

SINGULAR TEMPERATURE DEPENDENCE OF THE EQUATION OF STATE OF SUPERCONDUCTORS WITH SPIN–ORBIT INTERACTION IN THE LOW-TEMPERATURE REGION

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The equation of state is investigated for a thin superconducting film in a longitudinal magnetic film and with strong spin–orbit interaction at the critical point. As a first step, the state with the maximal value of the magnetic field for a given value of spin–orbit interaction at $T = 0$ is chosen. This state is investigagted in the low-temperature region. The temperature contribution to the equation of state is weakly singular.

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1. INTRODUCTION

The interest in the effects of spin–orbit interaction on superconductivity is connected, in particular, with experimental progress in the preparation of monoatomic or monomolecular layers deposited on a substrate [1, 2] with a noticeable spin–orbit interaction of the general nature [3, 4]. In such systems, pairing with a finite momentum occurs [5, 6]. For $T = 0$, spin–orbit interaction leads to the appearance of several critical points. In two of them, the superconducting state is homogeneous, and in six other points, the state is inhomogeneous. We investigate the state with the maximal value of the critical magnetic field at $T = 0$, in the low-temperature region. The considered solution has an equation of state with weakly singular temperature corrections. On the way, we obtain an infinite number of nontrivial integral expressions connected by the Euler ψ-function. The weak singularity is a result of these equations.

2. EQUATIONS FOR THE SUPERCONDUCTING ORDER PARAMETER

We consider very thin superconducting films deposited on a substrate [1,2]. In such a sample, spin–orbit interaction can be taken in the form [3, 4]

$$\hat{V}_{so} = \frac{-i\hbar^2}{2m^2c^2}\left[\frac{\partial U}{\partial \mathbf{r}} \times \frac{\partial}{\partial \mathbf{r}}\right] \cdot \hat{\mathbf{S}}, \tag{1}$$

where $\hat{\mathbf{S}}$ is the spin operator of an electron, m is the mass, c is the speed of light, U is a self-consistent potential. In normal metals, the Green's function is a 2×2 matrix and satisfies the equation

$$\left(\frac{\partial}{\partial \tau} + \hat{L}\right)\hat{G} = \delta(\mathbf{r}-\mathbf{r}')\delta(\tau-\tau'), \tag{2}$$

where τ is an imaginary "time" and

$$\hat{L} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial \mathbf{r}^2} + \hat{V}_{so} - \mu - \frac{e\hbar}{mc}\mathbf{H}\cdot\mathbf{S}. \tag{3}$$

The last term is the Zeeman energy, μ is the chemical potential, and $\mathbf{H}$ is an external magnetic field.

We seek eigenfunctions of the operator $\hat{L}$ in the form

$$\begin{pmatrix} \psi_1^{(n)} \\ \psi_2^{(n)} \end{pmatrix} = \exp\left(\frac{ip_x}{\hbar}x + \frac{ip_y}{\hbar}y\right)\chi_n(z) \times \begin{pmatrix} f_1^{(n)} \\ f_2^{(n)} \end{pmatrix}, \tag{4}$$

where $\chi_n(z)$ are normalized eigenfunctions of the operator

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + U(z)\right)\chi_n(z) = \epsilon_n\chi_n(z), \quad n = 0, 1, 2, \ldots \tag{5}$$

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In the subspace with given values of p_x, p_y, and n, the vector $(f_1^{(n)}, f_2^{(n)})$ is a solution of the equation

$$\left[-\frac{\hbar^2}{4m^2c^2}\,\alpha_n \begin{pmatrix} 0 & ip_x+ip_y \\ p_x-ip_y & 0 \end{pmatrix} - \right.$$
$$\left. - \begin{pmatrix} 0 & h_x-ih_y \\ h_x+ih_y & 0 \end{pmatrix}\right]\begin{pmatrix} f_1^{(n)} \\ f_2^{(n)} \end{pmatrix} =$$
$$= \lambda^{(n)}\begin{pmatrix} f_1^{(n)} \\ f_2^{(n)} \end{pmatrix} \quad (6)$$

with

$$\alpha_n = \int\limits_{-\infty}^{\infty} dz \frac{\partial U}{\partial z}\left|\chi_n(z)\right|^2, \qquad h = \mu_B \mathbf{H}, \quad \mu_B = \frac{e\hbar}{2mc}. \quad (7)$$

The eigenvalues $\lambda^{(n)}$ are

$$\lambda_{\pm}^{(n)} = \pm\left\{\frac{\tilde{\alpha}_n^2}{4}v_F^2p^2 + h^2 + (h_xp_y - h_yp_x)\,v_F\tilde{\alpha}_n\right\}^{1/2}, \quad (8)$$

where

$$\tilde{\alpha}_n = \alpha_n\frac{\hbar}{2m^2c^2v_F}. \quad (9)$$

The eigenfunctions $f_{\pm}^{(n)}$ are

$$f_+^{(n)} = \frac{1}{\sqrt{2}}\times \left[1; -\frac{1}{\lambda_+^{(n)}}\left(\frac{\tilde{\alpha}v_F(p_y-ip_x)}{2} + (h_x+ih_y)\right)\right],$$
$$f_-^{(n)} = \frac{1}{\sqrt{2}}\times \left[-\frac{1}{\lambda_-^{(n)}}\left(\frac{\tilde{\alpha}v_F}{2}(p_y+ip_x) + (h_x-ih_y)\right); 1\right]. \quad (10)$$

In a normal metal, the Green's function decomposes into four blocks

$$\begin{pmatrix} \hat{G}_{++}^{(0)} & ; & 0 \\ 0 & ; & \hat{G}_{--}^{(0)} \end{pmatrix}, \quad \begin{pmatrix} \frac{\partial}{\partial\tau}+\hat{L} & ; & 0 \\ 0 & ; & \frac{\partial}{\partial\tau}-\hat{L} \end{pmatrix}\times \begin{pmatrix} \hat{G}_{++}^{(0)} \\ \hat{G}_{--}^{(0)} \end{pmatrix} = \delta(\mathbf{r}-\mathbf{r}')\delta(\tau-\tau'). \quad (11)$$

From Eqs. (2), (8), (10), and (11), we obtain

$$\hat{G}_{++}^{(0)} = T\sum_{\omega}\sum_{n}\chi_n(z)\chi_n(z')\sum_{\lambda_+^{(n)}}\frac{1}{(2\pi\hbar)^2}\times \int\limits_{-\infty}^{\infty} dp_xdp_y\left(\left(f(\mathbf{p})^{(n)}f(\mathbf{p})^{(n)}\right)^{\dagger}\right)_+ \times \frac{\exp\left(\frac{i\mathbf{p}}{\hbar}\cdot(\mathbf{r}-\mathbf{r}')\right)\exp\left(-i\omega(\tau-\tau')\right)}{-i\omega + p^2/2m + \epsilon_n - \mu + \lambda_+^{(n)}(p)}. \quad (12)$$

In the thin film under consideration, the energy difference between levels in Eq. (5) is large and it is possible to keep only one lowest level in the sum over levels ($n = 0$). Equation (10) forms a complete basis, in terms of which we can express the electron–electron interaction as [7]

$$H_{int} = \frac{1}{2}\int d\mathbf{r}\,d\mathbf{r}'V(\mathbf{r}-\mathbf{r}')\times \left(\psi_{\nu}^{\dagger}(\mathbf{r})\left(\psi_{\mu}^{\dagger}(\mathbf{r}')\psi_{\mu}(\mathbf{r}')\right)\psi_{\nu}(\mathbf{r})\right), \quad (13)$$

where $\nu, \mu = \pm$. We assume in what follows that the interaction is δ-function-like: $V(\mathbf{r}-\mathbf{r}') = V_0\delta(\mathbf{r}-\mathbf{r}')$. We consider Cooper pairing of the type $\langle\psi_+(\mathbf{r})\psi_-(\mathbf{r}')\rangle$.

In a superconductor, the Green's function is a 4×4 matrix. It is the solution of the system of equations

$$\begin{pmatrix} \frac{\partial}{\partial\tau}+\hat{L} & ; & \hat{\Delta} \\ \hat{\Delta}^+ & ; & \frac{\partial}{\partial\tau}-\hat{L} \end{pmatrix}\begin{pmatrix} \hat{G}_{++} & ; & \hat{F} \\ \hat{F^+} & ; & \hat{G}_{--} \end{pmatrix} = \delta(\mathbf{r}-\mathbf{r}')\delta(\tau-\tau'). \quad (14)$$

The operator $\hat{\Delta}(\mathbf{r})$ is [8]

$$\hat{\Delta}(\mathbf{r}) = \hat{\Delta}(\mathbf{r})\begin{pmatrix} 1;0 \\ 0;1 \end{pmatrix}\mathrm{Sp}, \quad (15)$$

where Sp denotes taking the trace. In the approximation linear in Δ, we use Eq. (14) to obtain

$$\hat{\Delta}^+\hat{G}_{++}^{(0)} + \left(\frac{\partial}{\partial\tau}-\hat{L}\right)_{--}\hat{F}^{\dagger} = 0, \qquad \hat{F}^{\dagger} = -\hat{G}_{--}^{(0)}\hat{\Delta}^+\hat{G}_{++}^{(0)}. \quad (16)$$

The self-consistency equation is [8]

$$\Delta^* = |V_0|\,\mathrm{Sp}\,\hat{F}^{\dagger}(\mathbf{r},\mathbf{r},\tau=\tau'). \quad (17)$$

Equations (14) and (17) are to be solved at the critical point with $T = 0$ [8]. Eight branches are then found. Two of them correspond to homogeneous states and six to inhomogeneous states. Free parameters in the considered problem are $\tilde{\alpha}$, h, $\mathbf{Q}$, and T. The vector $\mathbf{Q}$ can be found from the extremum condition for the magnetic field value for a given spin–orbit interaction. Four orientations are possible: $\mathbf{Q}\cdot\mathbf{H}=0$ and $\mathbf{Q}$ oriented along the magnetic field $\mathbf{H}$. The last two orientations cannot be realized. Two orientations with $\mathbf{Q}\cdot\mathbf{H}=0$ are degenerate and give the same value for the critical magnetic field. We can introduce two dimensionless quantities:

$$\gamma_1 = \left(\frac{2h}{v_F Q}\right)^2, \quad \gamma_2 = \left(\frac{\tilde{\alpha}\epsilon_F}{h}\right)^2. \tag{18}$$

Two quantities $R_{1,2}$ are related to $\gamma_{1,2}$:

$$R_{1,2} = \gamma_1\sqrt{\gamma_2} \pm \sqrt{\gamma_1^2\gamma_2 + \gamma_1\gamma_2 + \gamma_1}. \tag{19}$$

On each branch of the plane $\gamma_1\gamma_2$, the functions γ_1 and γ_2 are explicit functions of only one parameter, R_2 or R_1.

It was found in [8] that for $T=0$, branches I and II are optimal. We have

$$\gamma_1 = -\frac{R_2}{2} + \frac{1}{2}\sqrt{R_2^2 - (1+R_2)^2}, \qquad \gamma_2 = \left(\frac{1+R_2}{2\gamma_2}\right)^2 \tag{20}$$

for branch I and

$$\gamma_1 = -\frac{R_2}{2} - \frac{1}{2}\sqrt{R_2^2 - (1+R_2)^2}, \qquad \gamma_2 = \left(\frac{1+R_2}{2\gamma_1}\right)^2 \tag{21}$$

for branch II. In both cases, $R_1 = 1$ and the interval for R_2 is $-1 < R_2 < -1/2$.

Integrating over ξ in Eq. (17) with the use of Eqs. (12) and (16), we obtain the result ($\xi = p^2/2m - \mu - \epsilon_0$)

$$1 = \frac{|V_0|}{2} T \sum_{\omega_n} \frac{m}{4\pi\hbar^2}\langle\chi_0^2\rangle \int_0^{2\pi} d\varphi \times \left\{ \frac{|\omega_n|}{\omega_n^2 + \left(\frac{\mathbf{p}\cdot\mathbf{Q}}{2m} + \lambda_+^{(0)}\right)^2} + \frac{|\omega_n|^2}{\omega_n^2 + \left(\frac{\mathbf{p}\cdot\mathbf{Q}}{2m} + \lambda_-^{(0)}\right)^2} \right\}. \tag{22}$$

On the considered branches I and II, we have $\mathbf{Q}\cdot\mathbf{H}=0$ and Eq. (22) takes the form

$$1 = \frac{|V_0|}{2} T \sum_{\omega} \frac{m}{4\pi\hbar^2}\langle\chi_0^2\rangle \times \int_0^{2\pi} d\varphi \left\{ \frac{|\omega_n|}{\omega_n^2 + a_+^2} + \frac{|\omega_n|}{\omega_n^2 + a_-^2} \right\} \tag{23}$$

with

$$a_\pm = \frac{p_F Q}{2m}\sin\varphi \pm \lambda, \quad \lambda \equiv \lambda_+.$$

3. LOW-TEMPERATURE CORRECTION IN THE EQUATION OF STATE

The sum over ω_n in Eq. (23) can be calculated explicitly, with the result

$$1 = \frac{|V_0| m\langle\chi_0^2\rangle}{16\pi^2\hbar^2} \int_0^{2\pi} d\varphi \left\{ \ln \frac{\omega_D^2}{(a_+^2+\pi^2T^2)(a_-^2+\pi^2T^2)} - \left[\psi\left(\frac{1}{2} + \frac{ia_+}{2\pi T}\right) + \psi\left(\frac{1}{2} - \frac{ia_+}{2\pi T}\right) - \ln\left(\frac{1}{2} + \frac{ia_+}{2\pi T}\right) - \ln\left(\frac{1}{2} - \frac{ia_+}{2\pi T}\right)\right] - \left[\psi\left(\frac{1}{2} + \frac{ia_-}{2\pi T}\right) + \psi\left(\frac{1}{2} - \frac{ia_-}{2\pi T}\right) - \ln\left(\frac{1}{2} + \frac{ia_-}{2\pi T}\right) - \ln\left(\frac{1}{2} - \frac{ia_-}{2\pi T}\right)\right]\right\}, \tag{24}$$

where ψ is the Euler di-gamma function and

$$a_+ = \frac{h}{\sqrt{\gamma_1}} \times \left\{\sin\varphi + (\gamma_1 + \gamma_1\gamma_2 + 2\gamma_1\sqrt{\gamma_2}\sin\varphi)^{1/2}\right\}, \qquad a_- = \frac{h}{\sqrt{\gamma_1}} \times \left\{\sin\varphi - (\gamma_1 + \gamma_1\gamma_2 + 2\gamma_1\sqrt{\gamma_2}\sin\varphi)^{1/2}\right\}. \tag{25}$$

In what follows, we consider the limit $T \ll T_{c0}$ with $T_{c0} \equiv T_c$ ($h=0$, $\tilde{\alpha}=0$). In this case, a parameter R_2 exists such that $\gamma_2(R_2) = (\tilde{\alpha}\epsilon_F/h)^2$ and R_2 can be found from Eq. (20) for branches I and II. The value of γ_1 is close to the value of $\gamma_1(R_2)$ and can be found from Eq. (24). For this, we set

$$\gamma_1 = \gamma_1(R_2) + \delta\gamma_1. \tag{26}$$

The expression $(a_+^2+\pi^2T^2)(a_-^2+\pi^2T^2)$ is a polynomial of degree four in $\sin\varphi$. Two roots of this polynomial are

$$(\sin\varphi)_{1,2} = \frac{i\pi T\sqrt{\gamma_1}}{h} + \gamma_1\sqrt{\gamma_2} \pm \left(\gamma_1 + \gamma_1\gamma_2 + \gamma_1^2\gamma_2 + \frac{2i\pi T}{h}\gamma_1\sqrt{\gamma_1\gamma_2}\right)^{1/2}. \tag{27}$$

Because the polynomial has real coefficients, two other roots, 3 and 4, are complex conjugate to roots 1 and 2:

$$(\sin\varphi)_{3,4} = (\sin\varphi)^*_{1,2}\,.$$

Using Eqs. (20) (branches I and II) and (27), we obtain

$$\begin{aligned}(\sin\varphi)_1 &= 1 + \frac{\delta\gamma_1}{\gamma_1(1-R_2)} + \frac{2i\pi T\sqrt{\gamma_1}}{h(1-R_2)},\\ (\sin\varphi)_2 &= R_2 - \frac{\delta\gamma_1}{\gamma_1}\,\frac{R_2^2}{1-R_2} - \frac{2i\pi T\sqrt{\gamma_1}R_2}{h(1-R_2)}.\end{aligned} \tag{28}$$

Equations (28) allow calculating the first term in the r.h.s. of Eq. (24). The second and third terms are located near zero of the quantities $a_\pm$. We should find an explicit expression for a_+ near the points $\sin\varphi = R_2$ and for a_- near the point $\varphi = \pi/2$. Near the point $\varphi_0 = -\arcsin|R_2|$, we put

$$\sin\varphi = R_2 + \sqrt{1-R_2^2}\,\delta\varphi, \tag{29}$$

and then

$$a_+ = \frac{h}{\sqrt{\gamma_1}}\left\{-\sqrt{1-R_2^2}\,\frac{1-R_2}{2R_2}\delta\varphi - \frac{\delta\gamma_1}{2\gamma_1}R_2\right\} \tag{30}$$

in the vicinity of φ_0.

Near the point $\varphi_1 = -\pi + \varphi_0$, we put

$$\sin\varphi = R_2 - \delta\varphi\sqrt{1-R_2^2}, \tag{31}$$

and then

$$a_+ = \frac{h}{\sqrt{\gamma_1}}\left\{\sqrt{1-R_2^2}\,\frac{1-R_2}{2R_2}\delta\varphi - \frac{\delta\gamma_1 R_2}{2\gamma_1}\right\} \tag{32}$$

in the vicinity of φ_1.

In the vicinity of the point $\varphi = \pi/2$, we put

$$\varphi = \frac{\pi}{2} + \delta\varphi, \quad \sin\varphi = 1 - \frac{(\delta\varphi)^2}{2}. \tag{33}$$

In the vicinity of the point $\varphi = \pi/2$, we have

$$a_- = -\frac{h}{2\sqrt{\gamma_1}}\left\{\frac{1-R_2}{2}(\delta\varphi)^2 + \frac{\delta\gamma_1}{\gamma_1}\right\}. \tag{34}$$

The point $\delta\gamma_1/\gamma_1 = 0$ is a deep-lying break point of the integral in the r.h.s. of Eq. (24). At negative value of $\delta\gamma_1$, there exists an external point over $\delta\gamma_1$ in Eq. (24). To find it, we put

$$\frac{\delta\gamma_1}{\gamma_1} = -\beta Z\frac{1-R_2}{2}, \quad Z = \frac{4\pi T\sqrt{\gamma_1}}{h(1-R_2)}. \tag{35}$$

The quantity β is a dimensionless function of Z. The equation for β is found below.

Using Eqs. (25) and (28)–(34), we reduce Eq. (24) to the form

$$1 = \frac{|V_0|m\left\langle\chi_0^2\right\rangle}{16\pi^2\hbar^2}\left\{4\pi\ln\left(\frac{\sqrt{\gamma_1}\omega_D}{h}\right)^2 + {} + I_0 + I_1 + I_2 + I_3 + I_4 + I_5\right\}, \tag{36}$$

where

$$\begin{aligned}
I_0 &= \int_0^{2\pi} d\varphi\ln\left\{\left|\left(\sin^2\varphi - \left(1+\frac{\delta\gamma_1}{\gamma_1(1-R_2)}\right)^2\right)\left(\sin^2\varphi - \left(R_2 - \frac{\delta\gamma_1}{\gamma_1}\,\frac{R_2^2}{1-R_2}\right)^2\right)\right|^{-1}\right\},\\
I_1 &= -\int_0^{2\pi} d\varphi\ln\left\{\frac{\left(\sin\varphi - 1 - \dfrac{\delta\gamma_1}{\gamma_1(1-R_2)}\right)^2 + \dfrac{4\pi^2T^2\gamma_1}{h^2(1-R_2)^2}}{\left(\sin\varphi - 1 - \dfrac{\delta\gamma_1}{\gamma_1(1-R_2)}\right)^2}\right\},\\
I_2 &= -\int_0^{2\pi} d\varphi\ln\left\{\frac{\left(\sin\varphi - R_2 + \dfrac{\delta\gamma_1}{\gamma_1}\,\dfrac{R_2^2}{1-R_2}\right)^2 + \dfrac{4\pi^2T^2\gamma_1R_2^2}{h^2(1-R_2)^2}}{\left(\sin\varphi - R_2 + \dfrac{\delta\gamma_1}{\gamma_1}\,\dfrac{R_2^2}{1-R_2}\right)^2}\right\},\\
I_3 &= -\sqrt{Z}\int_{-\infty}^{\infty} dt\left\{\psi\left(\frac12 + \frac{i}{2}(t^2-\beta)\right) + \psi\left(\frac12 - \frac{i}{2}(t^2-\beta)\right) - \ln\left(\frac14 + \frac{(t^2-\beta)^2}{4}\right)\right\},\\
I_4 &= -Z\sqrt{\frac{R_2^2}{1-R_2^2}}\int_{-\infty}^{\infty} dx\left\{\psi\left(\frac12 + ix\right) + \psi\left(\frac12 - ix\right) - \ln\left(\frac14 + x^2\right)\right\},\\
I_5 &= I_4.
\end{aligned} \tag{37}$$

With the help of Eqs. (33)–(35), we obtain

$$I_1 = -4\sqrt{Z}\int_{-\infty}^{\infty} dt \frac{t^2}{(t^2-\beta)((t^2-\beta)^2+1)} = = -4\sqrt{Z}\frac{\pi}{(1+\beta^2)^{1/4}}\left[\cos\left(\frac{1}{2}\arctan\frac{1}{\beta}\right) - \beta\sin\left(\frac{1}{2}\arctan\frac{1}{\beta}\right)\right]. \quad (38)$$

This integral is taken in the sense of principal value. The integral entering the expression for $\{I_4, I_5\}$ is equal to

$$\int_0^{\infty} dx \left\{\psi\left(\frac{1}{2}+ix\right)+\psi\left(\frac{1}{2}-ix\right)-\ln\left(\frac{1}{4}+x^2\right)\right\} = = -\frac{\pi}{2}. \quad (39)$$

More complicated for calculation is the integral I_3. After a simple transformation, we obtain

$$I_3 = -2\sqrt{Z}\left\{-\frac{\pi}{\sqrt{\beta}} - \int_{\beta/2}^{\infty} dx\left(\frac{2}{\sqrt{\beta}}-\frac{1}{\sqrt{2x+\beta}}\right) \times \left(\psi\left(\frac{1}{2}+ix\right)+\psi\left(\frac{1}{2}-ix\right)-\ln\left(\frac{1}{4}+x^2\right)\right) + \int_0^{\beta/2} dx\left(\left(\frac{1}{\sqrt{\beta-2x}}-\frac{1}{\sqrt{\beta}}\right)-\left(\frac{1}{\sqrt{\beta}}-\frac{1}{\sqrt{\beta+2x}}\right)\right) \times \left(\psi\left(\frac{1}{2}+ix\right)+\psi\left(\frac{1}{2}-ix\right)-\ln\left(\frac{1}{4}+x^2\right)\right)\right\}. \quad (40)$$

The integral I_2 can easily be found from Eq. (37). We obtain

$$I_2 + I_4 + I_5 = 0. \quad (41)$$

The value of the parameter β can be found from the extremum condition

$$\frac{\partial}{\partial\beta}\left(I_1 + I_3 - 2\pi(1-R_2)\beta Z\right) = 0. \quad (42)$$

We now check that the quantity $I_1 + I_3$ has a deep minimum at the point $\beta = 0$. From Eq. (38), we obtain

$$I_1(0) = -2\sqrt{2}\,\pi\sqrt{Z}. \quad (43)$$

From Eq. (37), we find

$$I_3(0) = -2\sqrt{Z}\int_0^{\infty} dt\left(\psi\left(\frac{1}{2}+\frac{it^2}{2}\right) + \psi\left(\frac{1}{2}-\frac{it^2}{2}\right) - \ln\left(\frac{1}{4}+\frac{t^4}{4}\right)\right) = = \sqrt{Z}\left[\left(\sqrt{2}-\frac{1}{2}\right)\pi\zeta\left(\frac{3}{2}\right) + 2\cdot 2\sqrt{2}\int_0^{\infty} dx\, x^{3/2} \times \sum_{k=0}^{\infty}\left(\frac{4k+3}{\left((k+1/2)^2+x^2\right)^3} - \frac{8(k+1)^3}{\left((k+1/2)^2+x^2\right)^3\left((k+3/2)^2+x^2\right)}\right)\right], \quad (44)$$

where $\zeta(x)$ is the Riemann zeta function. Numerical calculation of the last integral in Eq. (44) gives

$$2\sqrt{2}\int_0^{\infty} dx\, x^{3/2}\sum_{k=0}^{\infty}\left(\frac{4k+3}{\left((k+1/2)^2+x^2\right)^3} - \frac{8(k+1)^3}{\left((k+1/2)^2+x^2\right)^3\left((k+3/2)^2+x^2\right)}\right) = = -1.208951, \quad (45)$$

and because $\zeta(3/2) = 2.612373$, we obtain

$$I_1(0) + I_3(0) = -\pi\cdot 1.209802409\sqrt{Z}. \quad (46)$$

Because $I_1(\beta) + I_3(\beta)$ tends to zero as β tends to infinity, Eq. (42) has a solution for β that is much larger than unity. The function $I_1(\beta)$ (see Eq. (38)) has a normal expansion in inverse powers of β:

$$I_1(\beta) = -\frac{4\pi\sqrt{Z}}{\sqrt{\beta}}\left\{\frac{1}{2} - \frac{1}{16\beta^2} + \frac{7}{256\beta^4} - \frac{33}{2048\beta^6} + \ldots\right\}. \quad (47)$$

The function $\phi(x)$ equal to

$$\phi(x) = \psi\left(\frac{1}{2}+ix\right) + \psi\left(\frac{1}{2}-ix\right) - \ln\left(\frac{1}{4}+x^2\right) \quad (48)$$

has only an asymptotic expansion in $1/x^2$:

$$\phi(x) = -\frac{1}{3x^2} + \frac{1}{60x^4} - \frac{13}{1008x^6} - \frac{7}{960x^8} - \frac{647}{42240x^{10}}\ldots \quad (49)$$

Using this expansion, we can present expression (40) for $I_3(\beta)$ in the form

$$
\begin{aligned}
I_3(\beta) = 2\sqrt{Z}\Bigg\{ & \frac{\pi}{\sqrt{\beta}} - \frac{3}{\beta^{5/2}}\int_0^\infty dx\, x^2\left(\phi(x)+\frac{1}{3x^2}\right) + \\
& + \frac{1}{3}\left[\int_0^{\beta/2}\frac{dx}{x^2}\left(\frac{1}{\sqrt{\beta-2x}}+\frac{1}{\sqrt{\beta+2x}}-\frac{2}{\sqrt{\beta}}\right) - \right. \\
& \left. - \int_{\beta/2}^\infty \frac{dx}{x^2}\left(\frac{2}{\sqrt{\beta}}-\frac{1}{\sqrt{\beta+2x}}\right)\right] - \\
& - \frac{35}{4\beta^{9/2}}\int_0^\infty dx\, x^4\left(\phi(x)+\frac{1}{3x^2}-\frac{1}{60x^4}\right) + \\
& + \frac{1}{60}\left[-\int_0^{\beta/2}\frac{dx}{x^4}\left(-\frac{3x^2}{\beta^{5/2}}+\left(\frac{1}{\sqrt{\beta-2x}} + \right.\right.\right. \\
& \left.\left. + \frac{1}{\sqrt{\beta+2x}}-\frac{2}{\sqrt{\beta}}\right)\right) + \\
& \left. + \int_{\beta/2}^\infty \frac{dx}{x^4}\left(\frac{3x^2}{\beta^{5/2}}+\left(\frac{2}{\sqrt{\beta}}-\frac{1}{\sqrt{\beta+2x}}\right)\right)\right] - \\
& - \frac{231}{8\beta^{13/2}}\int_0^\infty dx\, x^6\left(\phi(x)+\frac{1}{3x^2}-\frac{1}{60x^4}+\frac{13}{1008x^6}\right) + \\
& + \frac{13}{1008}\left[\int_0^{\beta/2}\frac{dx}{x^6}\left(-\frac{3x^2}{\beta^{5/2}}-\frac{35x^4}{4\beta^{9/2}} + \right.\right. \\
& \left. + \left(\frac{1}{\sqrt{\beta-2x}}+\frac{1}{\sqrt{\beta+2x}}-\frac{2}{\sqrt{\beta}}\right)\right) - \\
& \left. - \int_{\beta/2}^\infty \frac{dx}{x^2}\left(\frac{35}{4\beta^{9/2}}+\frac{3}{\beta^{5/2}x^2}+\frac{1}{x^4}\left(\frac{2}{\sqrt{\beta}}-\frac{1}{\sqrt{\beta+2x}}\right)\right)\right] + \\
& + \int_{\beta/2}^\infty dx\left(\phi(x)+\frac{1}{3x^2}-\frac{1}{60x^4}+\frac{13}{1008x^6}\right)\times \\
& \times\left(\frac{3x^2}{\beta^{5/2}}+\frac{35x^4}{4\beta^{9/2}}+\frac{231x^6}{8\beta^{13/2}}+\left(\frac{2}{\sqrt{\beta}}-\frac{1}{\sqrt{\beta+2x}}\right)\right) - \\
& - \int_0^{\beta/2} dx\left(\phi(x)+\frac{1}{3x^2}-\frac{1}{60x^4}+\frac{13}{1008x^6}\right)\times \\
& \times\left(-\frac{3x^2}{\beta^{5/2}}-\frac{35x^4}{4\beta^{9/2}}-\frac{231x^6}{8\beta^{13/2}} + \right. \\
& \left. + \left(\frac{1}{\sqrt{\beta-2x}}+\frac{1}{\sqrt{\beta+2x}}-\frac{2}{\sqrt{\beta}}\right)\right)\Bigg\}. \qquad (50)
\end{aligned}
$$

It follows from Eq. (50) that the function $I_3(\beta)$ has an asymptotic expansion in a series in $\beta^{-1/2-k}$ with different structures of the coefficients for odd and even k. The coefficients with odd k are equal to the expansion coefficient of $\phi(x)$ at x^{-2k} times the quantity in square brackets in Eq. (40), proportional to $\beta^{-(2k-1/2)}$. This factor has a simple structure and depends only on the functions $(\beta \pm 2x)^{-1/2}$.

The expansion coefficients with even k are equal to the product of the expansion coefficient of the function

$$\frac{1}{\sqrt{\beta-2x}}+\frac{1}{\sqrt{\beta+2x}}-\frac{2}{\sqrt{\beta}}$$

in x for $|x| < \beta/2$ at x^{2k} and the quantity

$$\int_0^\infty dx\, x^{2k}\left(\phi(x)-[\phi(x)]_{2k}\right), \qquad (51)$$

where $[\phi(x)]_{2k}$ is the sum of the first k terms in asymptotic expansion (49).

Equation (50) gives the first three expressions for the expansion coefficients at the function I_3 for powers $\beta^{-1/2-(2k+1)}$ and $\beta^{-1/2-2k}$, $k = 1, 2, 3$ in explicit form. With the same method, all other expansion coefficients can be found from Eq. (50). The coefficients $\beta^{-1/2-(2k+1)}$ can be calculated analytically. The first three coefficients at the powers $3/2$, $7/2$, and $11/2$ are equal to zero. To calculate integrals (51), we use the numerical method. With the accuracy of 10^{-6}, we found that the coefficients at powers $5/2$, $9/2$, and $13/2$ compensate the values estimated in Eq. (47) for I_1. Hence, the first seven expansion coefficients of the function $I_1 + I_3$ are equal to zero. Our conjecture is that the function $I_1 + I_3$ tends exponentially to zero as $\beta \to \infty$. We prove this conjecture below.

In what follows, it is convenient to use the following presentation of the function $I_3(\beta)$:

$$
\begin{aligned}
I_3(\beta) = 2\sqrt{Z}\int_0^\infty dt\, t^2(t^2-\beta)\Bigg\{ & -\frac{4}{(t^2-\beta)^2+1} + \\
& + \sum_{k=0}^\infty \frac{2k+1}{\left[(k+1/2)^2+\left((t^2-\beta)/2\right)^2\right]^2}\Bigg\}. \qquad (52)
\end{aligned}
$$

From Eq. (52), we obtain

$$I_3(\beta) = 2\sqrt{Z}\left(\sum_{k=0}^{\infty} - \int_0^{\infty} dk\right) \times$$
$$\times \int_0^{\infty} dt \frac{2k+1}{(k+1/2)^2 + \left((t^2-\beta)/2\right)^2} =$$
$$= 2\sqrt{Z}\int_0^{\infty} d\rho \left\{2\arctan\left(\frac{1}{\rho^2+\beta}\right) - \right.$$
$$\left. - \pi\left[1 - \text{th}\left(\frac{\pi\left(\rho^2+\beta\right)}{2}\right)\right]\right\}. \quad (53)$$

Simple calculation gives

$$\int_0^{\infty} d\rho \arctan\left(\frac{1}{\rho^2+\beta}\right) = \int_0^{\infty} d\rho \frac{2\rho^2}{(\rho^2+\beta)^2+1} =$$
$$= \pi\left(\beta^2+1\right)^{1/4}\sin\left(\frac{1}{2}\arctan\frac{1}{\beta}\right). \quad (54)$$

The comparison with Eq. (38) leads to the result

$$I_1(\beta) + I_3(\beta) =$$
$$= -2\pi\sqrt{Z}\int_0^{\infty} d\rho \left(1 - \text{th}\left(\frac{\pi(\rho^2+\beta)}{2}\right)\right). \quad (55)$$

Equation (55) means that all expansion coefficients of the function $I_1 + I_3$ in $\beta^{-k-1/2}$ (see Eq. (50)) are equal to zero. We hence obtain exact values for an infinite set of quantities in Eq. (51). As an example, we give the first three such equations:

$$\int_0^{\infty} dx\, x^2 \left(\phi_{(x)} + \frac{1}{3x^2}\right) - \frac{\pi}{24} = 0,$$
$$\int_0^{\infty} dx\, x^4 \left(\phi_{(x)} + \frac{1}{3x^2} - \frac{1}{60x^4}\right) + \frac{\pi}{160} = 0,$$
$$\int_0^{\infty} dx\, x^6 \left(\phi_{(x)} + \frac{1}{3x^2} - \frac{1}{60x^4} + \frac{13}{1008x^6}\right) - \frac{\pi}{896} = 0.$$

From Eq. (55), we obtain

$$I_1(\beta) + I_3(\beta) = 2\pi\sqrt{Z}\sum_{k=1}^{\infty}(-1)^k \frac{\exp(-\pi k\beta)}{\sqrt{k}}. \quad (56)$$

Inserting the function $I_1 + I_3$ into Eq. (42), we obtain the extremal point β_0:

$$\exp(-\pi\beta_0) = \frac{1-R_2}{\pi}\sqrt{Z},$$
$$\beta_0 = \frac{1}{\pi}\ln\frac{\pi}{(1-R_2)\sqrt{Z}}. \quad (57)$$

The final equation for the value in the magnetic field at a critical point in the low-temperature region acquires the form

$$1 = \frac{|V_0| m \left\langle\chi_0^2\right\rangle}{16\pi^2\hbar^2}\left\{4\pi\ln\left(\frac{\sqrt{\gamma_1}\omega_D}{h}\right)^2 + 8\pi\ln 2 - \right.$$
$$\left. - 2(1-R_2)Z\left(1 + \ln\frac{\pi}{(1-R_2)\sqrt{Z}}\right)\right\}, \quad (58)$$

where ω_D is the Debye frequency. We see from Eq. (58) that $T = 0$ is a weakly singular point in the expansion in T in Eq. (23). Subtracting the equation for the critical magnetic field H_{cr} for $T = 0$ from Eq. (58), we obtain

$$\ln\frac{H_{cr}(0)}{H_{cr}(T)} = \frac{1-R_2}{4\pi}Z\left(1 + \ln\frac{\pi}{(1-R_2)\sqrt{Z}}\right). \quad (59)$$

4. CONCLUSIONS

Investigation of the low-temperature behavior of the superconductor with spin–orbit interaction shows that it is of a nontrivial nature. There probably exists an intrinsic symmetry leading to a strong suppression of temperature corrections to the equation of state. It is much smaller than can be expected. In addition, the temperature correction has a weak singularity. Simultaneously, we obtained an infinite set of equations connected by the Euler ψ-function and leading to a suppression of temperature corrections to the equation of state.

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