ЖЭТФ, 2019, том 156, вып. 4 (10), стр. 667–670

CLASSICAL AND QUANTUM INTEGRABLE SIGMA MODELS. RICCI FLOW, "NICE DUALITY" AND PERTURBED RATIONAL CONFORMAL FIELD THEORIES

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Received March 10, 2019,
revised version March 10, 2019
Accepted for publication March 12, 2019

Contribution for the JETP special issue in honor of I. M. Khalatnikov's 100th anniversary

DOI: 10.1134/S0044451019100080

Duality plays an important role in the analysis of statistical, quantum field, and string theory systems. Usually, it maps a weak coupling region of one theory to the strong coupling region of the other and makes it possible to use perturbative, semiclassical, and renormalization group methods in different regions of the coupling constant. For example, the well known duality between Sine-Gordon and massive Thirring models [1, 2] together with integrability plays an important role for the justification of exact scattering matrix [3] in these theories. Another well known example of the duality in two dimensional integrable systems is the weak-strong coupling flow from affine Toda theories to the same theories with dual affine Lie algebra [4–6]. The phenomenon of electric-magnetic duality in four dimensional $N = 4$ supersymmetric gauge theories conjectured in [7,8] and developed for $N = 2$ theories in [9] (and in many subsequent papers) opens the possibility for the non-perturbative analysis of the spectrum and phase structure in supersymmetric gauge field theories. The remarkable field/string duality [10,11] leads to the unification of the ideas and methods for the analysis of these seemingly different quantum systems.

While known for many years the phenomenon of duality in quantum field theory still looks rather mysterious and needs to be further analyzed. Such analysis crucially simplifies for two-dimensional integrable relativistic systems. These theories besides the Lagrangian formulation possess also an unambiguous definition in terms of factorized scattering theory, which contains all information about off- shell data of quantum theory. These data allow the use of non-perturbative methods for the calculation of observables in integrable field theories. The comparison of the observables calculated from the scattering data and from the perturbative, semiclassical or renormalization group analysis based on the Lagrangian formulation makes it possible in some cases to justify the existence of two different (dual) Lagrangian representations of a quantum theory.

The two particle factorized scattering matrix is a rather rigid object. It is constrained by the global symmetries, factorization equation and unitarity and crossing symmetry relations. After solving of these equations the scattering matrix S can contain one (or more) free parameter. At some value of this parameter $\lambda = \lambda_0$ the scattering matrix $S(\lambda_0)$ becomes the identity matrix and has a regular expansion around this point. In many cases this expansion can be associated with the perturbative expansion of some Lagrangian theory with parameter b near some free point. Sometimes there is a second point $\lambda = \lambda_1$ where $S(\lambda)$ reduces to identity matrix and admits a regular expansion in $(\lambda - \lambda_1)$. If this expansion can be associated with the perturbative expansion of another local Lagrangian at small coupling $\gamma = \gamma(b)$, then the two different Lagrangians describe

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the same theory, which has two different (dual) perturbative regimes.

A more interesting situation occurs when $S(\lambda)$ has a regular expansion in $(\lambda - \lambda_0)$ which agrees with perturbative expansion in b of some field theory with local action $\mathcal{A}(b)$, but at the point λ_1 the S-matrix tends to some "rational" scattering matrix corresponding to the S-matrix of a non-linear sigma model on a symmetric space. Near the point λ_1 it can be considered as a deformation of a symmetric scattering. In this case it is natural to search for the dual theory as sigma model with target space looking as a deformed symmetric space. The metric and other characteristics of sigma model on the manifold is subject to very rigid conditions, namely non-linear renormalization group (RG) equations [12]. If one has found the solution of RG equations which gives the observables in the sigma model theory, coinciding with those derived from the factorized S-matrix theory one can conclude that field theory with the action $\mathcal{A}(b)$ is dual to a sigma model on the deformed symmetric space. The short distance behaviour of such theory can be studied by RG and conformal field theory (CFT) methods. The agreement of the CFT data, derived from the action $\mathcal{A}(b)$ (considered as a perturbed CFT) with the data derived from RG for sigma model gives an additional important test for the duality ("nice" duality).

The analysis of integrable quantum SMs on the deformed symmetric spaces and their dualities started in the papers [13–15]. Later in the papers [16, 17] the general classical SMs on the deformed groups and cosets manifolds have been constructed. Unfortunately, not all these SMs, integrable classically, are integrable in quantum case. In particular it happens for the cosets having $U(1)$ group in denominator (see for example [18]). In many cases this situation can be improved by introduction of additional quantum degrees of freedom, which are invisible in the classical limit.

We say that an integrable SM has the "nice" duality if the dual integrable QFT has the weak coupling region. This implies that we can study this theory by different methods (perturbation theory, RG and CFT analysis) in different regimes. Note that the SMs with the property of "nice" duality form a very small subspace in the space of all integrable quantum SMs on the deformed symmetric spaces and such SMs with additional quantum degrees freedom.

A simple (but rather non-trivial) example is provided by the $CP(n-1)$ SM. This model is integrable classically but non-integrable (for $n > 2$) at the quantum level. After adding the massless fermion interacting with $U(1)$ gauge field on $CP(n-1)$ (massless axion, linearly coupled with the density of topological charge), this SM becomes integrable and its deformed version has the "nice" duality property. We study this theory in the main part of this paper. Here, we say a few words about non-integrable $CP(n-1)$ SMs.

These models were intensively studied during 70-80s due to their's similarity with four-dimensional $SU(n)$ gauge QFTs. Namely, the $CP(n-1)$ SMs and $SU(n)$ gauge theories are asymptotically free, possess instantons, and manifest the phenomenon of confinement[1]. It is natural that $CP(n-1)$ SMs served as baby-laboratory for the analysis of $SU(n)$ gauge theory, in particular, for analysis of instanton contributions [19] and lattice simulations [20].

The spectrum of the $CP(n-1)$ SMs in $1/n$ approach was studied in [21]. It was shown that besides the basic particles, which are the only particles in the integrable version of this model, one has also the particles which are their bound states, confined by Coulomb forces. The addition of fermion (axion) produces an essential restructure of the spectrum. Of course, the influence of the axion on the spectrum of gauge theory is a more interesting and much more complicated problem.

The spectrum of the deformed non-integrable $CP(n-1)$ SMs seems to be qualitatively the same as that of the undeformed models and can not be studied by perturbative methods. Due to the "nice" duality, in the integrable $CP(n-1)$ SMs with axion there is a weak coupling region. In this region the basic particles also form the bound state, which disappears from the spectrum outside the perturbative region. It is possible however that in a non-integrable QFT they survive in the strong coupling (SM) regime. We hope to return to this problem in a future publication.

This main paper (see JETP **129**, № 10) is organized as follows. In section 2, we describe the basic CFTs, which can be formulated in terms of $2n-1$ bosonic fields, and their primary fields are the exponents of these fields. We calculate the reflection amplitudes in these CFTs which are important for the calculation of UV asymptotics in perturbed CFTs. These amplitudes serve also for identification of CFTs in different representations. In particular, for justification of dual SM representations.

In section 3, we explain the general properties of deformed $CP(n-1)$ SMs with fermion and write the action of perturbed CFTs, constructed in section 2. We

[1] For $n > 2$, both theories have non-topological classical solutions, the role of which is not clear at the moment.

conjecture that these QFTs provide a dual description of deformed $CP\,(n-1)$ SMs with fermion.

In section 4, we represent the action of dual QFT in the form suitable for the perturbation theory in parameter b. We provide non-local integrals of motion which form the Borel subalgebra of $SU(n)_{\rm q}$ and generate $SU(n)_{\rm q}$ symmetry of the scattering theory. We describe the spectrum and scattering theory of this QFT.

In section 5, we use the Bethe Ansatz approach to derive the exact relations between the parameters of action and scattering theory in this QFT. We calculate the observables, which can be compared with the observables calculated using the dual SM description of our QFT.

In section 6, we consider classical and quantum integrable SMs on the deformed symmetric spaces. We discuss Ricci flows in these SMs and the relation between the parameters of integrable SMs with the parameters of their scattering theory. We calculate the observables in integrable deformed $CP\,(n-1)$ SMs and show that in the scaling (one loop) approximation they coincide with the observables calculated in the Bethe Ansatz approach.

In section 7, we consider the conformal limit of the deformed $CP\,(n-1)$ SMs. For simplicity of equations we consider the case $n=3$. We calculate the reflection amplitudes associated with this conformal SM and show that they coincide with reflection amplitudes calculated in section 2, i. e., these CFTs are dual. We use these amplitudes to get the UV asymptotics of effective central charge in the deformed $CP\,(n-1)$ SMs on the circle of length R.

In section 8, we study the second integrable perturbation of CFTs described in section 2. We study the scattering theory of these QFTs and see that at small and large values of the coupling constant b they can be studied by perturbation theory in b and $1/b$, i. e., it has two different dual representations. In the strong coupling regime they can be described by the action with the SM part coinciding with the conformal limit of the deformed $CP\,(n-1)$ SMs and the potential part described by tachyon. After a simple analytical continuation in the coupling constant the SM part of the actions become singular. These actions describe integrable classical models but after the quantization these QFTs are well defined only for discrete values of coupling constant.

In section 9, we discuss the rational CFTs, which are closely related with conformal SMs with singular actions and the discrete values of coupling constant. We show that these CFTs perturbed by proper fields describe non-integrable deformed $CP\,(n-1)$ SMs with topological parameter and integrable deformed $CP\,(n-1)$ with fermion (axion). We consider more general CFTs represented by the cosets $G_m \times G_l / G_{m+l}$ and study their deformations by different fields in different regions of integers m, l, and h (Coxeter number of G). We study their RG propertied and show that these QFTs provide an independent description of a large variety of SMs on the deformed symmetric spaces. Finally, we give a simple inequality which provides a necessary condition for a sigma model to have "nice" duality.

The full text of this paper is published in the English version of JETP.

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