frac{1}{B_0 r}\right],$$

and c_1, c_2 are arbitrary constants. Equation (24) shows static and nonstatic distributions leading to stable and unstable phases of a gravitating system. For instability analysis, we assume that when the instability phase begins, the system is in complete hydrostatic equilibrium ($t = -\infty$, $T(-\infty) = 0$). Using this assumption in Eq. (24), we have $c_2 = 0$, whereas $c_1 = -1$ can be chosen arbitrarily. The corresponding result is described by

$$T(t) = -\exp(\gamma_{\Sigma^{(e)}} t). \tag{25}$$

For a real instability regime, we assume only positive values of v/u.

For the investigation of instability ranges, we use an equation of state involving adiabatic index Γ [22] given by

$$\bar{p}_j = \Gamma \frac{p_{j0}}{\rho_0 + p_{j0}} \bar{\rho}. \tag{26}$$

The adiabatic index expresses the variation of principal stresses (pressures) with respect to density and represents rigidity of the gravitating fluid. We consider Γ to be constant throughout the stability analysis of the fluid. Equations (22) and (26) lead to

$$\bar{p}_r = -\Gamma\left[\frac{b}{B_0}p_{r0} + \frac{c}{r}\frac{\rho_0 + p_{\phi0}}{\rho_0 + p_{r0}}p_{r0} + \frac{p_{r0}}{\rho_0 + p_{r0}}\frac{\Phi\rho_0}{\phi_0} + \frac{p_{r0}A_0\phi_0}{\rho_0 + p_{r0}}\bar{K}_1\right]T, \tag{27}$$

$$\bar{p}_\phi = -\Gamma\left[\frac{b}{B_0}\frac{\rho_0 + p_{r0}}{\rho_0 + p_{\phi0}}p_{\phi0} + \frac{cp_{\phi0}}{r} + \frac{p_{\phi0}\Phi\rho_0}{(\rho_0 + p_{\phi0})\phi_0} + \frac{p_{\phi0}A_0\phi_0}{\rho_0 + p_{\phi0}}\bar{K}_1\right]T. \tag{28}$$

Using Eqs. (51), (22), (27), and (28) in (23), we construct a hydrostatic equation given by

$$\begin{aligned}
&\Gamma\left[p_{r0}\left(\frac{bT}{B_0} + \frac{\rho_0 + p_{\phi0}}{\rho_0 + p_{r0}}\frac{cT}{r} + \frac{1}{\rho_0 + p_{r0}}A_0\phi_0\bar{K}_1T\right)\right]' - \frac{\Gamma}{r}\left[p_{r0}\left(\frac{bT}{B_0} + \frac{\rho_0 + p_{\phi0}}{\rho_0 + p_{r0}}\frac{cT}{r} + \frac{p_{r0}}{\rho_0 + p_{r0}}A_0\phi_0\bar{K}_1T\right)\right] + \\
&+ \frac{p_{\phi0}}{r}\Gamma\left[p_{\phi0}\left(\frac{bT}{B_0}\frac{\rho_0 + p_{r0}}{\rho_0 + p_{\phi0}} + \frac{cT}{r} + \frac{1}{\rho_0 + p_{\phi0}}A_0\phi_0\bar{K}_1T\right)\right] - \\
&- \Gamma\left[p_{r0}\left(\frac{bT}{B_0} + \frac{\rho_0+p_{\phi0}}{\rho_0+p_{r0}}\frac{cT}{r} + \frac{1}{\rho_0 + p_{r0}}A_0\phi_0\bar{K}_1T\right)\right] \times \\
&\times \frac{A_0'}{A_0} - \left[\frac{bT}{B_0}(\rho_0+p_{r0}) + (\rho_0 + p_{\phi0})\frac{cT}{r} + A_0\phi_0\bar{K}_1T\right] \times \\
&\times \frac{A_0'}{A_0} - \Gamma\left[p_{r0}\left(\frac{bT}{B_0} + \frac{\rho_0 + p_{\phi0}}{\rho_0 + p_{r0}}\frac{cT}{r} + \frac{1}{\rho_0 + p_{r0}}A_0\phi_0\bar{K}_1T\right)\right]\frac{\phi_0'}{\phi_0 B_0} + \phi_0 B_0 \bar{K}_2 = 0.
\end{aligned} \tag{29}$$

This represents the general form of the collapse equation that describes the instability of hydrostatic equilibrium of gravitating filaments in BD gravity.

3.1. Isotropic fluid

Here, we analyze instability ranges of an isotropic fluid in Newtonian and pN limits. In an isotropic fluid, all principal stresses are equal ($p_r = p_\phi = p_z$). Using this condition in Eq. (29), we obtain the corresponding collapse equation

$$\begin{aligned}
&\Gamma\left[p_{r0}\left(\frac{bT}{B_0} + \frac{cT}{r} + \frac{1}{\rho_0 + p_{r0}}A_0\phi_0\bar{K}_1T\right)\right]' - \\
&- \Gamma\left[p_{r0}\left(\frac{bT}{B_0} + \frac{cT}{r} + \frac{1}{\rho_0 + p_{r0}}A_0\phi_0\bar{K}_1T\right)\right] \times \\
&\times \frac{A_0'}{A_0} - \left[(\rho_0 + p_{r0})\left(\frac{bT}{B_0} + \frac{cT}{r}\right) + A_0\phi_0\bar{K}_1T\right] \times \\
&\times \frac{A_0'}{A_0} - \Gamma\left[p_{r0}\left(\frac{bT}{B_0} + \frac{cT}{r} + \frac{1}{\rho_0+p_{r0}}A_0\phi_0\bar{K}_1T\right)\right] \times \\
&\times \frac{\phi_0'}{\phi_0 B_0} + \phi_0 B_0\bar{K}_2 = 0.
\end{aligned} \tag{30}$$

3.1.1. Newtonian limit

The Newtonian limit in the BD theory leads to the relations

$$\begin{aligned}
&\rho_0 \gg p_{r0}, \quad \rho_0 \gg p_{\phi0}, \quad B_0 = 1, \\
&A_0 = 1 - \frac{m_0}{rc^2}, \quad \phi_0 = \text{const}, \quad V_0 = \bar{V} = 0.
\end{aligned} \tag{31}$$

With these limits used together with (25), the collapse condition turns out to be

$$\Gamma\left[(p_{r0}Z_N)_{,1}-\frac{m_0}{r^2c^2}p_{r0}Z_N\right]-\rho_0 Z_N\frac{m_0}{r^2c^2}+K_3<0,$$

which gives

$$\Gamma<\frac{\rho_0 Z_N\dfrac{m_0}{r^2c^2}-K_3}{(p_{r0}Z_N)_{,r}-\dfrac{m_0}{r^2c^2}p_{r0}Z_N}. \qquad (32)$$

Here,

$$Z_N=b+\frac{c}{r},$$

$$K_3=-\left[-\frac{a'm_0}{r^2c^2\phi_0}+\frac{\Phi}{\phi_0}\left(1-\frac{m_0}{rc^2}\right)-2p_{r0}\frac{\Phi}{\phi_0}\right].$$

This shows that the adiabatic index depends on dynamical properties such as density, pressure, and the scalar field. To preserve a difference between configurations of the pressure gradient and gravitational forces, we assume $\Gamma>0$. Thus, the celestial objects remain unstable until (32) is satisfied, which leads to

$$\frac{\rho_0 Z_N\dfrac{m_0}{r^2c^2}-K_3}{(p_{r0}Z_N)_{,r}-\dfrac{m_0}{r^2c^2}p_{r0}Z_N}>\Gamma>0. \qquad (33)$$

This leads to the following possibilities:

1) $\rho_0 Z_N\dfrac{m_0}{r^2c^2}-K_3=\left[(p_{r0}Z_N)_{,r}-\dfrac{m_0}{r^2c^2}p_{r0}Z_N\right]$;

2) $\rho_0 Z_N\dfrac{m_0}{r^2c^2}-K_3<\left[(p_{r0}Z_N)_{,r}-\dfrac{m_0}{r^2c^2}p_{r0}Z_N\right]$;

3) $\rho_0 Z_N\dfrac{m_0}{r^2c^2}-K_3>\left[(p_{r0}Z_N)_{,r}-\dfrac{m_0}{r^2c^2}p_{r0}Z_N\right]$.

The first and second cases along with (33) show that the isotropic system becomes unstable for $0<\Gamma<1$. The corresponding expressions lead to

$$p_{r0}=Z_N^{-1}\int\limits_{r_0}^{r}Z_N\left((\rho_0-1)\frac{m_0}{r^2c^2}-Z_N^{-1}K_3\right)dr', \qquad (34)$$

$$p_{r0}<Z_N^{-1}\int\limits_{r_0}^{r}Z_N\left((\rho_0-1)\frac{m_0}{r^2c^2}-Z_N^{-1}K_3\right)dr'. \qquad (35)$$

These are the constraint expressions for a collapsing cylindrical isotropic filamentary structure with $0<\Gamma<1$. In the third case, the denominator is less than the numerator and hence Γ can be taken greater than 1 in (33). The corresponding instability constraint is given by

$$p_{r0}>Z_N^{-1}\int\limits_{r_0}^{r}Z_N\left((\rho_0-1)\frac{m_0}{r^2c^2}-Z_N^{-1}K_3\right)dr', \qquad (36)$$

for which $\Gamma>1$ and an isotropic cylindrical system becomes unstable. It is obvious that if the system is unstable for $\Gamma>1$, then it is also unstable for $0<\Gamma<1$.

3.1.2. Post-Newtonian limit

The pN regimes are found up to the order c^{-4} by taking

$$A_0=1-\frac{m_0}{rc^2}+\frac{m_0^2}{r^2c^4},\quad B_0=1+\frac{\alpha m_0}{rc^2},\quad \phi_0=\text{const},\quad V_0=\bar V=0, \qquad (37)$$

where

$$\alpha=\frac{1+\omega_{BD}}{2+\omega_{BD}}.$$

Using the pN limits along with Eq. (25) in (29), we obtain

$$0<\Gamma<\frac{[(\rho_0+p_{r0})X_{pN}]\left(\dfrac{m_0}{r^2c^2}-2\dfrac{m_0}{r^3c^4}\right)-K_5}{\left[[p_{r0}X_{pN}]'-[p_{r0}X_{pN}]\left(\dfrac{m_0}{r^2c^2}-2\dfrac{m_0}{r^3c^4}\right)\right]}, \qquad (38)$$

where

$$X_{pN}=\left[\left(b\left(1-\frac{\alpha m_0}{rc^2}\right)+\frac{c}{r}\right)+\frac{K_4}{\rho_0+p_{r0}}\right],$$

and K_4 and K_5 are given in Appendix. Expression (38) describes a condition for the instability of a cylindrical filamentary structure in the pN limit. Similarly to the Newtonian case, the system collapses for $0<\Gamma<1$ with the following constraints:

$$1)\ p_{r0}=X_{pN}^{-1}\exp\left\{2r\left(\frac{m_0}{r^2c^2}-\frac{2m_0^2}{r^3c^4}\right)+\int\limits_{r_0}^{r}Y_{pN}dr'\right\}\times \int\limits_{r_0}^{r}X_{pN}dr'\exp\left\{-2r\left(\frac{m_0}{r^2c^2}-\frac{2m_0^2}{r^3c^4}\right)-\int\limits_{r_0}^{r}Y_{pN}dr'\right\}\times \left(\rho_0\left(\frac{m_0}{r^2c^2}-\frac{2m_0^2}{r^3c^4}\right)+X_{pN}^{-1}\left[\frac{a'}{\phi_0}\left(1+\frac{2m_0}{rc^2}(1-\alpha)\right)+\frac{m_0^2}{r^2c^4}(1+4\alpha)\right]\right)-X_{pN}^{-1}\gamma^2_{\Sigma^{(e)}}\times \left(\frac{m_0}{r^2c^2}-\frac{4m_0^2}{r^3c^4}+\frac{2\alpha m_0^2}{r^4c^4}\right),$$

$$2)\; p_{r0} < X_{pN}^{-1} \exp\left\{2r\left(\frac{m_0}{r^2c^2}-\frac{2m_0^2}{r^3c^4}\right)+\int_{r_0}^{r} Y_{pN}dr'\right\}\times$$
$$\times\int_{r_0}^{r} X_{pN}dr' \exp\left\{-2r\left(\frac{m_0}{r^2c^2}-\frac{2m_0^2}{r^3c^4}\right)-\int_{r_0}^{r} Y_{pN}dr'\right\}\times$$
$$\times\left(\rho_0\left(\frac{m_0}{r^2c^2}-\frac{2m_0^2}{r^3c^4}\right)+X_{pN}^{-1}\left[\frac{a'}{\phi_0}\left(1+\frac{2m_0}{rc^2}(1-\alpha)\right)+\right.\right.$$
$$\left.\left.+\frac{m_0^2}{r^2c^4}(1+4\alpha)\right]\right)-X_{pN}^{-1}\gamma^2_{\Sigma^{(e)}}\times$$
$$\times\left(\frac{m_0}{r^2c^2}-\frac{4m_0^2}{r^3c^4}+\frac{2\alpha m_0^2}{r^4c^4}\right).$$

In the third case, $\Gamma > 1$ leads to an unstable configuration with the constraint

$$p_{r0} > X_{pN}^{-1} \exp\left\{2r\left(\frac{m_0}{r^2c^2}-\frac{2m_0^2}{r^3c^4}\right)+\int_{r_0}^{r} Y_{pN}dr'\right\}\times$$
$$\times\int_{r_0}^{r} X_{pN}dr' \exp\left\{-2r\left(\frac{m_0}{r^2c^2}-\frac{2m_0^2}{r^3c^4}\right)-\int_{r_0}^{r} Y_{pN}dr'\right\}\times$$
$$\times\left(\rho_0\left(\frac{m_0}{r^2c^2}-\frac{2m_0^2}{r^3c^4}\right)+X_{pN}^{-1}\left[\frac{a'}{\phi_0}\left(1+\frac{2m_0}{rc^2}(1-\alpha)\right)+\right.\right.$$
$$\left.\left.+\frac{m_0^2}{r^2c^4}(1+4\alpha)\right]\right)-X_{pN}^{-1}\gamma^2_{\Sigma^{(e)}}\times$$
$$\times\left(\frac{m_0}{r^2c^2}-\frac{4m_0^2}{r^3c^4}+\frac{2\alpha m_0^2}{r^4c^4}\right),$$

where

$$Y_{pN} = X_{pN}^{-1}\left[\frac{a'}{\phi_0}\left(1+\frac{2m_0}{rc^2}(1-\alpha)\right)+\frac{m_0^2}{r^2c^4}(1+4\alpha)\right].$$

In this case, $0 < \Gamma < 1$ is also an instability range.

3.2. Anisotropic fluid

Here, we have $p_{r0} \neq p_{\phi_0} \neq p_z$ and hydrostatic equilibrium is described by Eq. (29).

3.2.1. Newtonian limit

Using Eq. (31) in (29), we obtain a condition for unstable anisotropic filaments as

$$0 < \Gamma < \frac{\dfrac{2p_{r0}\Phi}{\phi_0}+\dfrac{p_{r0}-p_{\phi 0}}{r}\left[\dfrac{c}{r}\right]'+\rho_0(Z_N+K_6)+K_7}{[p_{r0}Z_N]'+\dfrac{p_{r0}-p_{\phi 0}}{r}Z_N-p_{r0}Z_N\dfrac{m_0}{r^2c^2}}, \qquad (39)$$

where K_6 and K_7 are given in Appendix. Similarly to the isotropic case, this implies that for

$$p_{r0} \le r^{-1}Z_N^{-1}\exp\left\{\frac{m_0}{r^2c^2}-\int_{r_0}^{r}\frac{\Phi}{\phi}+\frac{1}{r}\left[\frac{c}{r}\right]'dr'\right\}\times$$
$$\times\left[\int_{r_0}^{r} rZ_N\exp\left\{-\left(\frac{m_0}{r^2c^2}-\int_{r_0}^{r}\frac{\Phi}{\phi}+\frac{1}{r}\left[\frac{c}{r}\right]'dr'\right)\right\}\times\right.$$
$$\left.\times\left[\frac{p_{\phi_0}}{r}+\rho_0(1+K_6)+K_7\right]dr'\right],$$

Γ lies in $(0,1)$ and the system collapses. If

$$p_{r0} > r^{-1}Z_N^{-1}\exp\left\{\left(\frac{m_0}{r^2c^2}-\int_{r_0}^{r}\frac{\Phi}{\phi}+\frac{1}{r}\left[\frac{c}{r}\right]'dr'\right)\right\}\times$$
$$\times\left[\int_{r_0}^{r} rZ_N\exp\left\{-\left(\frac{m_0}{r^2c^2}-\int_{r_0}^{r}\frac{\Phi}{\phi}+\frac{1}{r}\left[\frac{c}{r}\right]'dr'\right)\right\}\times\right.$$
$$\left.\times\left[\frac{p_{\phi_0}}{r}+\rho_0(1+K_6)+K_7\right]dr'\right],$$

the system becomes unstable for $\Gamma > 1$.

3.2.2. Post-Newtonian limit

The collapse condition of anisotropic cylindrical filaments in the pN regime is

$$\Gamma < \frac{\dfrac{p_{r0}}{r}U_{pN}\left(\dfrac{m_0}{r^2c^2}-\dfrac{m_0^2}{r^3c^4}\right)-(p_{r0}+\rho_0)\,[U_{pN}]-K_9}{[p_{r0}U_{pN}]'+\dfrac{p_{r0}}{r}U_{pN}-\dfrac{p_{\phi 0}}{r}V_{pN}}, \qquad (40)$$

where

$$U_{pN} = \left[b\left(1-\alpha\frac{m_0}{rc^2}\right)+\frac{p_{\phi 0}+\rho_0}{p_{r0}+\rho_0}\frac{c}{r}+\frac{1}{(p_{r0}+\rho_0)}K_8\right],$$
$$V_{pN} = \left[b\left(1-\alpha\frac{m_0}{rc^2}\right)\frac{p_{r0}+\rho_0}{p_{\phi 0}+\rho_0}+\frac{c}{r}+\frac{1}{p_{\phi 0}+\rho_0}K_8\right].$$

The values of K_8 and K_9 are given in Appendix. The system becomes unstable for the instability range $0 < \Gamma < 1$ if

- $p_{r0}rU_{pN}\left(\dfrac{m_0}{r^2c^2}-\dfrac{m_0^2}{r^3c^4}\right)-(p_{r0}+\rho_0)\,[U_{pN}]-K_9 = \left[[p_{r0}U_{pN}]'+\dfrac{p_{r0}}{r}U_{pN}-\dfrac{p_{\phi 0}}{r}V_{pN}\right],$

- $$p_{r0}rU_{pN}\left(\frac{m_0}{r^2c^2}-\frac{m_0^2}{r^3c^4}\right)-(p_{r0}+\rho_0)\left[U_{pN}\right]-K_9 < \\ < \left[[p_{r0}U_{pN}]'+\frac{p_{r0}}{r}U_{pN}-\frac{p_{\phi 0}}{r}V_{pN}\right],$$

and becomes unstable for $\Gamma > 1$ if

- $$p_{r0}rU_{pN}\left(\frac{m_0}{r^2c^2}-\frac{m_0}{r^3c^4}\right)-(p_{r0}+\rho_0)\left[U_{pN}\right]-K_9 > \\ > \left[[p_{r0}U_{pN}]'+\frac{p_{r0}}{r}U_{pN}-\frac{p_{\phi 0}}{r}V_{pN}\right].$$

4. CONCLUDING REMARKS

The study of structure formation in modified gravity is an important issue. Cosmic filamentary structures with cylindrical symmetry arise in astrophysics on both large (cosmic web) and small (tidal tails) scales. The behavior of these structures has an important role in the formation of structures in the universe. In this paper, we have investigated instability ranges of anisotropic cylindrically symmetric collapsing filaments in the BD theory. We have used contracted Bianchi identities to obtain two dynamical equations of a collapsing filamentary system. By applying a perturbative technique to the BD and dynamical equations, we separate the unperturbed (static) and perturbed (nonstatic) distributions of all dynamical relations. We have developed a hydrostatic equation (collapse equation) through a perturbed configuration of the second dynamical equation.

The equation of state involving an adiabatic index controls the ranges of instability for a collapsing filamentary structure. We have used the collapse equation along with the equation of state to investigate the instability ranges of both isotropic and anisotropic BD fluid in Newtonian and pN limits. It is concluded that in both approximations, the adiabatic index depending upon dynamical properties (energy density, pressure, scalar field terms, and some constraints) controls the instability ranges. We have constructed constraints on the static radial matter pressure under the effects of the scalar field. It is found that the cylindrical filamentary structures always remain unstable for $0 < \Gamma < 1$, while $\Gamma > 1$ is the instability range in a special case. We would like to mention here that the instability ranges for spherical and cylindrical distributions in GR depend upon $\Gamma < 4/3$ and $\Gamma < 1$. In $f(R)$ and $f(T)$ theories, physical variables such as density, pressure, and the corresponding modified dark terms provide the instability ranges. The instability range of a spherically symmetric isotropic BD fluid is $\Gamma > 4/3$, while an anisotropic spherical BD fluid always remains unstable for $0 < \Gamma < 1$, and $\Gamma > 1$ leads to a collapse only in the special case.

APPENDIX

The nonzero components of the BD equations for the interior region are

$$G_{00}=\frac{1}{\phi}(T_{00}^m+T_{00}^\phi)= \\ =\frac{1}{\phi}\left(\rho A^2+\frac{\omega_{BD}}{2\phi}\left(\dot\phi^2+\frac{A^2\phi'^2}{B^2}\right)\right)- \\ -\frac{\dot\phi}{\phi}\left(\frac{2\dot A}{A}+\frac{\dot B}{B}+\frac{\dot C}{C}\right)+\frac{\phi' A^2}{\phi B^2}\left(\frac{B'}{B}+\frac{C'}{C}\right)+ \\ +\frac{A^2\phi''}{B^2\phi}-\frac{A^2V(\phi)}{2\phi}, \quad (41)$$

$$G_{01}=\frac{1}{\phi}(T_{01}^m+T_{01}^\phi)=\frac{\omega_{BD}}{\phi^2}(\dot\phi\phi')+ \\ +\frac{1}{\phi}\left(\dot\phi'-\frac{A'\dot\phi}{A}-\frac{\dot B\phi'}{B}\right), \quad (42)$$

$$G_{11}=\frac{1}{\phi}(T_{11}^m+T_{11}^\phi)= \\ =\frac{1}{\phi}\left(p_rB^2+\frac{\omega_{BD}}{2\phi}\left(\phi'^2+\frac{B^2\dot\phi^2}{A^2}\right)\right)+\frac{\dot B\ddot\phi}{A^2\phi}+ \\ +\frac{B^2\dot\phi}{A^2}\left(\frac{\dot A}{A}+\frac{\dot C}{C}\right)-\frac{\phi'}{\phi}\left(\frac{A'}{A}+\frac{C'}{C}\right)+\frac{B^2V(\phi)}{2\phi}, \quad (43)$$

$$G_{22}=\frac{1}{\phi}(T_{22}^m+T_{22}^\phi)= \\ =\frac{1}{\phi}\left(p_\perp C^2+\frac{\omega_{BD}}{2\phi}\left(\frac{C^2\dot\phi^2}{A^2}-\frac{C^2\phi'^2}{B^2}\right)\right)+\frac{\ddot\phi C^2}{A^2\phi}+ \\ +\frac{C^2\dot\phi}{A^2\phi}\left(\frac{\dot A}{A}+\frac{\dot B}{B}\right)-\frac{C^2\phi'}{B^2\phi}\left(\frac{A'}{A}+\frac{B'}{B}\right)- \\ -\frac{C^2\phi''}{B^2\phi}+\frac{C^2V(\phi)}{2\phi}, \quad (44)$$

$$G_{33}=\frac{1}{\phi}(T_{33}^m+T_{33}^\phi)=p_z+\frac{\omega_{BD}}{2\phi^2B^2}\left[\frac{\dot\phi^2}{A^2}-\frac{\phi'^2}{B^2}\right]+\frac{\ddot\phi}{A^2\phi}+ \\ +\frac{\dot\phi}{A^2\phi}\left[\frac{\dot A}{A}+\frac{\dot B}{B}+\frac{\dot C}{C}\right]-\frac{\phi'}{B^2\phi}\left[\frac{A'}{A}+\frac{B'}{B}+\frac{C'}{C}\right]- \\ -\frac{\phi''}{B^2\phi}+\frac{V(\phi)}{2\phi}, \quad (45)$$

and Eq. (3) becomes

$$\dot{\phi}\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{A^2B}-\frac{\dot{C}}{A^2B}\right)+\frac{\ddot{\phi}}{A^2}+ + \phi'\left(\frac{A'}{AB^2}-\frac{B'}{B^3}-\frac{C'}{CB^2}\right)-\frac{\phi''}{B^2}=\frac{1}{2\omega_{BD}+3}\times \times\left[(\rho+3p_r+p_\phi+p_z)+\left(\phi\frac{dV}{d\phi}-2V\right)\right]. \quad (46)$$

The scalar terms K_1 and K_2 in Eqs. (12) and (13) are

$$K_1=\left(T^{\phi}_{00}\right)_{,t}A^{-1}-\left(T^{\phi}_{01}\right)_{,r}A^{-1}B^{-2}+ +\left(\rho A^{-1}+T^{\phi}_{00}A^{-2}\right)\phi^{-2}\dot{\phi}+ +T^{\phi}_{01}A^{-1}B^{-2}\phi^2\phi'-2T^{\phi}_{01}B^3AB'-T^{\phi}_{01}A^{-2}B^{-2}A',$$

$$K_2=T^{\phi}_{11}B^{-1}\phi'\phi^{-2}+\left(\rho+T^{\phi}_{01}\right)A^{-2}B^{-1}\phi^{-2}\dot{\phi}- -\left(T^{\phi}_{01}A^{-2}B^{-2}\right)_{,t}B-\left(T_{11}B^{-2}\right)_{;r}B.$$

The static distribution of the BD field equations is

$$\frac{\rho_0}{\phi_0}+\frac{\omega_{BD}\phi_0'^2}{2B_0^2\phi_0^2}+\frac{B_0'\phi_0'}{B_0^3\phi_0}+\frac{2\phi_0'}{B_0^2r\phi_0}+ +\frac{\phi_0''}{B_0^2\phi_0}-\frac{V_0}{2\phi_0}=\frac{1}{B_0^2r}\frac{B_0'}{B_0}, \quad (47)$$

$$\frac{p_{r0}}{\phi_0}+\frac{\omega_{BD}\phi_0'^2}{2B_0^2\phi_0^2}-\frac{A_0'\phi_0'}{A_0B_0^2\phi_0}-\frac{2\phi_0'}{B_0^2r\phi_0}+\frac{V_0}{2\phi_0}= =\frac{1}{B_0^2r}\frac{A_0'}{A_0}, \quad (48)$$

$$\frac{p_{\phi0}}{\phi_0}-\frac{\omega_{BD}\phi_0'^2}{2B_0^2\phi_0^2}-\frac{B_0'\phi_0'}{B_0^3\phi_0}-\frac{A_0'\phi_0'}{A_0B_0^2\phi_0}- -\frac{\phi_0''}{B_0^2\phi_0}+\frac{V_0}{2\phi_0}=\frac{1}{B_0^2}\left[\frac{A_0''}{A_0}+\frac{A_0'}{A_0}\frac{B_0'}{B_0}\right], \quad (49)$$

$$\frac{p_z}{\phi_0}-\frac{\phi_0'A_0'}{B_0^2A_0\phi_0}-\frac{\phi_0'B_0'}{B_0^3\phi_0}-\frac{\phi_0'}{B_0^2r\phi_0}- -\frac{\phi_0''}{B_0^2}-\frac{\omega_{BD}\phi_0'^2}{2\phi_0^2B_0^2}+\frac{V_0}{2\phi_0}= =\left(\frac{A_0'}{r}+A_0''\right)\frac{1}{A_0B_0^2}-\frac{B_0'}{B^3}\left(\frac{1}{r}+\frac{A_0'}{A_0}\right). \quad (50)$$

The unperturbed wave equation is

$$\frac{\phi_0'A_0'}{A_0}-\frac{\phi_0'B_0'}{B_0^2}+\frac{\phi_0'}{rB_0}=\frac{-1}{2\omega_{BD}+3}\times \times\left[(\rho_0+3p_{r0}+p_{\phi0}+p_{z0})+(\phi_0V_0-2V_0)\right].$$

The static distribution of Eq. (12) is identically satisfied in a static background, while (13) turns out to be

$$\frac{p_{r0}'}{B_0\phi_0}+\frac{\phi_0'p_{r0}}{\phi_0^2B_0^2}+(\rho_0+p_{r0})\frac{A_{0'}}{A_0B_0\phi_0}+ +\frac{1}{B_0\phi_0r}(p_{r0}-p_{\phi_0})-K_2'=0, \quad (51)$$

where

$$K_2'=T^{\phi}_{11unp}\frac{\phi_0'}{\phi_0B_0^2}-\frac{p_{r0}'}{\phi_0B_0}-(p_{r0}+\rho_0)\frac{A_0'}{A_0B_0\phi_0}- -\frac{p_{r0}+p_{\phi0}}{B_0r\phi_0}+2T^{\phi}_{11unp}\frac{B_0'}{B_0^2}-\frac{(T^{\phi}_{11unp})_{;r}}{B_0}$$

and the term T^{ϕ}_{11unp} represents the unperturbed form of the energy tensor due to the scalar field.

The static part of Eq. (10) is

$$p_{r0}\overset{\Sigma^{(e)}}{=}\frac{-V_0}{2}. \quad (52)$$

The perturbed form of the BD field equations is

$$-\frac{2T}{B_0^2}\left[\left(\frac{c}{r}\right)''-\frac{1}{r}\left(\frac{b}{B_0}\right)'-\frac{B_0'}{B_0}\left(\frac{c}{r}\right)'\right]= =-\frac{\bar{\rho}}{\phi_0}-\frac{T\rho_0\Phi}{\phi_0^2}+\frac{T\omega_{BD}\phi_0'^2b}{\phi_0^2B_0^3}-\frac{T\omega_{BD}\bar{\Phi}\phi_0'^2}{B_0^3\phi_0^3}+ +\frac{\omega_{BD}T\bar{\Phi}'\phi_0'}{B_0^2\phi_0^2}+\frac{T\phi_0'}{\phi_0B_0^2}\left(\frac{c}{r}\right)'+\frac{T\phi_0'}{\phi_0B_0^2}\left(\frac{\phi_0b}{B_0}\right)'- -\frac{2Tb\phi_0'}{\phi B_0^3}\frac{1}{r}+\left[\frac{T}{B_0^2r}+\frac{TB_0'}{B_0^3}\right]\left[\frac{\Phi}{\phi_0}\right]'+\frac{T\Phi''}{\phi_0B_0^2}- -\frac{2Tb\phi_0''}{B_0^3\phi_0}-\frac{T\phi_0'\Phi}{B_0^2\phi_0^2}-\frac{TV_0\Phi}{2\phi_0^2}-\frac{T\bar{V}}{2\phi_0}, \quad (53)$$

$$-\frac{c'}{c}+\frac{A_0'}{A_0}+\frac{b}{cB_0}=\frac{\omega_{BD}\dot{T}\Phi'\dot{\phi}}{\phi_0^2}-\frac{\omega_{BD}\dot{T}\Phi\phi_0'}{\phi_0}+ +\frac{A_0'\dot{T}c}{rA_0}-\frac{\dot{T}c'}{r}+\frac{\dot{T}b}{rB_0}, \quad (54)$$

$$-\frac{2\ddot{T}c}{rA_0^2}+\frac{T}{B_0^2r}\left[\left(\frac{a}{A_0}\right)'+r\frac{A_0'}{A_0}\left(\frac{c}{r}\right)'\right]-\frac{2bA_0'}{rA_0B_0^3}= =\frac{\bar{p}_r}{\phi_0}-\frac{Tp_{r0}\Phi}{\phi_0^2}-\frac{T\omega_{BD}}{\phi_0^2}\times \times\left[\frac{\phi_0'^2b}{B_0^3}+\Phi'-\frac{\Phi\phi_0'^2}{\phi_0}\right]-\frac{T\phi_0'}{\phi_0B_0^2}\left[\left(\frac{a}{A_0}\right)'+\left(\frac{c}{r}\right)'\right]+ +\frac{2Tb\phi_0'}{\phi B_0^3}\left[\frac{A_0'}{A_0}-\frac{1}{r}+\frac{V_0}{2B_0}\right]-\left[\frac{T}{B_0^2r}+\frac{TA_0'}{B_0^2A_0}-\frac{TB_0'}{B_0^3}\right]\times \times\left[\frac{\Phi}{\phi_0}\right]'-\frac{\dot{T}b\phi_0'}{\phi_0B_0^2}+\frac{\ddot{T}\Phi}{A_0^2\phi_0}+\frac{TV_0\Phi}{2\phi_0^2}+\frac{T\bar{V}}{2\phi_0}, \quad (55)$$

$$-\frac{b\ddot{T}}{A_0^2B_0}+\frac{T}{A_0B_0^2}\left[\left(\frac{a}{A_0}\right)''+\left(\frac{c}{r}\right)''+\right.$$
$$+\left(\frac{2A_0'}{A_0}-\frac{B_0'}{B_0}+\frac{1}{r}\right)\left(\frac{a}{A_0}\right)'-$$
$$\left.-\left(\frac{A_0'}{A_0}+\frac{1}{r}\right)\left(\frac{b}{B_0}\right)'+\left(\frac{A_0'}{A_0}-\frac{B_0'}{B_0}+\frac{2}{r}\right)\left(\frac{c}{r}\right)'\right]=$$
$$=-\frac{\bar{p}_\phi}{\phi_0}-\frac{Tp_{\phi0}\Phi}{\phi_0^2}-\frac{T\omega_{BD}\phi_0'^2b}{2\phi_0^2B_0^3}-\frac{T\omega_{BD}\Phi'}{2\phi_0^2B_0^2}-$$
$$-\frac{T\phi_0'}{\phi_0B_0^2}\left[\left(\frac{a}{A_0}\right)'+\frac{T}{r^2\phi_0}\left(\frac{br^2\phi_0'}{B_0}\right)'\right]+$$
$$+\frac{2Tb\phi_0'}{\phi B_0^3}\frac{A_0'}{A_0}-\left[\frac{TB_0'}{B_0^3}+\frac{TA_0'}{B_0^3A_0}\right]\left[\frac{\Phi}{\phi_0}\right]'-$$
$$-\frac{T}{B_0^2\phi_0}\left[\frac{\Phi'}{B_0}\right]'-\frac{T\Phi}{\phi_0B_0}\left[\frac{\phi_0'}{B_0}\right]'+\frac{2Tc}{r^2}\times$$
$$\times\left[\frac{\phi_0'B_0'}{B_0^3}+\frac{V_0^2}{2}\right]+\frac{2Tb\phi_0''}{B_0^3\phi_0}+\frac{T\phi_0'\Phi}{B_0^2\phi_0^2}+\frac{TV_0\Phi}{2\phi_0^2}+\frac{T\bar{V}}{2\phi_0}, \quad (56)$$

$$-T\left[\frac{2bA_0''}{A_0B_0^3}+\frac{A_0''}{A_0^2B_0^2}+\frac{a''}{A_0B_0^2}+\frac{A_0'}{B_0^3A_0}\left(\frac{b}{B_0}\right)'-\right.$$
$$-\frac{2bA_0'B_0'}{A_0B_0^3}+\frac{1}{B_0^2}\left(\frac{a}{A_0}\right)'-\frac{B_0'}{B_0^3}\left(\frac{c}{r}\right)'+\frac{2bB_0}{rB_0^3}-$$
$$-\frac{1}{r}\left(\frac{b}{B_0}\right)'+\frac{2bA_0'}{A_0rB_0^3}+\frac{A_0'}{A_0}\left(\frac{c}{r}\right)'+\frac{1}{rB_0^2}\left(\frac{a}{A_0}\right)'+$$
$$\left.+\frac{2bA_0'}{A_0B_0^3r}+\frac{A_0'}{A_0B_0^2}\left(\frac{c}{r}\right)'+\frac{1}{rB_0^2}\left(\frac{a}{A_0}\right)'\right]-$$
$$-\frac{b\ddot{T}}{A_0^2B_0}-\frac{c\ddot{T}}{rA_0^2}=\bar{p}_z-\frac{T}{\phi_0}\left[\frac{b\phi_0'}{B_0}\right]'-$$
$$-\frac{T\phi_0'}{\phi_0B_0^2}\left[\frac{a}{A_0}\right]'-\frac{TA_0'}{B_0^2A_0}\left[\frac{\Phi}{\phi_0}\right]'-\frac{TB_0'}{\phi_0B_0^3}\left[\frac{\Phi}{\phi_0}\right]'+$$
$$+\frac{2Tb\phi_0'}{B_0^3\phi_0}\left[\frac{A_0'}{A_0}+\frac{1}{r}\right]-\frac{\omega_{BD}T}{\phi_0^2B_0^2}\left[\frac{\Phi'\phi_0'}{\phi_0^2}-\frac{\Phi\phi_0'^2}{\phi_0^3}-\right.$$
$$\left.-\frac{\omega_{BD}b\phi_0'^2}{B_0^3\phi_0^2}\right]+\frac{\Phi''}{B_0\phi_0}-\frac{T\phi_0''\Phi}{B_0^2\phi_0^2}-\frac{TV_0\Phi}{\phi_0^2}, \quad (57)$$

and the perturbed wave equation is given by

$$\frac{\phi_0'}{B_0}\left[\frac{TaA_0'}{A_0}-Ta'+\frac{TbB_0'}{B_0}-Tb'+\frac{Tc}{r}-Tc'\right]+$$
$$+2\frac{Tb\phi_0''}{B_0}-\frac{T\Phi''}{B_0^2}=\frac{1}{2\omega_{BD}+3}\times$$
$$\times\left[\bar{\rho}+3\bar{p}_r+\bar{p}_\phi+\bar{p}_z+T\Phi V_0-2\bar{V}\right]. \quad (58)$$

The perturbed configuration in Eq. (10) is

$$-\bar{p}_r\overset{\Sigma^{(e)}}{=}-\frac{T\Phi p_{r0}}{\phi_0}-\frac{T\Phi V_0}{2\phi_0}+\frac{T\bar{V}}{2\phi_0}. \quad (59)$$

The perturbed terms $\bar{K}_1$ and $\bar{K}_2$ in Eqs. (22) and (23) are described as

$$\bar{K_1}=\dot{T}\left[\frac{\rho_0\Phi}{A_0\phi_0^2}+T^{\phi}_{00(p)}A_0^{-1}+\left(T^{\phi}_{01(p)}\right)'A^{-1}B^{-2}-\right.$$
$$\left.-T^{\phi}_{01(p)}A_0'A^{-2}B^{-2}-T^{\phi}_{01(p)}B_0'A^{-1}B^{-3}\right],$$

$$\bar{K_2}=-T^{\phi}_{11p}\frac{\phi_0'}{\phi_0^2B_0^2}+\left[-2T\Phi\phi_0+T\Phi'-\frac{2Tb\phi_0'}{\phi_0}\right]\times$$
$$\times\frac{T^{\phi}_{11unp}}{\phi_0^2B_0^2}-\frac{\bar{p}_r'}{\phi_0B_0}+\frac{p_{r0}'T}{B_0}\left[\frac{b}{B_0}+\frac{\Phi}{\phi_0}\right]+\frac{\bar{p}_r-\bar{p}_\phi}{rB_0\phi_0}+$$
$$+\left[\frac{b\Phi}{B_0^2r\phi_0^2}+\frac{1}{rB_0\phi_0^2}\left(\frac{c}{r}\right)'\right]T(p_{r0}-p_{\phi0})-\frac{(T^{\phi}_{10})_{,t}}{A_0B_0^2}+$$
$$+\frac{4bB_0'}{B_0^3}T^{\phi}_{11unp}-2T^2_{11unp}\frac{Tb'}{B_0^2}+T^{\phi}_{11unp}\frac{Tb}{B_0^2}-$$
$$-\frac{2T^{\phi}_{11unp}B_0'}{B_0^2}-\frac{(T^{\phi}_{11p})'}{B_0^2}+\left[\frac{b}{B_0}+\frac{\Phi}{\phi_0}+\frac{1}{B_0\phi_0}\left[\frac{a}{A_0}\right]'\right]\times$$
$$\times\frac{(p_{r0}+\rho_0)A_0'}{A_0B_0\phi_0}+(\bar{\rho}+\bar{p}_r)\frac{A_0'}{A_0B_0\phi_0}+\frac{T_{11p}B_0'}{B_0^2},$$

where $T^{\phi}_{\mu\nu(unp)}$ and $T^{\phi}_{\mu\nu(p)}$ indicate unperturbed and perturbed distributions of the BD energy part.

The values of K_4 and K_5 in (38) are given by

$$K_4=\rho_0\frac{\Phi}{\phi_0}+\Phi\frac{m_0}{r^3c^2}+\Phi'\left(\frac{m_0}{r^2c^2}+(3+2\alpha)\frac{m_0^2}{r^3c^4}\right)+$$
$$+\Phi\left[1+\frac{2m_0}{rc^2}+\frac{m_0^2}{r^2c^4}-\frac{2\alpha m_0}{rc^2}+(4\alpha+1)\frac{m_0^2}{r^2c^4}\right],$$

$$K_5=-p_{r0}\frac{\Phi}{\phi_0}\left(1-\frac{2\alpha}{r^2c^2}\right)+(\rho_0+p_{r0})\times$$
$$\times\left[\frac{a'}{\phi_0}\left(1+\frac{2m_0}{rc^2}(1-\alpha)\right)+\frac{m_0^2}{r^2c^4}(1+4\alpha)\right]-$$
$$-\gamma^2_{\Sigma^{(e)}}\left(\frac{m_0}{r^2c^2}-\frac{4m_0^2}{r^3c^4}+\frac{2\alpha m_0^2}{r^4c^4}\right).$$

The scalar field terms in (39) are

$$K_6=\gamma_{\Sigma^{(e)}}\left[\rho_0\Phi+\left(1+\frac{\Phi'}{\phi_0}\right)+\right.$$
$$\left.+m_0\left(\frac{\Phi\phi_0}{r^3c^2}\right)\right]+\frac{\phi_0}{r}\left(1-\frac{m_0}{rc^2}\right), \quad (60)$$
$$K_7=\gamma^2_{\Sigma^{(e)}}\frac{\Phi m_0^2\phi_0}{r^2c^2}-\frac{\rho_0a'm_0r}{c^2}.$$

The values of K_8 and K_9 in (40) are

$$K_8 = \gamma_{\Sigma^{(e)}}\rho_0\Phi + \left[\frac{c}{r}\right]'\left(1+\frac{m_0^2}{r^2c^4}-\frac{2\alpha m_0}{rc^4}\right)+ \\ +\left(1+\frac{m_0^2}{r^2c^4}-\frac{3\alpha m_0}{rc^2}\right)\frac{\Phi'}{\phi_0}+ \\ +\frac{\phi_0}{r}\left(1-\frac{m_0}{rc^2}-\frac{2\alpha m_0}{r^2c^2}\right)+\Phi\phi_0\left(\frac{m_0^2}{r^2c^2}-2\frac{m_0^2}{r^3c^4}\right)- \\ -\Phi\phi_0\frac{m_0^2}{r^2c^4}-2\Phi\alpha\frac{m_0^2}{r^2c^4},$$

$$K_9 = 1+\frac{\Phi'}{\phi_0}+\frac{\phi_0}{r}\left(1-\frac{m_0}{rc^2}\right)+\phi_0\frac{m_0}{r^3c^3}+ \\ +\gamma^2_{\Sigma^{(e)}}\frac{\Phi\phi_0 m_0^2}{r^2c^2}.$$

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