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KINETIC PROCESSES IN FERMI–LUTTINGER LIQUIDS

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Abstract. The present work discusses extensions of the pioneering analysis by Dzyaloshinskii and Larkin [Sov. Phys. JETP **38**, 202 (1974)] of correlation functions for one-dimensional Fermi systems, focusing on the effects of quasiparticle relaxation enabled by nonlinear dispersion. Throughout the work we employ both, the weakly interacting Fermi gas picture and nonlinear Luttinger liquid model to describe attenuation of excitations and explore the fermion-boson duality between both approaches. A special attention is devoted to the role of spin-exchange processes, effects of interaction screening, and integrability. Thermalization rates for electron- and hole-like quasiparticles, as well as the decay rate of collective plasmon excitations and the momentum space mobility of spin excitations are calculated for various temperature regimes. The phenomenon of spin-charge drag is considered and the corresponding momentum transfer rate is determined. In the context of transport properties, momentum relaxation due to several competing mechanism, viz. triple electron collisions, electron-phonon scattering, and long-range inhomogeneities is addressed. Energy transfer facilitated by plasmons is highlighted from the perspective of inhomogeneous Luttinger liquid. The full matrix of thermoelectric coefficients is found at the quantum critical point of the first conductance plateau transition.

1. Introduction. The concept of quasiparticles plays a central role in the condensed matter physics of strongly interacting many-body quantum systems [1, 2]. For instance, in the context of electrons in conductors, one typically views the quasiparticle states as those evolving from the free electron gas to a Fermi liquid when adiabatically turning on the interaction. In accordance with Landau theory [3], quasiparticles inherit some of the basic quantum numbers of bare electrons such as spin, charge, and momentum. Their respective dispersion relations as well as thermodynamical and kinetic properties may, however, differ significantly due to interaction-induced renormalizations. A crucial advantage of the quasiparticle picture is that residual interactions are assumed to be weak, and can be systematically and controllably addressed by means of perturbation theory. The central question related to the validity of the quasiparticle description concerns their lifetime τ_{qp}. Indeed, in the process of scattering quasiparticles decay and their mere notion is meaningful only if attenuation is weak enough and they can be considered as sufficiently long-lived collective excitations. In Fermi systems, the Pauli principle severely limits the phase space available for quasiparticle collisions. The low temperature decay rate can then be estimated from the Golden rule as

$$\tau_{qp}^{-1}(\varepsilon, T) \propto (\nu V_0)^2 \frac{\varepsilon^2 + \pi^2 T^2}{\varepsilon_F}. \tag{1}$$

In this expression, the excitation energy $\varepsilon = v_F(p - p_F)$ of a quasiparticle with momentum p is counted from the Fermi energy ε_F, ν is the density of states and V_0 is the characteristic strength of the short-range re-

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pulsive interaction[1]. The dominant microscopic scattering channel leading to Eq. (1) involves quasiparticle decaying into three: another quasiparticle and a particle-hole excitation. The amplitude for this process is proportional to V_0, hence, the dimensionless factor of $(\nu V_0)^2$ in the scattering probability entering Eq. (1). The factor ε^2 is the phase space volume for scattering of a quasiparticle with energy ε compatible with the conservation of total energy and momentum. At finite temperatures the smearing of states in the energy strip of order $\sim T$ per particle leads to the corresponding T^2 dependence of τ_{qp}^{-1}. Higher-order processes involving $2n+1$ quasiparticles, namely $n>1$ electron-hole pairs, are usually neglected as their respective rate scales with higher powers of energy. In particular, at zero-temperature the rate for relaxation processes of a quasiparticle with energy ε involving n particle-hole pairs vanishes as $\tau_{qp}^{-1} \propto \varepsilon^{2n}$. One notable property of Eq. (1) is that it predicts the same relaxation time for particle-like and hole-like excitations. Another property is that the crossover from zero-temperature to finite-temperature relaxation is governed only by one scale, viz. when the excitation energy compares to the temperature itself $\varepsilon \sim T$.

In addition to the quasiparticle relaxation, which is often viewed as an out-scattering rate from a particular quantum state, one may address a more general question of relaxation of a nonequilibrium quasiparticle distribution function. In kinetic theory such problem is typically analyzed in the framework of the linearized Boltzmann equation. The eigenvalues of the corresponding collision operator define relaxation times of different distribution function modes. In three dimensional Fermi liquids this problem is exactly solvable [4, 5] and one finds that all these rates are parametrically the same, scaling respectively as $\propto T^2$. In contrast, in two-dimensional Fermi liquids, kinematics of head-on collisions leads to a parametrically distinct relaxation of odd and even momentum harmonics of the distribution function, in particular $\tau_{even}^{-1} \propto T^2/\varepsilon_F$ while $\tau_{odd}^{-1} \propto T^4/\varepsilon_F^3$ [6, 7].

The role of dimensionality in quasiparticle relaxation becomes the most dramatic in one-dimension (1D). This special case of electron liquids can be experimentally realized in quantum wires of GaAs/AlGaAs heterostructure [8] or carbon nanotubes [9] when particle density is such that only the lowest sub-band of transversal modes is occupied. It further requires that temperature is sufficiently low and sample purity is sufficiently high, so that thermally- and disorder-induced transitions to higher sub-bands are suppressed. In addition, edge modes formed at the boundaries of a 2D electron gas when placed in a strong magnetic field in the integer or fractional quantum Hall regime [10, 11], or edge states of 2D quantum spin Hall topological-insulators [12], provide other distinct examples of, respectively, chiral and helical quantum 1D electron liquids.

In principle, all these systems can be successfully described within the framework of Luttinger liquid theory [13–15], which builds out of the Tomonaga–Luttinger (TL) model [16, 17]. As is known form pioneering works [18–20], in the asymptotic low-energy limit $\varepsilon/\varepsilon_F \ll 1$, the key properties of the TL model are manifestly non-Fermi liquid like. A power-law anomaly manifests in the suppression of the single particle density of states

$$\nu(\varepsilon) = \nu_0 \left(\frac{|\varepsilon|}{v_F p_\Lambda}\right)^{2g} \frac{\sin(\pi g)}{\pi g}\Gamma(1-2g), \qquad (2)$$

and collapse of the quasiparticle residue in the distribution function. At $T \to 0$ that is

$$n(\varepsilon) = \frac{\Gamma(1/2+g)}{2\sqrt{\pi}\Gamma(1+g)} \times \\ \times \left[1 - \frac{\Gamma(1/2-g)}{\Gamma(1/2+g)} \left(\frac{|\varepsilon|}{v_F p_\Lambda}\right)^{2g} \mathrm{sgn}(\varepsilon)\right], \qquad (3)$$

where $\nu_0 = 1/(2\pi v_F)$, $\Gamma(z)$ is the Euler's gamma function, and p_Λ is the momentum cutoff of the model (parametrically $p_\Lambda \sim p_F$). In the simplest spinless version of the TL-model with short-ranged interaction, a single dimensionless coupling constant,

$$g = \frac{1}{2}\left[\frac{1+\nu_0 V_0}{\sqrt{1+2\nu_0 V_0}} - 1\right], \qquad (4)$$

can be related to the zero-momentum Fourier component of the bare interaction potential V_0. The limit of weak interaction corresponds to $g \ll 1$ and Eqs. (2), (3) are valid for $g < 1/2$[2]. However, a direct attempt to apply Luttinger liquid theory to the question of quasiparticle lifetime meets formidable challenges. In a fermionic representation of the TL-model, elaborated explicitly by Dzyaloshinskii and Larkin [19], the electron self-energy vanishes on the mass shell in all orders of perturbation theory and, consequently, correlation functions assume power-law tails. These results,

[1] Throughout the paper we use units with Planck and Boltzmann constants set to unity $\hbar = k_B = 1$.

[2] In Ref. [19] the limit of strong interactions, $g > 1/2$, was also considered, including the scenario when coupling between fermions of the same chirality is different from coupling between fermions of different chirality. For additional details on the derivation of Eq. (3) see also Ref. [21].

and the absence of relaxation, can be alternatively understood from the Mattis and Lieb [18], and Luther and Peschel [20] bosonization construction, which maps interacting 1D fermions to a collection of decoupled harmonic modes of charge-density and spin-density oscillations. Notably, in both approaches the exact solution relies heavily on the linearization of the fermionic dispersion relation.

One is then left with the natural puzzle whether incorporating curvature of the dispersion relation into the TL-model would cure the issue and yield a finite lifetime of excitations, thus possibly restoring Fermi liquid like properties of the system. This line of reasoning can be also corroborated within the fermionic picture, noting that spectrum nonlinearity softens phase space restrictions for quasiparticle scattering, thus making their relaxation possible.

Similarly, at the level of the bosonic description, nonlinear terms of the dispersion relation couple charge and spin modes thus enabling their decay. However, it was quickly recognized that curvature cannot be included perturbatively, and a naive expansion leads to spurious divergences. These and other related questions to 1D kinetics, including the connection between the two pictures of the fermion-boson duality, attracted significant recent interest. This has lead to the development of the nonlinear Luttinger liquid theory, also referred to as Fermi–Luttinger liquid (FLL) theory (see Refs. [22, 23] for comprehensive reviews and references herein). Specifically for the problem of quasiparticle relaxation in quantum wires, various scattering rates were calculated within different interaction models for both, spinless [24–33] and spin-1/2 fermions [34–40]. In parts of the present work we review and extend these results.

On the experimental forefront the hallmark signatures of Luttinger liquid behavior have been observed by means of various spectroscopic techniques. Namely, power-law anomalies in the density of states, tunneling conductance, and current-voltage characteristics [9, 10, 41, 42], spin-charge separation [43, 44], and charge fractionalization [45, 46]. Besides GaAs quantum wires, carbon nanotubes, and edge modes, clear features of Luttinger liquid physics have been identified in many other systems such as bundles of $NbSe_3$ [47] and MoSe [48] nanowires, polymer nanofibers [49] and conjugated polymers at high carrier densities [50], as well as atomically controlled chains of gold atoms on Ge surfaces [51], just to name a few distinct examples. In the most recent report [52], relaxation processes in quantum wires were captured and bounds on the corresponding timescales were determined, thus providing measurements of quasiparticle properties beyond the parading of linear Luttinger liquid theory. In a parallel line of developments [53–56], cooling of nonequilibrium quasiparticles in quantum Hall edge fluids was measured and corresponding lengths scales of thermalization processes were quantified.

The focus of this communication is on the description of elementary kinetic processes inducing relaxation in nonlinear Luttinger liquids and their emergent transport properties. Keeping forward scattering electron-electron interactions and accounting for nonlinear contributions to the electron dispersion, this theory is beyond the Dzyaloshinskii–Larkin theorem. The latter relax kinematic constraints and open phase space for multi-loop corrections to the electron self-energy, thereby providing a variety of inelastic processes which affect equilibrium as well as nonequilibrium properties of the 1D quantum electron liquids. The rest of this work is structured as follows (see the full text). Section 2 focuses on the hierarchy of relaxation times in Fermi–Luttinger liquids. We present results beyond parametric estimates, including detailed computations of a number of experimentally relevant interaction models[3)]. The complimentary kinetic equation approach, applied to the quasiparticle picture of a weakly interacting Fermi gas, and spin- and charge-excitations of a Luttinger liquid, are explored concurrently. We present numerical estimates for experimentally measured relaxation rates and provide detailed comparison to previous results. In Sec. 3, the temperature dependence of kinetic coefficients is calculated, accounting for extrinsic mechanisms of momentum relaxation due to phonons or long-range inhomogeneities. The contribution to heat transport mediated by plasmons in the inhomogeneous Luttinger liquid is elucidated. Finally, we devote parts of the discussion to the thermoelectric properties at the first plateau transition of the quantum conductance. In Sec. 4, we provide a summary of main findings and open questions, sketching a broader picture and commenting on related topics relevant for chiral, helical, and spiral versions of 1D quantum fluids. Several Appendices accompany our presentation in the main text, providing additional technical details of the presented analysis and formalism.

2. Hierarchy of relaxation processes. The physics of quasiparticle relaxation in 1D quantum electron liquids is perhaps a surprisingly rich and complicated problem. In part this has to do with the fact that, in contrast to their higher dimensional counter-

[3)] In part this material was summarized in Sec. IV of the extensive review in Ref. [23].

parts, two-particle collisions, namely scattering processes with the emission of a single particle-hole excitation, do not result in finite relaxation rates. This statement pertains to generic dispersion relations, i. e. including curvature, and not only applies to models with linear dispersion. Indeed, kinematics of two-particle scattering in 1D is such that particles either keep or swap their momenta, but neither of these options causes relaxation. To allow for the redistribution of momenta and, at the same time, to comply with restrictions of conservations laws one necessarily needs to consider triple electron collisions, or alternatively, assume some extrinsic mechanism.

The analysis of 1D kinematics of multi-particle collisions resolving energy and momentum conservations reveals a plethora of possible scattering events. They ultimately lead to a hierarchy of relaxation stages in the system and an emergent asymmetry between the relaxation of particle-like and hole-like excitations. All processes can be broken down into several distinct classes. First are the forward scattering processes with soft momentum transfer that involve either (i) all particles from the same branch, or (ii) particles from both branches such that all initial and final states are near the Fermi energy. Second are processes involving states deeper in the band. These latter are relevant for (iii) the drift of quasiholes and (iv) backscattering processes that change the number of right and left moving excitations before and after the collision. We will refer to thermalization when discussing relaxation processes that proceed without backscattering. These processes determine the lifetime of quasiparticles associated to the redistribution of excess energy, and affect thermal transport properties of the system. In contrast, the notion of equilibration will be used to refer to relaxation processes involving the backscattering of quasiparticles, which ultimately govern electrical transport properties.

2.1. Quasiparticle interaction model. In the picture of a weakly nonideal Fermi gas, the probabilities of particle collisions can be calculated perturbatively in the interaction, employing the usual $\hat{T}$-matrix formalism [57]. Within the Golden Rule, the scattering rate

$$W = 2\pi|A|^2\delta(E-E')\delta_{P,P'} \tag{5}$$

is expressed in terms of the scattering amplitude A of the corresponding process. Here $E(E')$ and $P(P')$ label total energy and momentum of initial (final) states, and the delta function $\delta(E-E')$ along with the Kronecker delta $\delta_{P,P'}$ enforce energy and momentum conservations. In the semiclassical limit, the three-particle amplitude A was considered in Ref. [58]. The generalization to the degenerate quantum limit was presented in the work of Ref. [59], and exchange terms were carefully examined in Refs. [35, 60]. The resulting amplitude takes the form

$$A = \frac{1}{L^2}\sum_{\mathbb{P}\mathbb{P}'}\operatorname{sgn}(\mathbb{P})\operatorname{sgn}(\mathbb{P}')\times \\ \times\frac{V_{p'_a-p_a}V_{p'_c-p_c}}{\varepsilon_{p_b}+\varepsilon_{p_c}-\varepsilon_{p_b+p_c-p'_c}}\,\Xi_{\sigma\sigma'}. \tag{6}$$

Here L is the system size and sums are over all possible permutations $\mathbb{P}$ of momenta p_i with $i=1,\,2,\,3$ starting from the direct scattering process $(p_1,p_2,p_3)\to$ $\to\ (p'_1,p'_2,p'_3)$ to all its exchange processes, with $\operatorname{sgn}(\mathbb{P})$ accounting for the sign of the particular permutation (using the convention that $\operatorname{sgn}(123)=+1$). Each permutation comes with a spin-dependent factor $\Xi_{\sigma\sigma'}=\delta_{\sigma_a\sigma'_a}\delta_{\sigma_b\sigma'_b}\delta_{\sigma_c\sigma'_c}$ reflecting particle exchange. In the spinless case, the amplitude has an identical structure to Eq. (6) with $\Xi_{\sigma\sigma'}\equiv 1$. The amplitude consists of 36 distinct terms that can be split into groups of 6, each representing one direct and five exchange scattering processes, respectively. Technically speaking Eq. (6) appears from the iteration of the $\hat{T}$-matrix, $\hat{T}=V+V\hat{G}_0\hat{T}$, to second order in the bare two-particle interaction potential V. Here $\hat{G}_0$ is the resolvent operator (viz. the free particle Green's function) and ε_p denotes the energy-momentum dispersion relation.

For practical applications to quasiparticle scattering in quantum wires, it is sufficient to assume the simple dispersion of a parabolic band $\varepsilon_p=p^2/2m^*$ with effective mass m^*, and use a Coulomb interaction potential. Effects of screening due to nearby gates can be modeled by a conducting plate placed at a distance d away from the wire. In this case the interaction potential is of the form

$$V(x)=\frac{e^2}{\varkappa}\left[\frac{1}{|x|}-\frac{1}{\sqrt{x^2+4d^2}}\right], \tag{7}$$

where $\varkappa$ is the dielectric constant of the host material. The diverging short-range behavior of this potential needs to be regularized in order to evaluate the small-momentum Fourier components V_p entering the amplitude in Eq. (6). To this end, we introduce the small width w of the quantum wire, $w\ll d$, and replace $1/|x|\to 1/\sqrt{x^2+4w^2}$. Upon 1D Fourier transform we then find

$$V_p=\frac{2e^2}{\varkappa}\left[K_0(2w|p|)-K_0(2d|p|)\right], \tag{8}$$

where $K_0(z)$ is the modified Bessel function of the second kind. Using the asymptotic expression of the Bessel function at $z \ll 1$,

$$K_0(z) \approx \ln\left(\frac{2}{ze^{\gamma_E}}\right) + \frac{z^2}{4}\ln\left(\frac{2}{ze^{\gamma_E-1}}\right),$$

with γ_E the Euler constant, one then finds the simplified form of the interaction potential

$$V_p \approx \frac{2e^2}{\varkappa}\left[\ln\left(\frac{d}{w}\right) - (pd)^2\ln\left(\frac{e^{1-\gamma_E}}{|p|d}\right)\right], \quad (9)$$

applicable to the screened limit of Coulomb interaction and valid for $p \ll 1/d$. In the opposite regime, $d^{-1} \ll p \ll w^{-1}$, the second term in Eq. (8) can be neglected since $K_0(z) \propto e^{-z}/\sqrt{z}$ at $z \gg 1$. One then arrives at the simplified form of the unscreened potential

$$V_p \approx \frac{2e^2}{\varkappa} \times \times \left[\ln\left(\frac{e^{-\gamma_E}}{|p|w}\right) + (pw)^2\ln\left(\frac{e^{1-\gamma_E}}{|p|w}\right)\right]. \quad (10)$$

A few comments are in order in relation to the interaction model presented in this section. (i) It should be noted that retaining numerical pre-factors of the order of unity under the logarithm in above expressions for V_p would exceed the accuracy of further calculations, so they will be dropped and simply set to unity. (ii) However, retaining the sub-leading corrections containing p^2 in the main log-series expansion of both Eqs. (9) and (10) is actually crucial. Indeed, in the spinless case, the model with contact interaction as well as the Calogero–Sutherland model, are known to be completely integrable [61]. This implies that all irreducible multi-particle scattering amplitudes must vanish identically for a constant V_p and $V_p \propto |p|$. Furthermore, the extended model of short-ranged interaction, $V_p \propto p^2$, corresponding to the real space potential $V(x) \propto \delta''(x)$, is also integrable. This is known as Cheon–Shigehara model [62]. It is only due to the additional logarithm $\propto p^2\ln|p|$ in Eq. (9), that there is partial non-cancellation between different terms in Eq. (6) and the amplitude remains finite. (iii) In the model of long-ranged Coulomb interaction the situation is more subtle. A priori this model is not known to be integrable. Nevertheless, the amplitude in Eq. (6) vanishes for pure logarithmic interaction $V_p \propto \ln|p|$, so that retaining an additional $p^2\ln|p|$ term in Eq. (10) is important to get a finite result.

The triple electron scattering rate from Eq. (5) generates the collision integral (Stosszahlansatz) of the corresponding Boltzmann equation

$$\mathrm{St}\{n\} = \sum_{\{p\},\{\sigma\}} W[n_{p_1'}(1-n_{p_1})n_{p_2'}(1-n_{p_2})n_{p_3'} \times \times (1-n_{p_3}) - n_{p_1}(1-n_{p_1'})n_{p_2}(1-n_{p_2'})n_{p_3}(1-n_{p_3'})]. \quad (11)$$

Here each pair of Fermi functions, $n_p(1-n_{p'})$, captures statistical occupation probabilities, whereas the two terms of the collision integral correspond to incoming and outgoing processes. At thermal equilibrium these terms nullify each other by virtue of the detailed balance condition. At weak disequilibrium, one can linearize $n_p = f_p + \delta n_p$ in the external perturbation δn_p around the equilibrium Fermi–Dirac distribution function f_p. The collision term can then be considered as a linear integral operator, acting on $\delta n_p = f_p(1-f_p)\psi$, and one can formulate the eigenvalue problem for this operator, $\mathrm{St}\{\psi_n\} = \omega_n\psi_n$. The spectrum of eigenvalues ω_n may be discrete or continuous, and captures all the information about the decay of different distribution function modes. As solving this problem exactly for triple collisions presents a daunting task [29, 40], we here follow a simpler more pragmatic approach. Setting, for instance, $\delta n_{p_1} = \delta_{p_1,p_F+\varepsilon/v_F}$ describes a quasiparticle with excess energy ε. Neglecting then secondary collisions, the Boltzmann equation reduces to the simple relaxation time approximation, $(\partial_t + \tau_{qp}^{-1})\delta n_p = 0$, with solution $\delta n_p \propto \exp(-t/\tau_{qp})$. It is natural to identify the corresponding timescale for decay with the quasiparticle life-time

$$\tau_{qp}^{-1} = -\frac{\partial \mathrm{St}\{n\}}{\partial n_p}, \quad (12)$$

which follows from Eq. (11) by only retaining the out-scattering contribution. Alternatively, one may project the collision operator (11) onto either momentum or energy modes and thus infer the relaxation time of interest. This approach is parametrically correct, however, may miss numerical factors of order unity when compared to the exact solution of the eigenvalue problem. We will employ both approaches in the forthcoming sections.

2.2. Quasiparticle decay rates. Owing to one-dimensionality of the problem, it is convenient to think of particles of different chirality, namely right-movers (R) and left-movers (L). It can be readily checked that strictly at zero-temperature quasiparticle relaxation is only possible if collisions involve both, right- and left-moving particles since otherwise conservation

laws cannot be satisfied. For this reason, consider first a process of relaxation that involves two right-moving particles, with initial momenta p_1, p_2, and a left-moving particle labeled by momentum p_3. The outgoing momenta after the collision, $p_i' = p_i + q_i$, will be labeled by momenta transfer q_i for each of the particle $i = 1, 2, 3$. In these notations, the momentum conservation becomes $q_1 + q_2 + q_3 = 0$, and the energy conservation, for a simple parabolic band, can be cast in the form

$$2(p_1q_1 + p_2q_2 + p_3q_3) + q_1^2 + q_2^2 + q_3^2 = 0.$$

These conditions set the phase-space constraints for collisions.

For an initial state with $p_1 = p_F + \varepsilon/v_F$, the quasiparticle life-time corresponding to an RRL-process is then

$$\tau_{qp}^{-1} = \\ = \sum_{\substack{p_2p_3 \\ p_1'p_2'p_3'}} W(1 - f_{p_1'})f_{p_2}(1 - f_{p_2'})f_{p_3}(1 - f_{p_3'}), \quad (13)$$

where we begin analysis from the spinless case. At this point it is convenient to shift momenta of left- and right-movers from the respective Fermi points, $p_{1,2} = p_F + k_{1,2}$ and $p_3 = -p_F + k_3$. In addition, it is sufficient to linearize the spectrum in the distribution functions, approximating

$$f_{\pm p_F + k} \to f_{\pm k} = \left[\exp\left(\pm\frac{v_F k}{T}\right) + 1\right]^{-1},$$

but not in the scattering probability W. Indeed, an analysis of the kinematic constraints suggests that $q_1 \approx -q_2$ and $q_3 \approx (q_1/p_F)(k_1 - k_2 + q_1)$, implying that $|q_3| \ll |q_{1,2}|$. In other words, relaxation occurs in incremental steps of momentum transfer $q_3 \sim \varepsilon^2/v_F^2 p_F$ from right-movers to left-movers. With these observations at hand, we next need the corresponding three-particle scattering amplitude. For the case of long-ranged Coulomb interaction Eq. (10), one finds from Eq. (6) after a laborious expansion

$$A \approx \frac{2(p_F w)^2}{L^2 \varepsilon_F}\left(\frac{2e^2}{\varkappa}\right)^2 \times \\ \times \left[1 - \frac{3}{4}\ln\left(\frac{1}{p_F w}\right)\right]\ln\left(\frac{q_1^2}{p_F|q_3|}\right). \quad (14)$$

This result is obtained to leading logarithmic accuracy using two small parameters $|q_1|/p_F \sim |q_3|/|q_1| \ll 1$ in the expansion. With the same level of accuracy the momentum and energy conservations in Eq. (5) can be simplified to

$$\delta_{P,P'}\delta(E - E') \approx \\ \approx \frac{1}{v_F}\delta\left(q_3 - \frac{q_1(k_1 - k_2) + q_1^2}{p_F}\right)\delta_{q_1,-q_2}. \quad (15)$$

These approximations enable one to complete all five momentum integrations. Two integrations are removed by delta functions which fix values of q_2 and q_3 in terms of $k_{1,2}$ and q_1. Furthermore, in the zero temperature limit, $T \to 0$, Fermi occupations become step-functions, $f_k \to \theta(-k)$. The integral over k_3 then becomes elementary, contributing by a pure phase space factor

$$\sum_{k_3} f_{-k_3}(1 - f_{-k_2-q_3}) = \frac{L}{2\pi}|q_3|\theta(-q_3).$$

The product of Fermi factors, $f_{k_2}(1 - f_{k_2-q_1})$, simply limits the domain of k_2 to the range $k_2 \in [-|q_1|, 0]$, while the remaining $1 - f_{k_1+q_1}$ dictates that $q_1 < k_1$. Finally, we recall that in this setting $k_1 = \varepsilon/v_F$. Putting everything together the RRL-process gives the life-time

$$\tau_{qp}^{-1} = c_1 \varepsilon_F g^4 \lambda_1^2(p_F w)\left(\frac{\varepsilon}{\varepsilon_F}\right)^4, \quad (16)$$

where $g = e^2/\varkappa v_F$ is the dimensionless interaction strength of the model and we introduced $\lambda_1(z) = z^2\ln(1/z)$. The numerical coefficient $c_1 = (15 - \pi^2)/32\pi^3$ is obtained with help of the following integral

$$\iint_0^1 x^2 g(x,y)\ln^2\left(\frac{x}{g(x,y)}\right)dx\,dy = \frac{15 - \pi^2}{72}, \quad (17)$$

where $g(x, y) = 1 - x(1 - y)$. Notice that the numerical factor in Eq. (16) differs from the one calculated in Refs. [26, 28] as different properties of the interaction potential were assumed[4)].

We see that finite decay rate emerges in forth order of the interaction strength. We also notice that the attenuation is inversely proportional to the cube of mass, $\tau_{qp}^{-1} \propto (m^*)^{-3}$, and vanishes as the limit $m^* \to \infty$ is taken at fixed band velocity. This limit corresponds to the situation considered by Dzyaloshinskii and Larkin. The energy scaling of the decay rate, $\propto \varepsilon^4$, is consistent with expectations based on the Fermi liquid picture for

[4)] In Appendix (see full text) we sketch derivation of Eq. (16) from the bosonization framework of an impurity scattering in Luttinger liquids.

a process involving two particle-hole excitations. However, this result is not universal. This becomes evident from repeating the above calculation for the model of screened short-range interaction, i.e. using the potential given by Eq. (9). Expanding the amplitude in Eq. (6) under the same conditions as above, one then finds instead of Eq. (14) the amplitude

$$A \approx -\frac{5(p_F d)^4}{3L^2\varepsilon_F}\left(\frac{2e^2}{\varkappa}\right)^2 \ln\left(\frac{1}{p_F d}\right) \times \\ \times \left[\frac{q_1^2}{4p_F^2}\left[1+6\ln\left(\frac{|q_1|}{p_F}\right)\right] - \right. \\ \left. - \frac{q_3^2}{q_1^2}\left[1+6\ln\left(\frac{|q_3|}{|q_1|}\right)\right]\right]. \quad (18)$$

The crucial difference here compared to Eq. (14) is the appearance of the additional small parameter $|q_1|/p_F \sim |q_3|/|q_1| \sim \varepsilon/\varepsilon_F \ll 1$, which can be related to the fact that this particular model is nearly integrable. A close inspection of the amplitude in Eq. (6) reveals that each term individually diverges as $1/q$ at small characteristic momentum transfer. However, all exchange terms combined together remove the singularity and partially cancel out all the way to $\sim q^2 \ln q$ order. The rest of the calculation carries through in exactly the same way as in the previous example, and one finds the decay rate

$$\tau_{qp}^{-1} = c_2 \varepsilon_F g^4 \lambda_2^2(p_F d)\left(\frac{\varepsilon}{\varepsilon_F}\right)^8 \ln^2\left(\frac{\varepsilon_F}{\varepsilon}\right) \quad (19)$$

with $c_2 = 2445/3584\pi^3$ and $\lambda_2(z) = z^4\ln(1/z)$. The four extra powers in the energy dependence, can be traced back to the different asymptotic form of the amplitude in Eq. (18). This demonstrates the high sensitivity of decay rates in 1D to details of the interaction. The result captured by Eq. (19) is of course perturbative. For a generic nonintegrable models with short-ranged interaction, it can be generalized to arbitrary interaction strength. It can further be shown that $\tau_{qp}^{-1} \propto \varepsilon^8$ remains valid, and the pre-factor can be expressed in terms of the exact spectrum [31].

As should be anticipated from the discussion above, electron spin plays a crucial role in the transition matrix element for the three-particle process, and should thus significantly affect the quasiparticle decay rate. Indeed, in the spinless case antisymmetry of the electron wave function dictates that its orbital component should be odd and therefore relevant exchange amplitudes are suppressed by Pauli exclusion. Mathematically, one sees this in a cancellation of various terms that lead to Eq. (18). In contrast, for spinful electrons singular parts of the amplitude do not cancel. They are dominated by $2p_F$ exchange-processes between branches, in which left-movers are scattered into right-movers [35]. Even though the strength of $2p_F$ exchange interaction is weaker than small momentum scattering, $V_{2p_F} \ll V_0$, for Coulomb interaction the relative reduction is only logarithmic. The gain in the amplitude, on the other hand, is more substantial and controlled by the large factor $\sim \varepsilon_F/v_F q \gg 1$. This statement can be verified explicitly from Eq. (6) where after spin summation one finds for the square of the amplitude for the RRL-process

$$\sum_{\substack{\sigma_2\sigma_3 \\ \sigma_1'\sigma_2'\sigma_3'}} |A|^2 = \frac{3V_{2p_F}^2(V_0 - V_{2p_F})^2}{32L^4\varepsilon_F^2}\left[\frac{q_1^2}{q_3^2} + \frac{4p_F^2}{q_1^2}\right]. \quad (20)$$

To obtain this result we approximated $V_{p_1-p_2\pm q_i} \approx V_0$ and $V_{p_{1,2}-p_3\pm q_i} \approx V_{2p_F}$ in all the relevant terms since $p_{1,2} - p_3 \approx 2p_F$ and $q_i \ll |p_i|$. Again, by repeating momentum integrations, the decay rate is found to be of the form

$$\tau_{qp}^{-1} = c_3 \varepsilon_F g^4 \lambda_3^2(p_F w)\left(\frac{\varepsilon}{\varepsilon_F}\right)^2 \ln^2\left(\frac{\varepsilon_F}{\varepsilon}\right), \quad (21)$$

with $c_3 = 45/32\pi^3$ and $\lambda_3(z) = \ln(1/z)$. To be consistent with the approximations that lead to Eq. (20), the difference $V_0 - V_{2p_F}$ should be understood as a weak logarithmic factor $\simeq (2e^2/\varkappa)\ln(\varepsilon_F/\varepsilon)$ for the Coulomb interaction potential. This was incorporated into Eq. (21). The singularity of the amplitude was compensated by phase space factors, and perhaps surprisingly this restores essentially the Fermi liquid form of the decay rate at $T = 0$. We note that up to model dependent pre-factors, the quadratic dependence of the relaxation rate on energy of spin-1/2 particles given by Eq. (21) is consistent with predictions of previous studies [35, 39].

We proceed with discussion of the effects of thermal broadening on relaxation processes. In the Fermi liquid picture one expects a simple crossover at excitation energies of the order of temperature $\varepsilon \sim T$. For 1D liquids this is not the case, as even at $T < \varepsilon$ there are intermediate regimes and relaxation shows nontrivial temperature dependence. Indeed, at finite temperatures each collision results in a typical momentum transfer $q_i \sim T/v_F$ allowed by thermal smearing of states near the Fermi energy. As RRL relaxation is controlled by the momentum transfer between the branches, one needs to compare phase spaces available to left movers. Since at zero temperature $q_3 \sim \varepsilon^2/v_F^2 p_F$, one deduces from comparison to $q_3 \sim T/v_F$ the crossover scale $\varepsilon_T \sim$

$\sim \sqrt{\varepsilon_F T}$. Technically, this argument can be also made clear by observing that

$$\sum_{k_3} f_{-k_3}(1 - f_{-k_3-q_3}) = \frac{L}{2\pi} q_3 \left[\exp\left(\frac{v_F q_3}{T}\right) - 1\right]^{-1},$$

and reducing to $LT/2\pi v_F$ as $q_3 \to 0$. These considerations suggest that Eqs. (16), (19), and (21) are valid for $T \ll \varepsilon^2/\varepsilon_F$. Above this threshold one finds

$$\tau_{qp}^{-1} = c_4 \varepsilon_F g^4 \lambda_1^2 (p_F w) \left(\frac{\varepsilon}{\varepsilon_F}\right)^2 \frac{T}{\varepsilon_F}, \qquad (22)$$

instead of Eq. (16) for the spinless Coulomb case. Similarly,

$$\tau_{qp}^{-1} = c_5 \varepsilon_F g^4 \lambda_2^2 (p_F d) \left(\frac{\varepsilon}{\varepsilon_F}\right)^6 \frac{T}{\varepsilon_F} \ln^2\left(\frac{\varepsilon_F}{\varepsilon}\right), \qquad (23)$$

instead of Eq. (19) for the spinless screened case, and finally

$$\tau_{qp}^{-1} = c_6 \varepsilon_F g^4 \lambda_3^2 (p_F w) \frac{T}{\varepsilon_F} \ln^2\left(\frac{\varepsilon_F}{\varepsilon}\right) \qquad (24)$$

instead of Eq. (21) for the spin-1/2 Coulomb case. The set of coefficients $c_{4,5,6}$ can be determined from numerical integrations, however, their specific values are of no particular significance here.

At elevated temperatures the above mechanism of relaxation competes with another process involving only particles of the same chirality. As indicated earlier, this RRR- (or equivalently LLL-) process is kinematically possible only at finite energies. It follows from the same amplitude Eq. (6), but admits different conditions on the involved momenta. In this process, a high-energy particle with excess energy ε can relax on two other comoving particles, which during the collision are scattered in opposite directions in energy. Namely, one is drifting slightly upwards in energy, whereas the other float downwards, closer to the Fermi energy. A detailed calculation in the spinless Coulomb model shows that the corresponding relaxation rate is given by

$$\tau_{qp}^{-1} = c_7 g^4 (p_F w)^4 \frac{T^3}{\varepsilon \varepsilon_F} \ln^2\left(\frac{\varepsilon_w}{\varepsilon}\right), \qquad (25)$$

where $\varepsilon_w = v_F/w$. This rate exceeds that given in Eq. (22), provided that temperature is higher than $\sim \varepsilon\sqrt{\varepsilon/\varepsilon_F}$. In the case of screened Coulomb interaction, the same mechanism is more strongly suppressed

$$\tau_{qp}^{-1} = c_8 g^4 (p_F d)^8 \frac{T^7}{\varepsilon \varepsilon_F^5} \ln^2\left(\frac{\varepsilon_d}{\varepsilon}\right) \ln^2\left(\frac{\varepsilon}{T}\right), \qquad (26)$$

where $\varepsilon_d = v_F/d$. In fact, $\propto T^7$ is a generic property for any non-integrable finite-range interaction model with a sufficient degree of analyticity at small momenta [32, 33]. Lastly, in the case of spin-1/2 chiral electrons one estimates the decay rate to be of the form

$$\tau_{qp}^{-1} = c_9 g^4 \frac{T \varepsilon_T^6}{\varepsilon^2 \varepsilon_d^4} \ln^4\left(\frac{d}{w}\right). \qquad (27)$$

In addition to relaxation of particles with the same chirality, thermal broadening allows for the relaxation of hot quasiholes, a process kinematically forbidden at zero temperature. The derivation of the corresponding decay rate τ_{qh}^{-1} proceeds in close analogy to that for the RRL-process. Crucial modifications are (i) the sign of q_3, (ii) a smaller phase space volume, now suppressed by an additional factor $\sim T/(\varepsilon^2/\varepsilon_F)$, and (iii) that it takes $\sim (\varepsilon/\varepsilon_T)^2$ steps to relax the excess energy. As a result, the quasihole relaxation rate e. g. for the spin-1/2 model,

$$\tau_{qh}^{-1} = c_{10} \varepsilon_F g^4 \lambda_3^2 (p_F w) \left(\frac{T}{\varepsilon}\right)^2 \ln^2\left(\frac{\varepsilon_F}{\varepsilon}\right), \qquad (28)$$

is by a factor $(\varepsilon_T/\varepsilon)^4$ smaller than τ_{qh}^{-1} defined in Eq. (21) when taken at the same energy. This pronounced asymmetry in the relaxation rates of electron-like and hole-like excitations is a direct consequence of the 1D kinematics of three-particle scattering with nonlinear spectrum. This feature marks a sharp distinction between the quantum 1D Fermi–Luttinger liquids and higher dimensional Fermi liquids.

We summarize in Table the discussed quasiparticle relaxation rates in the different regimes.

2.3. Distribution imbalance rates. Another common technique in kinetic theory applied to the determination of relaxation rates is to project the collision integral onto specific modes of interest, to infer their corresponding decay times. For instance, in the context of the present problem, one can look at the thermal imbalance relaxation between left- and right-movers. This amounts to projecting the collision term onto the energy mode of the distribution function n_p, which is even in momentum.

To see the practical implementation of this method, consider a situation in which right-movers are hotter than left-movers. The goal is then to derive an equation which describes the relaxation of the difference in temperatures $\Delta T = T^R - T^L$ of left- and right-moving electrons. It should be noted that the physical setting with imbalanced temperature is justified in 1D: while three-particle collisions generate both right- and

Table. Energy and temperature dependencies of quasiparticle relaxation rates (only the leading parametric behavior is indicated and logarithmic terms are omitted for brevity). First two rows summarize results for spinless electrons interacting via Coulomb and screened short-range interaction models, respectively, and the last row gives the result for the spin-1/2 model. The first two columns describe processes involving particles of both chiralities (e. g. the RRL-process), and the last column describes the relaxation of comoving particles with only same chirality (e. g. the RRR-process). In all cases $T_1 \sim \varepsilon^2/\varepsilon_F$, while $T_2 \sim \varepsilon\sqrt{\varepsilon/\varepsilon_F}$ in the Coulomb model, $T_2 \sim \varepsilon \sqrt[6]{\varepsilon/\varepsilon_F}$ in the screened model, and $T_2 \sim \varepsilon(\varepsilon_d/\varepsilon_F)\sqrt[3]{\varepsilon_d/\varepsilon}$ in the spinful model

τ_{qp}^{-1}	$T < T_1$	$T_1 < T < T_2$	$T_2 < T < \varepsilon$
Coulomb	$\varepsilon^4/\varepsilon_F^3$	$T\varepsilon^2/\varepsilon_F^2$	$T^3/\varepsilon\varepsilon_F$
Screened	$\varepsilon^8/\varepsilon_F^7$	$T\varepsilon^6/\varepsilon_F^6$	$T^7/\varepsilon\varepsilon_F^5$
Spin-1/2	$\varepsilon^2/\varepsilon_F$	T	$T\varepsilon_T^6/\varepsilon^2\varepsilon_d^4$

left-moving particle-hole pairs the intrabranch relaxation induced by these processes is faster, while interbranch is a slow.

We start from the Boltzmann equation, multiply both sides by $\varepsilon_{p_1} - \varepsilon_F$, and sum over $p_1 > 0$

$$\sum_{p_1>0} (\varepsilon_{p_1} - \varepsilon_F)\partial_t n_{p_1} = \sum_{p_1>0} (\varepsilon_{p_1} - \varepsilon_F)\mathrm{St}\{n\}, \quad (29)$$

where, as above, momentum p_1 is that of a right-moving particle. We then assume n_{p_1} to be of Fermi–Dirac form with nonequilibrium temperature $T^R = T + \Delta T$ of right-moving excitations, and linearize in the left-hand-side with respect to ΔT,

$$\partial_t n_{p_1} = \partial_T n_{p_1} \partial_t \Delta T = \frac{(\varepsilon_{p_1} - \varepsilon_F)\partial_t \Delta T}{4T^2 \operatorname{ch}^2\left(\dfrac{\varepsilon_{p_1} - \varepsilon_F}{2T}\right)}. \quad (30)$$

When computing integral over p_1 it is convenient to shift momentum to the respective Fermi point, $p_1 = p_F + k_1$. Linearizing further the dispersion relation in k_1, $\varepsilon_{p_1} - \varepsilon_F \approx v_F k_1$, one may use that the integral is peaked at p_F and rapidly converging. Noting that

$$\int_{-\infty}^{+\infty} z^2 dz/\operatorname{ch}^2(z) = \pi^2/6,$$

one readily finds

$$\sum_{p_1>0} (\varepsilon_{p_1} - \varepsilon_F)\partial_t n_{p_1} = \frac{\pi L T}{6 v_F}\, \partial_t \Delta T. \quad (31)$$

The next step is to also linearize the right-hand-side of Eq. (29) in ΔT. To accomplish this task we parametrize $n_p = f_p + f_p(1-f_p)\psi_p$, which allows to conveniently take advantage of the detailed balance condition in the collision integral $\mathrm{St}\{n\}$. For the thermal imbalance $\psi_p = (\varepsilon_p - \varepsilon_F)\Delta T/T^2$, and one finds upon expansion in ΔT

$$\sum_{p_1>0} (\varepsilon_{p_1} - \varepsilon_F)\mathrm{St}\{n\} = -\frac{\Delta T}{T^2} \sum_{\{k,q,\sigma\}} (v_F k_1)(v_F q_3)\mathcal{W}. \quad (32)$$

Here

$$\mathcal{W} = W f_{k_1}(1 - f_{k_1+q_1}) f_{k_2}(1 - f_{k_2+q_2}) \times \times f_{-k_3}(1 - f_{-k_3-q_3}), \quad (33)$$

and at intermediate steps we made use of the energy conservation implicit in W, and approximated $\varepsilon_{p_1} - \varepsilon_F \approx v_F k_1$ and $\varepsilon_{p_3'} - \varepsilon_{p_3} \approx -v_F q_3$. It is now evident that Eq. (29) can be cast in form of the usual relaxation time approximation,

$$\partial_t \Delta T = -\Delta T/\tau_{th}, \quad (34)$$

where we introduced the corresponding thermalization time. For the kinematics of the RRL-process, the latter evaluates to

$$\tau_{th}^{-1} = c_{11}\varepsilon_F g^4 \lambda_3^2(p_F w)\left(\frac{T}{\varepsilon_F}\right)^2 \ln^2\left(\frac{\varepsilon_F}{T}\right). \quad (35)$$

In a similar fashion one can find the relaxation rate for the odd part of the imbalanced distribution. For this purposes one may consider a boosted frame of reference, $\varepsilon_p - pu$, and derive the relaxation equation for u by projecting the collision integral onto the momentum mode. Kinematics of the respective collision is different though, and will be considered in the next section.

To get an idea of the order of magnitude of the different timescales, it is instructive to consider the following estimates for GaAs quantum wires using experimental parameters of Ref. [52]. For $v_F \sim 2 \cdot 10^5$ m/s and $\varkappa \sim 10$, the interaction parameter is just within the applicability criterion of the perturbative expressions $g \sim 1$. For the typical electron density we use $p_F \sim 10^8$ m^{-1}, $w \sim 10$ nm, and $\varepsilon_F \sim 1$ meV. Then for $\varepsilon \sim \varepsilon_F/4$, which is a typical excess energy of injected particles in tunneling experiments, and $T \sim 0.25$ K one is securely in the regime $T \ll \varepsilon^2/\varepsilon_F$. For this set of parameters $\tau_{qp}^{-1} \sim 10^{11}$ s^{-1}, $\tau_{qh}^{-1} \sim 10^9$ s^{-1}, and $\tau_{th}^{-1} \sim 10^6$ s^{-1}.

2.4. Backscattering hole mobility rates. Relaxation processes of low-energy excitations leading to the decay of quasiparticles near the Fermi energy do not change the numbers of right- and left-moving particles. Thus they are chirality conserving. It turns out that it is also possible to have backscattering processes. The kinematics of these collisions involves states deep in the Fermi sea, and for this reason it is useful to consider the mobility of holes at the bottom of the band. These processes are commonly considered from the perspective of mobile impurities in a Luttinger liquid [63–69]. Here we will continue using the kinetic equation approach for their description. The idea is then to single out hole states at the bottom of the band with small momenta, and to derive an effective kinetic equation capturing their dynamics and allowing the calculation of corresponding backscattering rates [70, 71].

For this purpose, let p_1 and p_1' be momenta near the band bottom, p_2 and p_2' lie near the right Fermi point $(+p_F)$, and p_3 and p_3' be taken near the left Fermi point $(-p_F)$. As before, the unprimed momenta correspond to incoming states whereas primed ones are associated with outgoing states. With these conventions, we introduce the hole distribution function, $h_{p_1} = 1 - n_{p_1}$, and the collision integral for holes, $\mathrm{St}\{h_{p_1}\} = -\mathrm{St}\{n_{p_1}\}$. Starting from Eq. (11), the latter can be cast in the form

$$\mathrm{St}\{h_{p_1}\} = \sum_{p_1'} \left[\mathcal{P}(p_1, p_1')h_{p_1'} - \mathcal{P}(p_1', p_1)h_{p_1}\right], \quad (36)$$

where

$$\mathcal{P}(p_1, p_1') = 12 \sum_{\{\sigma\}} \sum_{\substack{p_2 p_3 \\ p_2' p_3'}} W f_{p_2}(1 - f_{p_2'}) f_{p_3}(1 - f_{p_3'}) \quad (37)$$

is the rate for a transition in which a hole scatters from some state p_1' into p_1, while $\mathcal{P}(p_1', p_1)$ denotes the rate for the inverse process. In the above sums, all momenta have been restricted to the discussed ranges, which explains the combinatorial overall factor of 12. Since both p_1 and p_1' lie near the bottom of the band, the distribution functions h_{p_1} and $h_{p_1'}$ are exponentially small $\propto e^{-\varepsilon_F/T}$ due to Pauli exclusion, and so is the collision integral of holes $\mathrm{St}\{h_p\}$. It is therefore unnecessary to account for additional exponentially small contributions in the transition rates $\mathcal{P}(p_1, p_1')$ and $\mathcal{P}(p_1', p_1)$, and this is why we replaced $f_{p_1} \simeq 1$ and $f_{p_1'} \simeq 1$ in both. As in the case of the forward scattering process, the typical scale for momentum change of all three particles in a hole backscattering is set by temperature, $q_i = p_i' - p_i \sim T/v_F$. At the same time, the typical momentum of a hole is $p_1 \sim \sqrt{m^* T}$ so that $q_1/p_1 \sim \sim \sqrt{T/\varepsilon_F} \ll 1$. This means that the net momentum change in each scattering event is small, and holes effectively drift through the bottom of the band. Thus relaxation occurs in multiple steps and the underlying dynamics is momentum space diffusion. Under these conditions, the mobile impurity falls into the universal class of problems described by a Fokker–Planck equation [72]. The collision integral Eq. (36) can then be simplified by expanding in the small momentum step $q_1 \ll p_1$, and maps to the differential operator

$$\mathrm{St}\{h_{p_1}\} \approx -\partial_{p_1}\left[\mathcal{A}(p_1)h_{p_1}\right] + \frac{1}{2}\partial_{p_1}^2\left[\mathcal{B}(p_1)h_{p_1}\right]. \quad (38)$$

Here we introduced

$$\begin{aligned} \mathcal{A}(p_1) &= -\sum_{q_1} q_1 \mathcal{P}_{q_1}(p_1), \\ \mathcal{B}(p_1) &= \sum_{q_1} q_1^2 \mathcal{P}_{q_1}(p_1), \end{aligned} \quad (39)$$

and used the short-hand notation $\mathcal{P}_{q_1}(p_1) = \mathcal{P}(p_1', p_1)$. The diffusion coefficient in momentum space $\mathcal{B}(p_1)$ is a function of the hole-momentum p_1 varying on a scale set by p_F. For holes at the bottom of the band, one may thus approximate $\mathcal{B}(p_1)$ by its value at $p_1 = 0$, in the following simply denoted by $\mathcal{B}$ without argument. Furthermore, the drift coefficient $\mathcal{A}(p_1)$ is readily obtained from noting that the collision integral (38) has to vanish for hole distributions of an equilibrium Boltzmann form. This condition leads to the relation $\mathcal{A}(p) = p\mathcal{B}/2m^*T$.

The rest of the calculation depends on the structure of the amplitude for the given kinematics of the three-particle process. In calculating A from Eq. (6) for the momentum configuration under consideration, and up to small corrections in $T/\varepsilon_F \ll 1$, it is sufficient to approximate $p_1 \approx 0$, $p_2 \approx +p_F$ and $p_3 \approx -p_F$. Momentum and energy conservations provide additional restrictions on the transferred momenta, enforcing that $q_2 \approx q_3 \approx -q_1/2$, again up to small corrections in $T/\varepsilon_F \ll 1$. As a result, the amplitude A can be parametrized only by a single momentum q_1. Expanding Eq. (6) and summing over spins one then finds

$$\sum_{\{\sigma\}} |A|^2 = \frac{6}{\varepsilon_F^2 L^4} V_{p_F}^2 (V_{p_F} - V_{2p_F})^2 \frac{p_F^2}{q_1^2}. \quad (40)$$

The singularity of A at small momenta is cancelled in the spinless case. Specifically, for the long-range interaction model with Eq. (10) one finds

$$A \approx \frac{9}{16\varepsilon_F^2 L^4}\left(\frac{2e^2}{\varkappa}\right)^4 \lambda_1^2(p_F w)\ln^2\left(\frac{p_F}{|q_1|}\right), \quad (41)$$

whereas for the screened model

$$A \approx \frac{9(\ln 4-1)^2}{\varepsilon_F^2 L^4}\left(\frac{2e^2}{\varkappa}\right)^4 \lambda_2^2(p_F d). \qquad (42)$$

In order to perform remaining momentum integrations implicit in the definition of $\mathcal{B}$, one can approximate delta functions in the scattering probability by

$$\delta_{P,P'}\delta(E-E') \approx \frac{1}{v_F}\delta(q_2-q_3)\delta_{q_2,-q_1/2}.$$

This removes two integrations out of five, and gives

$$\mathcal{B} = \frac{12L}{v_F}\sum_{q_1 k_2 k_3} q_1^2 \sum_{\{\sigma\}} |A|^2 f_{k_2-q_1/2}(1-f_{k_2}) \times$$

$$\times f_{k_3+q_1/2}(1-f_{k_3}), \quad (43)$$

where we shifted momenta $p_{2,3}$ to the respective Fermi points, $\pm p_F + k_{2,3}$, and linearized the dispersion relation in all Fermi occupation functions. Finally, using the tabulated integral

$$\sum_k f_{k+q}(1-f_k) = \frac{L}{2\pi} q b_q, \quad b_q = \frac{1}{e^{v_F q/T}-1}, \quad (44)$$

where b_q is the equilibrium Bose distribution, we arrive at the general expression

$$\mathcal{B} = \frac{6\pi}{v_F}\left(\frac{L}{2\pi}\right)^3 \sum_{q_1} q_1^4 \sum_{\{\sigma\}} |A|^2 b_{q_1/2}(1+b_{q_1/2}). \quad (45)$$

A notable feature of this expression is that it is entirely expressed in terms of bosonic modes. In essence, this is a manifestation of bosonization at the level of fermionic kinetic theory, as the occupation of an electron-hole pair near one of the Fermi points integrated over the center of mass momentum is equivalent to a collective boson emitted/absorbed in a course of hole diffusion. It will be shown in the subsequent section that structurally the same expression for $\mathcal{B}$ can be obtained from a purely bosonic formulation of the problem. Finally, inserting Eq. (40) into Eq. (45) one finds the momentum space diffusion coefficient of spin-1/2 holes

$$\mathcal{B} = \frac{768\ln^2(2)}{\pi}\, g^4 \lambda_3^2(p_F w)\left(\frac{T}{\varepsilon_F}\right)^3 p_F^2 \varepsilon_F. \qquad (46)$$

The corresponding backscattering relaxation rate can be found from Einstein relation adopted to diffusion in momentum space, $\Delta p^2 = \mathcal{B}\tau_{dh}$. The notation τ_{dh} is meant to emphasize kinetics of a deep hole as opposed to earlier notation τ_{qh} describing quasiholes near Fermi energy. Thus for $\Delta p^2 \simeq m^* T$ the result is (omitting numerical factor for brevity)

$$\tau_{dh}^{-1} \simeq g^4 \lambda_3^2(p_F w)\left(\frac{T}{\varepsilon_F}\right)^2. \qquad (47)$$

Finally we recall that the mobility of particles μ is related to the diffusion constant by the simple kinetic formula $\mu = T/\mathcal{B}$, and therefore $\mu \propto 1/T^2$.

The result is different in the spinless case. From Eqs. (41), (42) and (45) one finds $\mathcal{B} \propto T^5$ in both cases, modulo a logarithmic factor $\ln^2 T$ in the Coulomb case, and thus $\tau_{dh}^{-1} \propto T^4$ and $\mu \propto 1/T^4$. The results discussed in this section are again perturbative in the interaction. The power laws in the temperature dependence of relaxation rates are, however, generic and also apply to the strongly interacting regime, as we further elaborate below, see also Refs. [67, 68].

2.5. Electron-phonon relaxation rates. Apart from the purely electronic mechanisms of relaxation electrons may scatter on phonons, disorder, and sample imperfections thus relaxing their energy and momentum. At extremely low temperatures phonons are not expected to be efficient at cooling the electronic sub-system. On the other hand, electron-phonon scattering has no such severe phase space restrictions like the three-particle collisions considered above. It is thus instructive to estimate the temperature dependence for the corresponding relaxation rate. Unlike the previous studies of electron-phonon relaxation in multichannel quantum wires [73, 74], and phonon-induced backscattering relaxation [75, 76], we focus on the complementary effect of soft collisions in a single-channel geometry of strictly 1D electrons and 3D phonons.

The coupling of electrons and phonons is described by the collision integral [77]

$$\mathrm{St}\{n_p, N_q\} =$$

$$= \sum_{p'q} W_-[n_{p'}(1-n_p)N_q - n_p(1-n_{p'})(1+N_q)] +$$

$$+ \sum_{p'q} W_+[n_{p'}(1-n_p)(1+N_q) - n_p(1-n_{p'})N_q], \quad (48)$$

where the scattering rate

$$W_\pm(p,p',q) = (2\pi)|A(q)|^2 \delta(\varepsilon_p - \varepsilon_{p'} \pm \omega_q)\delta_{p=p'\pm q_x} \quad (49)$$

describes phonon emission and absorption processes with an amplitude

$$A(q) = \sqrt{\frac{1}{2\varrho\mathcal{V}\omega_q}}(D|q| + i\Lambda).$$

Here we took into account that at the level of the leading Born approximation, the probabilities of scattering for direct and reverse processes are the same. In the amplitude we include both deformation (D) and piezoelectric (Λ) couplings, ϱ is the mass density, q_x the phonon wave-vector along the wire, and $\mathcal{V}$ is the system volume. For simplicity we assume only a single acoustic branch $\omega_q = s|q|$, with sound velocity s.

For equilibrium Fermi and Bose distribution functions of electrons and phonons respectively, $n_p \to f_p$ and $N_q \to b_q$, the collision integral in Eq. (48) vanishes due to detailed balance condition. As in the above example of the distribution imbalance relaxation, we then assume that electrons are hot, that is, at an excess temperature $T + \Delta T$ with respect to the temperature T of lattice phonons. Electron-phonon collisions tend to relax ΔT, and the corresponding rate for relaxation can be found by projecting the collision integral onto the energy mode,

$$\sum_p \epsilon_p \dot{n}_p = -\sum_p \epsilon_p \mathrm{St}\{n_p, N_q\},$$

with $\epsilon_p = \varepsilon_p - \varepsilon_F$. To linear order in ΔT one finds from the phonon emission processes of hot electrons, $\partial_t \Delta T = -\Delta T/\tau_{ep}$, where

$$\tau_{ep}^{-1} = -\frac{6v_F}{\pi T^3 L} \times \\ \times \sum_{pp'q} W \epsilon_p \omega_q f_{\varepsilon_p}(1 - f_{\varepsilon_p})(f_{\varepsilon_p + \omega_q} + b_{\omega_q}). \quad (50)$$

Upon completion of the remaining momentum integrations, we then find to leading order in T

$$\tau_{ep}^{-1} = \frac{9\zeta(3)}{8\pi^3} T(\Lambda^2/s^2 v_F \varrho). \quad (51)$$

The scattering rate due to the deformation potential is parametrically weaker, scaling as $\tau_{ep}^{-1} \propto T^3$. The backscattering mechanism results in an activated temperature dependence $\propto e^{-T_A/T}$ with $T_A = 2sp_F$. It is straightforward to generalize Eq. (51) to the case when electronic relaxation occurs via several acoustic branches. Notice also that the piezoelectric potential may have complicated angular dependence in case of wires oriented arbitrarily with respect to the crystallographical axis of the sample. A proper angular averaging would change then numerical factors in Eq. (51) where we took the simplest geometry. Luttinger liquid effects lead to renormalization of the linear-T behavior and transform it into a power-law with interaction dependent exponent $\propto T^K$, where $K = v_F/u$ is the ratio of Fermi and plasmon velocities. In the TL model $u = v_F\sqrt{1 + V_0/\pi v_F}$.

2.6. Spin-charge scattering rates. The applicability of the Born approximation, used to construct the quantum amplitude for triple particle processes captured by Eq. (6), requires that incoming spin-1/2 quasiparticles have sufficiently high energy compared to the typical scale of interparticle interaction $\varepsilon \gg m^* v_F V_0$.

In the generic interacting environment of a 1D quantum fluid, quasiparticle excitations break down into spin and charge modes. At the level of linear Luttinger liquid theory, spin-charge separation is an exact property of the model [14]. At weak coupling, the splitting between velocities of collective spin (v_σ) and charge (v_ρ) density waves is related to the forward scattering component of the interaction $v_\rho - v_\sigma \sim V_0$ (recall that for repulsive interactions $v_\rho > v_\sigma$). Assuming then thermal excitations with $\varepsilon \sim T$, the Born condition can be equivalently formulated as $T/(m^* v_F) \gg v_\rho - v_\sigma$. In other words, for fermionic quasiparticles to preserve their integrity the excitation energy (or temperature) should be bigger than the energy scale of spin-charge separation.

The interplay of spectrum nonlinearities and interactions leads to spin-charge coupling [78, 79]. Although irrelevant in the renormalization group sense, the newly emerging higher order operators capture the attenuation of quasiparticles. The kinetic properties of 1D quantum liquids with spin-charge coupling are not fully understood. There are basically two possible approaches one may pursue. The first is to refermionize the nonlinear bosonic theory to obtain an effective description in terms of dressed quasiparticles: holons and spinons. Holon relaxation was considered in Refs. [36, 37] based on non-Abelian bosonization [80]. The advantage of this complex theory is that, in principle, it allows to go beyond the weakly interacting limit for spinful fermions. Alternatively, one may choose to continue working in bosonic language. In the limit of weak backscattering one can then account for spin-charge interaction perturbatively in the basis of well-defined spin and charge modes. This second procedure is limited to weak interactions $V_{2p_F} \ll V_0 \ll v_F$. To complement previous studies, we follow in this section the second path. In part this will enable us to explore the fermion-boson duality. We delegate technical details of bosonization to the Appendix (see full text) and elucidate here the impact of spin-charge scattering on various decay rates.

The lowest order nonlinearity, compatible with SU(2) symmetry of the problem is cubic. It contains

one charge and two spin operators. Treating this term in a perturbative expansion generates a collision kernel that describes the decay of a plasmon into two spin modes $\rho \to \sigma\sigma$. It reads

$$\mathrm{St}\{N^\rho, N^\sigma\} = -\sum_{q_1 q_2} W \left[N_q^\rho(1+N_{q_1}^\sigma)(1+N_{q_2}^\sigma) - \right.$$
$$\left. - (1+N_q^\rho)N_{q_1}^\sigma N_{q_2}^\sigma\right], \quad (52)$$

where $N^{\rho/\sigma}$ are the bosonic occupations of charge (ρ) and spin (σ) excitations. The scattering probability

$$W = 2\pi|A|^2\delta_{q=q_1+q_2}\delta(\omega_q^\rho - \omega_{q_1}^\sigma - \omega_{q_2}^\sigma) \quad (53)$$

contains an amplitude scaling cubically with momenta of the bosons $|A|^2 = (\pi^3/8L)|q||q_1||q_2|\Gamma_{\rho\sigma\sigma}^2$. The perturbative result for the coupling constant is $\Gamma_{\rho\sigma\sigma} = V'_{2p_F}/\sqrt{2}\pi^2$, where the prime denotes the derivative with respect to p_F. Note that it thus vanishes for the integrable case of constant interaction. At smallest momenta the dispersion relations are linear $\omega^{\rho/\sigma} = v_{\rho/\sigma}|q|$. The kinematics of this process uniquely fixes momenta in the final state. Indeed, for concreteness let $q > 0$, then $q_1 = q(v_\rho + v_\sigma)/2v_\sigma$ and $q_2 = -q(v_\rho - v_\sigma)/2v_\sigma$, which means that spin waves are counterpropagating. From dimensional analysis it becomes apparent that $\mathrm{St}\{N^\rho, N^\sigma\}$ defines the decay rate of a plasmon, and one can introduce the characteristic rate

$$\tau_\rho^{-1} = \sum_{q_1 q_2} W \simeq q^3 (V'_{2p_F})^2 \frac{v_\rho^2 - v_\sigma^2}{v_\sigma^3}.$$

For the sake of an estimate, one may now take $V'_{2p_F} \sim V_{2p_F}/p_F$ and replace $v_{\rho/\sigma} \sim v_F$, except in their difference where $v_\rho - v_\sigma \sim V_0$, and finds the life-time

$$\tau_\rho^{-1} \sim \varepsilon_F \frac{V_0}{v_F}\left(\frac{V_{2p_F}}{v_F}\right)^2\left(\frac{q}{p_F}\right)^3. \quad (54)$$

Notice the nonanalytic dependence of interaction $\propto V^3$. For thermal plasmons the relaxation rate can be calculated from Eq. (52) by a projection onto an energy mode. We observe that as $|q_2| \ll q$ the relaxation occurs by small energy transfer from right-movers to left-movers (or vise versa) so that interbrach processes are slow. Assuming that right-moving excitations are hotter by ΔT, and in complete analogy to the fermionic case, we find

$$\sum_{q>0}\omega_q^\rho \partial_t N_q^\rho = \sum_{q>0}\omega_q^\rho \mathrm{St}\{N^\rho, N^\sigma\}. \quad (55)$$

The left-hand-side is straightforward to evaluate further, noting that

$$\partial_t N_q^\rho = \partial_T N_q^\rho \partial_t \Delta T = \frac{\omega_q^\rho \partial_t \Delta T}{4T^2 \,\mathrm{sh}^2(\omega_q^\rho/2T)}, \quad (56)$$

which after momentum integration gives a factor of $(\pi LT/6v_\rho)\partial_t\Delta T$. The right-hand-side can be linearized with the usual substitution $N_q = b_q + b_q(1+b_q)\phi_q$, where $\phi_q = \omega_q\Delta T/T^2$ for the case of a thermal imbalance. After some algebra one finds

$$\sum_{q>0}\omega_q^\rho \mathrm{St}\{N^\rho, N^\sigma\} =$$
$$= -\Delta T \sum_{q q_1 q_2} (\omega_q^\rho/T)^2 W(1+b_q^\rho) b_{q_1}^\sigma b_{q_2}^\sigma, \quad (57)$$

where we repeatedly used energy conservation and the detailed balance condition. Performing the final integrations, we then arrive at

$$\tau_\rho^{-1} = \frac{3\pi}{16}\Gamma_{\rho\sigma\sigma}^2 (T^3/v_\sigma^4) F(v_\sigma/v_\rho), \quad (58)$$

where the dimensionless function reads

$$F(\kappa) = \kappa(1-\kappa^2)\int_0^\infty dz\, z^5 (1+b_z) b_{z_+} b_{z_-} \quad (59)$$

with $b_z = (e^z - 1)^{-1}$ and $z_\pm = z(1 \pm \kappa)/2$. One can readily check that $F \to 32\pi^4/15$ in the limit $\kappa \to 1$.

The same scattering process can be alternatively viewed as a mutual spin-charge friction. Physically, this is analogous to the electron-phonon drag effect, typically studied in the context of thermoelectricity, or Coulomb drag in double-layers [81] and spin Coulomb drag [82]. In each of these examples momentum transfer between interactively coupled systems leads to dragging of one sub-system by the flow of the other. For instance, in the context of spin physics in Luttinger liquids, generation of spin current is possible by Coulomb drag [83]. To estimate the spin-charge drag rate, one can consider a boosted frame of reference for spin and charge excitations with mismatched boost velocities $u_{\rho/\sigma}$. The scattering leads to momentum exchange between spins and charge and, as a result, to relaxation $\partial_t u_\rho = -(u_\rho - u_\sigma)/\tau_{\rho\sigma}$. To capture this effect, we linearize the collision integral for $N(\omega_q - uq)$ with respect to u, for both spin and charge occupations, and then project onto the momentum mode to calculate the rate of momentum loss by (e. g.) charge modes

$$\partial_t P_\rho = \sum_q q\mathrm{St}\{N^\rho, N^\sigma\} =$$
$$= -\frac{\Delta u}{T}\sum_{q q_1 q_2} q^2 W(1+b_q^\rho) b_{q_1}^\sigma b_{q_2}^\sigma. \quad (60)$$

When we compare this to $\partial_t P_\rho = (\pi L T^2/3v_\rho^3)\partial_t u_\rho$, we find that thus defined drag relaxation rate $\tau_{\rho\sigma}^{-1}$ coincides with Eq. (58) up to a constant factor. It is perhaps useful to note that $\tau_{\rho\sigma}^{-1} \propto T^3$ is consistent with the expectation that Coulomb drag transresistivity between double quantum wires due to interwire momentum transfer from spin-charge coupling at zero magnetic field scales as $\rho_D \propto T^5$ [37]. Indeed, this rate is accompanied by two thermal phase space factors $\sim T$ per wire, thus leading to T^5. In the drag problem, the factor q^3 results from the width of the dynamic charge structure factor and the underlying scattering that gives rise to q^3 is precisely the decay of a charge boson into two spin bosons.

The next in complexity is a quartic nonlinearity in spin-charge coupling which leads to two-boson scattering $\rho\sigma \to \rho\sigma$. In particular, we consider backscattering of spin excitations on plasmons. Such scattering processes correspond to the diffusion of spin excitations near the spectral edge, and the goal is to calculate the corresponding diffusion constant. As alluded to earlier, the discussion parallels the previous calculation of the backscattering of a deep hole in the fermionic language. The corresponding collision integral reads

$$\mathrm{St}\{N^\rho, N^\sigma\} = \\ = -\sum_{q_2 q_1' q_2'} W \Big[N^\sigma_{q_1}(1+N^\sigma_{q_1'}) N^\rho_{q_2}(1+N^\rho_{q_2'}) - \\ - N^\sigma_{q_1'}(1+N^\sigma_{q_1}) N^\rho_{q_2'}(1+N^\rho_{q_2}) \Big]. \quad (61)$$

The scattering rate for this process is given by

$$W = 2\pi |A|^2 \delta_{Q,Q'} \delta(E-E') \quad (62)$$

with the amplitude $|A|^2 = (\Gamma_{\rho\sigma}/8L)^2 |q_1 q_1' q_2 q_2'|$, where the coupling constant at the perturbative level reads $\Gamma_{\rho\sigma} = V''_{2p_F}$. The notations for momentum and energy conservation here are $Q = q_1 + q_2$ and $E = \omega^\sigma_{q_1} + \omega^\rho_{q_2}$. Let momenta q_1 and q_1' correspond to the initial and final states of the spin excitation near the spectral edge. Kinematically each momentum is of the order of the Fermi momentum, $q_1 \sim q_1' \sim p_F$, while their difference, $q_1' - q_1 \sim T/v_F$, is small. This corresponds to a small momentum change in each collision, which is accompanied by the excitation of plasmons at low momenta $q_2 \sim q_2' \sim T/v_F$. For this reason the low-energy description based on Eq. (61) is sufficient to capture this physics. Under the specified conditions and n complete analogy with the fermionic case, we can convert the collision integral into a Fokker–Planck differential operator, thus describing the diffusion of spins. Indeed, for momenta $q \sim p_F$ the occupation is small, $N^\sigma_q \propto \propto e^{-\varepsilon_\sigma/T} \ll 1$, and correspondingly $1 + N^\sigma \approx 1$, where ε_σ is the band width of spin excitations. The latter is parametrically of the order of the spin exchange coupling. Viewing Eq. (61) as the collision integral for spins, we thus write

$$\mathrm{St}\{N^\sigma_{q_1}\} = \sum_{q_1'} [\mathcal{P}(q_1, q_1') N^\sigma_{q_1'} - \mathcal{P}(q_1', q_1) N^\sigma_{q_1}], \quad (63)$$

where

$$\mathcal{P}(q_1', q_1) = \sum_{q_2 q_2'} W N^\rho_{q_2'}(1+N^\rho_{q_2}) \quad (64)$$

is the transition rate for spin scattering processes. More specifically, it describes a collision with momentum transfer δq, in which a spin is scattered out of the initial state q_1. It can thus be rewritten as $\mathcal{P}(q_1', q_1) = \mathcal{P}_{\delta q}(q_1)$, and following the same prescription, the transition rate for the inverse process reads $\mathcal{P}(q_1, q_1') = \mathcal{P}_{-\delta q}(q_1 + \delta q)$. Performing then a small-momentum expansion,

$$\mathcal{P}(q_1, q_1') N^\sigma_{q_1'} \approx \mathcal{P}_{-\delta q}(q_1) N^\sigma_{q_1} + \\ + \delta q \partial_{q_1}[\mathcal{P}_{-\delta q}(q_1) N^\sigma_{q_1}] + \frac{\delta q^2}{2} \partial^2_{q_1}[\mathcal{P}_{-\delta q}(q_1) N^\sigma_{q_1}], \quad (65)$$

the collision integral of spin excitations takes the simplified form

$$\mathrm{St}\{N^\sigma_{q_1}\} = -\partial_{q_1}[\mathcal{A}_{\rho\sigma}(q_1) N^\sigma_{q_1}] + \\ + \frac{1}{2}\partial^2_{q_1}[\mathcal{B}_{\rho\sigma}(q_1) N^\sigma_{q_1}], \quad (66)$$

where

$$\mathcal{A}_{\rho\sigma} = -\sum_{\delta q} \delta q \mathcal{P}_{\delta q}(q_1), \quad \mathcal{B}_{\rho\sigma} = \sum_{\delta q} \delta q^2 \mathcal{P}_{\delta q}(q_1). \quad (67)$$

At this stage we focus on the derivation of $\mathcal{P}_{\delta q}(q_1)$. The momentum conservation implicit in W removes the q_2' integration. We then notice that distribution functions limit the typical momentum transfer and momenta of plasmons to $q_2 \sim \delta q \sim T/v_F$. At the same time, the typical momentum of spins at the spectral edge is $q_1 \sim \sim p_F$ and it is sufficient to calculate $\mathcal{P}_{\delta q}(p_F)$. With these observations at hand, we can now approximate energy conservation by

$$\delta(E - E') \approx \frac{1}{v_\rho}\delta(q_2 - \delta q/2).$$

This removes the q_2 integral, and we thus arrive at

$$\mathcal{P}_{\delta q} = \frac{V^2_{\rho\sigma}}{1024 L v_\rho} \frac{(\delta q/p_F)^2}{\mathrm{sh}^2(v_\rho \delta q/4T)}, \quad (68)$$

with the notation $V_{\rho\sigma} = p_F^2 \Gamma_{\rho\sigma}$. Finally, this defines the diffusion coefficient of spins in momentum space associated to $\rho\sigma \to \rho\sigma$ scattering channel

$$\mathcal{B}_{\rho\sigma} = \frac{\pi^3}{30} \left(\frac{V_{\rho\sigma}}{v_\rho}\right)^2 \left(\frac{T}{p_F v_\rho}\right)^5 p_F^3 v_\rho. \tag{69}$$

In addition to spin-charge scattering, nonlinearities also allow for spin-spin scattering. Importantly for the momentum space diffusion, scattering processes with spin-flips are enhanced. They are thus described by a different scaling of the probability with momentum as compared to Eq. (68). That is,

$$\mathcal{P}_{\delta q} = \frac{V_{\sigma\sigma}^2}{8\pi^2 L v_\sigma} \frac{1}{\mathrm{sh}^2(v_\sigma \delta q/4T)}, \tag{70}$$

and this crucial detail is technically speaking traced back to the non-commutativity of spin operators when calculating the corresponding amplitude. The importance of spin flips is also apparent at the level of fermions. Indeed, the ratio of scattering rates between spinless and spinful cases has exactly the same parameter $(q/p_F)^2 \ll 1$ as the ratio between probabilities in Eqs. (68) and (70). The resulting diffusion constant in the spin-spin channel is then

$$\mathcal{B}_{\sigma\sigma} = \frac{4\pi}{15} \left(\frac{V_{\sigma\sigma}}{v_\sigma}\right)^2 \left(\frac{T}{p_F v_\sigma}\right)^3 p_F^3 v_\sigma. \tag{71}$$

A microscopic calculation of the respective coupling constants for the different scattering channels is a challenging task. Known approaches include weak coupling results obtained via mobile impurity model [69], results for Kondo polarons [66], and calculations in the strong interaction limit within the non-Abelian bosonization framework [37], as well as a model departing from the Wigner crystal limit [84].

Two-boson processes also contribute to the thermalization rates [33,85,86]. For charge excitations this results in a subleading correction to Eq. (58). In the spin sector the situation is, however, different since at the cubic level of nonlinearities spins are kinematically forbidden to scatter. In both cases nonlinearity of the bosonic spectrum plays an important role to open phase space for such collisions. In order to generalize the present model, consider first the charge sector and assume a weakly anharmonic dispersion of plasmons, $\omega_q^\rho \approx v_\rho |q|(1-(\xi q)^2)$. Assume now that a right-moving boson with momentum $q_1 \gtrsim T/v_\rho$ is injected into the Luttinger liquid. For this setting the collision term from Eq. (61), with replacement $N^\sigma \to N^\rho$, dictates that the dominant process limiting the lifetime of the injected boson is due to scattering with inter-branch momentum transfer. Indeed, for $q_1, q_2, q_1' > 0$ momentum conservation implies that q_2' is order q^3, since energy conservation fixes $q_2' \approx -(3\xi^2/2) q_1 q_2 q_1'$. Curiously, even though a finite ξ is crucial to resolve the kinematic constraints it drops out from the corresponding rate provided that $q_1 \ll \sqrt[3]{T/v_\rho \xi^2}$. In this regime $v_\rho |q_2'| \ll T$, implying that $N^\rho_{q_2'} \approx T/\omega^\rho_{q_2'}$ and q_2' cancels out from W. The decay rate then scales parametrically as $\tau_\rho^{-1} \propto T q^4$. This estimate is applicable as long as $T/v_\rho \lesssim q \ll \sqrt[3]{T/v_\rho \xi^2}$.

For thermal plasmons, this rate can be estimated more accurately by projecting the collision integral onto the energy mode. Assuming that the boson with momentum q_1 is "hotter" by a temperature difference ΔT, one finds upon repeating the steps from the previous similar calculations

$$\tau_\rho^{-1} = \frac{6 v_\rho}{\pi L T} \sum_{\substack{q_1 q_2 \\ q_1' q_2'}} \frac{\omega_{q_1}\omega_{q_2}}{T^2} \times \\ \times W N^\rho_{q_1} N^\rho_{q_2} (1+N^\rho_{q_1'})(1+N^\rho_{q_2'}). \tag{72}$$

For the kinematics of the process specified above, one sum is removed by momentum conservation setting $q_1' = q_1 + q_2$. Energy conservation removes another integral, setting $q_2' = -(3\xi^2/2) q_1 q_2 (q_1+q_2)$. The remaining integrals can, after rescaling of momentum variables in units of temperature, be brought to a dimensionless double-integral. This results in

$$\tau_\rho^{-1} = \frac{3 c_{12}}{(4\pi)^4} \left(\frac{V_{\rho\rho}}{v_\rho}\right)^2 T \left(\frac{T}{p_F v_\rho}\right)^4, \tag{73}$$

where the coefficient

$$c_{12} = \int_0^\infty \frac{x^2 y^2 (x+y) e^{x+y} dx\, dy}{(e^x-1)(e^y-1)(e^{x+y}-1)}$$

and $V_{\rho\rho} = p_F^2 \Gamma_{\rho\rho}$.

The two-spin scattering can be analyzed in the same way, starting out from Eq. (61) by changing $N^\rho \to N^\sigma$. The crucial difference is in the momentum dependence of the scattering rate, which is enhanced by spin-flip processes. The resulting spin wave thermalization rate due to two-boson scattering processes reads

$$\tau_\sigma^{-1} \sim \left(\frac{V_{\sigma\sigma}}{v_\sigma}\right)^2 T \left(\frac{T}{p_F v_\sigma}\right)^2. \tag{74}$$

This final estimate exhausts all possible scattering processes emerging from the quartic corrections to the linear Luttinger liquid model.

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