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QUASI-ISOTROPIC EXPANSION FOR A TWO-FLUID COSMOLOGICAL MODEL CONTAINING RADIATION AND STRING GAS

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The quasi-isotropic solution of the Einstein equations near a cosmological singularity was first found by Lifshitz and Khalatnikov [1] for the Universe filled by radiation with the equation of state $p = \varepsilon/3$ in the early 1960th. In the paper [2], we presented its generalization to the case of an arbitrary one-fluid cosmological model. Then this solution was further generalized to the case of the Universe filled by two ideal barotropic fluids [3].

As is well known, modern cosmology deals with many very different types of matter. In comparison with the old standard model of the hot radiation dominated Universe (dubbed the Big Bang), the situation has been dramatically changed, first, with the development of inflationary cosmological models which contain an inflaton effective scalar field or/and other exotic types of matter as an important ingredient, and second, with the understanding that the main part of the non-relativistic matter in the present Universe is non-baryonic — cold dark matter. Furthermore, the appearance of brane and M-theory cosmological models and the discovery of the cosmic acceleration suggests that matter playing an essential role at different stages of cosmological evolution is multi-component generically, and these components may obey very different equations of state. Moreover, the very notion of the equation of state appears to be not fundamental; it has only a limited range of validity as compared to a more fundamental field-theoretical description. From this general point of view, the generalization of the quasi-isotropic solution to the case of two ideal barotropic fluids with constant but different p/ε ratios seems to be a natural and important next logical step.

To explain the physical sense of the quasi-isotropic solution, let us remind that it represents the most generic spatially inhomogeneous generalization of the Friedmann–Lemaître–Robertson–Walker (FLRW) model in which the space-time is locally FLRW-like near the cosmological singularity $t = 0$ (in particular, its Weyl tensor is much less than its Riemann tensor). On the other hand, generically it is very inhomogeneous globally and may have a very complicated spatial topology. As was shown in [2, 4], such a solution contains three arbitrary functions of spatial coordinates. From the FLRW point of view, these degrees of freedom represent the growing (non-decreasing in terms of metric perturbations) mode of adiabatic perturbations and the non-decreasing mode of gravitational waves (with two polarizations) in the case when deviations of a space-time metric from the FLRW one are not small. So, the quasi-isotropic

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solution is not a generic solution of the Einstein equations with a barotropic fluid. Therefore, one should not expect this solution to arise in the course of generic gravitational collapse (in particular, inside a black hole event horizon). The generic solution near a space-like curvature singularity (for $p < \varepsilon$) has a completely different structure consisting of the infinite sequence of anisotropic vacuum Kasner-like eras with space-dependent Kasner exponents [5–7].

For this reason, the quasi-isotropic solution had not attracted much interest for about twenty years. Its new life began after the development of successful inflationary models (i.e., with "graceful exit" from inflation) and the theory of generation of perturbations during inflation, because it had immediately become clear that generically (without fine tuning of initial conditions) scalar metric perturbations after the end of inflation remain small in a finite (though still very large compared to the presently observable part of the Universe) region of space which is much less than the whole causally connected space volume produced by inflation. It appears that the quasi-isotropic solution can be used for a global description of a part of space-time after inflation which belongs to "one post-inflationary universe". The latter is defined as a connected part of space-time where the hyper-surface $t = t_f(\mathbf{r})$ describing the moment when inflation ends is space-like and, therefore, can be made the surface of constant (zero) synchronous time t by a coordinate transformation. This directly follows from the derivation of perturbations generated during inflation given in [8] which is valid in case of large perturbations, too. Thus, when used in this context, the quasi-isotropic solution represents an intermediate asymptotic regime during expansion of the Universe after inflation. The synchronous time t appearing in it is the proper time since the end of inflation, and the region of validity of the solution is from $t = 0$ up to a moment in future when spatial gradients become important. For sufficiently large scales, the latter moment may be rather late, even of the order or larger than the present age of the Universe. Note also analogues of the quasi-isotropic solution related to the Universe behavior before the end of inflation which are produced by generic globally inhomogeneous late-time asymptotic solutions of the Einstein equations either with a cosmological constant [9] or with a scalar field having the exponential potential (power-law inflation) [10]. These solutions can be smoothly matched across the hypersurface of the end of inflation to a post-inflationary quasi-isotropic solution of the type we are studying (of course, the matter content has to be changed beyond this hypersurface, too).

A slightly different versions of the quasi-isotropic expansion were independently developed during last decades which are known under the names of long-wave expansion, gradient expansion, or the separate universe approach. However, the specific of our approach is that we consider only solutions having the local FLRW behavior near singularity at $t = 0$.

Originally the quasi-isotropic expansion was developed as a technique of generation of some kind of perturbative expansion in the vicinity of cosmological singularity at $t = 0$, where the synchronous time t serves as a small parameter. However, the more general treatment of the quasi-isotropic expansion is possible if one notices that the next order of the quasi-isotropic expansion contains higher orders of spatial derivatives of metric coefficients. Thus, it is possible to construct a natural generalization of the quasi-isotropic solution of the Einstein equations which would be valid not only in the vicinity of cosmological singularity, but in the full time range. In this case simple algebraic equations, which one resolves to find higher orders of the quasi-isotropic approximation in the vicinity of singularity are substituted by differential equations, where the time dependence of the space-time metric can be rather complicated in contrast to the power-law behavior of its coefficients of the original quasi-isotropic expansion.

In the present paper we construct this expansion for a relatively simple two-fluid cosmological model containing radiation and the cosmic string gas (see e.g. [11]). Such a model has a technical advantage: the corresponding Friedmann equation is exactly solvable in terms of the synchronous time t and, hence, t is a natural parameter for constructing the quasi-isotropic solution. In the second section of the paper (full text) we explicitly construct next order terms of the quasi-isotropic expansion for the metric tensor in the synchronous reference frame, energy densities and velocities of two fluids, and determine their asymptotic behavior at early and late times. The last section contains some concluding remarks. In the Appendix we apply the developed formalism to the case of a one-fluid cosmological model. In this case the solutions valid in the full time range coincide with those valid in the vicinity of singularity [2].

Summing up, we can say that we have calculated explicitly the next order terms in the quasi-isotropic solution for the metric tensor,

$$ds^2 = dt^2 - \gamma_{\alpha\beta} dx^\alpha dx^\beta, \tag{1}$$

$$\gamma_{\alpha\beta} = a_{\alpha\beta}(x)(t + b(x)t^2) + c_{\alpha\beta}(x,t), \tag{2}$$

$$c = -\bar{P}\frac{t^2}{2} + \frac{b_{|\alpha}{}^{\alpha}}{b^3}\left(b^2t^2 - bt + 2bt(1+bt)\ln(bt+1) + (2bt+1)\ln(2bt+1) - \right.$$
$$\left. - (2bt+1)\sqrt{b^2t^2+bt}\,\mathrm{Arch}(2bt+1) + \frac{1}{4}\mathrm{Arch}^2(2bt+1)\right) + \frac{(2bt+1)b_{,\alpha}b^{\alpha}_{,}}{b^4}\left(\frac{-19b^2t^2-85bt}{24(2bt+1)} + \frac{4(bt+1)\ln(bt+1)}{2bt+1} - \right.$$
$$- \frac{9\ln(bt+1)}{2} + \frac{61\ln(2bt+1)}{48} - \frac{3\,\mathrm{Arch}^2(2bt+1)}{4(2bt+1)} + \frac{2\,\mathrm{Arch}(2bt+1)\sqrt{b^2t^2+bt}}{2bt+1} + \frac{1}{8}\mathrm{Arch}^2(2bt+1)\ln\frac{bt}{bt+1} -$$
$$- \mathrm{Arch}(2bt+1)\left(\mathrm{Li}_2(e^{-\mathrm{Arch}(2bt+1)}) - \frac{1}{4}\mathrm{Li}_2(e^{-2\,\mathrm{Arch}(2bt+1)})\right) -$$
$$\left. - \left(\mathrm{Li}_3(e^{-\mathrm{Arch}(2bt+1)}) - \frac{1}{8}\mathrm{Li}_3(e^{-2\,\mathrm{Arch}(2bt+1)})\right) + \frac{7}{8}\zeta_R(3)\right), \quad (3)$$

$$\tilde{c}_{\alpha\beta} = \left(\frac{bt^2+t}{b} - \frac{(2bt+1)\sqrt{b^2t^2+bt}}{2b^2}\mathrm{Arch}(2bt+1)\right)\tilde{\bar{P}}_{\alpha\beta} +$$
$$+ \frac{bt^2+t}{b^2}\left(8\ln(1+bt) + 6 - \frac{3(2bt+1)\,\mathrm{Arch}(2bt+1)}{\sqrt{b^2t^2+bt}}\right)\left(b_{;\alpha\beta} - \frac{1}{3}a_{\alpha\beta}b^{\gamma}_{;\gamma}\right) +$$
$$+ \frac{bt^2+t}{b^3}\left(-\frac{2bt}{1+bt} - 16\ln(1+bt) + \zeta_R(3) - \mathrm{Li}_3(e^{-2\,\mathrm{Arch}(2bt+1)}) - 2\,\mathrm{Arch}^2(2bt+1)\mathrm{Li}_2(e^{-2\,\mathrm{Arch}(2bt+1)}) - \right.$$
$$- \frac{4}{3}\mathrm{Arch}^3(2bt+1) - \frac{(2bt+1)\,\mathrm{Arch}^3(2bt+1)}{3\sqrt{b^2t^2+bt}} + \mathrm{Arch}^2(2bt+1)\ln(bt) + 4\,\mathrm{Arch}^2(2bt+1)\ln 2 +$$
$$\left. + \mathrm{Arch}^2(2bt+1)\ln(1+bt) + \frac{\mathrm{Arch}^2(2bt+1)}{2(b^2t^2+bt)} + \frac{29(2bt+1)\,\mathrm{Arch}(2bt+1)}{4\sqrt{b^2t^2+bt}} - \frac{33}{2}\right)\left(b_{,\alpha}b_{,\beta} - \frac{1}{3}a_{\alpha\beta}b_{,\gamma}b^{\gamma}_{,}\right), \quad (4)$$

and for energy-densities

$$\varepsilon_R^{(1)} = \frac{\bar{P}t^2}{4(bt^2+t)^3} + \frac{b^{\alpha}_{|\alpha}}{2(b^2t^2+bt)^3}\left(-b^2t^2 + bt - bt(1+bt)\ln(1+bt) - (1+2bt)\ln(1+2bt) + \right.$$
$$\left. + (1+2bt)\sqrt{b^2t^2+bt}\,\mathrm{Arch}(2bt+1) - \frac{1}{4}(2b^2t^2+2bt+1)\,\mathrm{Arch}^2(2bt+1)\right) + \frac{b_{,\alpha}b^{\alpha}_{,}}{b^4(bt^2+t)^3}\times$$
$$\times\left(\frac{115b^2t^2+85bt}{48} + \left(\frac{5}{2}bt+\frac{1}{4}\right)\ln(1+bt) - \frac{192b^2t^2+318bt+61}{96}\ln(1+2bt) - \frac{5}{2}bt\sqrt{b^2t^2+bt}\,\mathrm{Arch}(2bt+1) - \right.$$
$$- \frac{2bt+1}{16}\mathrm{Arch}^2(2bt+1)\ln\frac{bt}{1+bt} + \frac{3(2b^2t^2+2bt+1)}{8}\mathrm{Arch}^2(2bt+1) + \frac{2bt+1}{2}\mathrm{Arch}(2bt+1)\times$$
$$\left.\times\left(\mathrm{Li}_2(e^{-\mathrm{Arch}(2bt+1)}) - \frac{1}{4}\mathrm{Li}_2(e^{-2\,\mathrm{Arch}(2bt+1)})\right) + \left(\mathrm{Li}_3(e^{-\mathrm{Arch}(2bt+1)}) - \frac{1}{8}\mathrm{Li}_3(e^{-2\,\mathrm{Arch}(2bt+1)})\right) - \frac{7}{8}\zeta_R(3)\right), \quad (5)$$

$$\varepsilon_S^{(1)} = \frac{\bar{P}bt^2}{2(bt^2+t)^2} + \frac{b^{\alpha}_{|\alpha}}{b^2(bt^2+t)^2}\left(b^2t^2 + 3bt - (1+2bt)\ln(1+2bt) - \frac{1}{4}\mathrm{Arch}^2(2bt+1)\right) + \frac{b_{,\alpha}b^{\alpha}_{,}}{b^3(bt^2+t)^2}\times$$
$$\times\left(\frac{18b^2t^2+37bt-6}{24} + \left(5bt+\frac{1}{2}\right)\ln(1+bt) - \frac{61(1+2bt)\ln(1+2bt)}{48} - \frac{1+2bt}{8}\mathrm{Arch}(2bt+1)\ln\frac{bt}{1+bt} + \right.$$
$$+ \frac{1-4bt-8b^2t^2}{4\sqrt{b^2t^2+bt}}\mathrm{Arch}(2bt+1) + \frac{12b^2t^2+12bt-1}{16b(bt^2+t)}\mathrm{Arch}^2(2bt+1) + (1+2bt)\,\mathrm{Arch}(2bt+1)\times$$
$$\times\left(\mathrm{Li}_2(e^{-\mathrm{Arch}(2bt+1)}) - \frac{1}{4}\mathrm{Li}_2(e^{-2\,\mathrm{Arch}(2bt+1)})\right) + (1+2bt)\left(\mathrm{Li}_3(e^{-\mathrm{Arch}(2bt+1)}) - \frac{1}{8}\mathrm{Li}_3(e^{-2\,\mathrm{Arch}(2bt+1)})\right) -$$
$$\left. - \frac{7(1+2bt)}{8}\zeta_R(3)\right), \quad (6)$$

and velocities

$$v_R = -\frac{\sqrt{bt^2+t}}{2b^{3/2}}\,\mathrm{Arch}(2bt+1)+\frac{t}{b}, \qquad (7)$$

$$v_S = -\frac{1}{2b^2}+\frac{1}{4b^{5/2}\sqrt{bt^2+t}}\,\mathrm{Arch}(2bt+1) \qquad (8)$$

of two fluids. Also we have found their asymptotic behavior for small and large values of the cosmic synchronous time t. As was easily predictable, the structure of the solution for small t is determined only by the radiation component, and it coincides with that found in the original paper [1]. However, the late time behavior of the metric tensor reveals an unusual feature: the anisotropic part of the metric grows essentially faster than the isotropic one and their ratio is $\propto \ln bt$. It seems that this effect is due to the two-fluid character of the model considered in this paper and to the specific property of the string gas equation of state which leads to the singular character of the quasi-isotropic expansion for the string gas alone.

It is not clear if the appearance of terms having non-power-law behavior is present in two-fluid quasi-isotropic models with other equations of state. To answer this question it is necessary to develop the formalism of building the quasi-isotropic expansion valid for the full range of time for arbitrary two-fluid models that is much more complicated technically.

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