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CYLINDRICAL GRAVITATIONAL PULSE WAVEGUIDE EXCITATIONS

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The foundations and principles of the theory of gravitation are based on the idea that space and time can be represented by a Riemannian (Lorentzian) variety, which consists in imposing purely geometrical requirements. Taking into account these constraints, Einstein's theory of relativity is recognized by relativists as the ideal in gravitation theory [1]. Although this theory predicts the existence of gravitational waves, many doubts arise on this subject where Einstein declares: "Together with a young collaborator I arrived at the interesting result that gravitational waves do not exist, though they had been assumed to be a certainty to the first approximation. This shows that nonlinear gravitational wave field equations tell us more or, rather, limit us more than we had believed up to now" [2]. The doubt emitted by Einstein concerning the theory of the gravitational waves was to know if the gravitational radiation has a real existence [3]. On this intriguing query, the numerical works of Piran and Stark [4] confirm the real existence of the gravitational radiation. This significant advance leads us to question about the exact solutions of the field equations and their physical interpretations concerning the relation between field and matter [1, 5]. Concerning the exact solutions of the field equations in relativity theory, the 1970's and late 1980's are considered as the legendary era with the appearance of the soliton generating transformation methods [5]. Among these various methods, we will make allegiance to the inverse scattering method (ISM) of Belinskii and Zahkarov [6] which rests on the integrability of the Einstein field equations in dimension two whose contruction gave rise to the "gravitational soliton." This new concept from ISM allows to put in evidence the phenomenon of temporal shift [7] and many other phenomena mentioned with success in this work [5]. We note that in this procedure emanating from the ISM, some solutions such as the gravitational cylindrical soliton offer the possibility to study the phenomenon of gravitational collapse [8]. In the same vein, this solution could serve as an interesting element in the application of quantum information theory [9]. The integrability in dimension five of the Einstein field equations allows the modification of the Belinskii and Zahkarov ISM [6] above, the improved Pomeransky ISM [10] which is a fundamental tool in the generation of black holes. This approach allows the construction of new gravitational solitons and their direct applications [11–13]. It permits the clarification of the studies on the applications of gravitational solitons [11, 14]. The real observation of the gravitational waves by the LIGO-Virgo science team [15] in a recent investigation, showing the typical profile of the propagating waves and therefore the existence of non zero energy densities of these structures, would actually foster the set-up of underlying analytical orientations to unite, from the exact solutions of the field equations, the points of convergences and divergences [3,4,16] concerning the waves of impulses or waves of Einstein and Rosen (ER) [16]. For that, we account for the relevant

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remark of Alekseev [17] which shows that the soliton can provide complete information during its propagation in the spacetime of Kasner.

In this paper, motivated by the above, we propose some underlying approach using the solution of two cylindrical pole-conjugate solitons generated by the improved Pomeransky ISM [10] while associating the numerical method of Piran and Stark [4], to construct the ER waves. In this procedure, we construct the explosion and implosion waves as described by Weber and Wheeler [3]. Then, we show the existence of different energy densities relative to the ER waves [14, 16]. In this context, we discuss the ER-metric [18] within the viewpoint of investigation of the soliton dynamics.

The organization of the paper is set as follows: in Sec. 2, we present the ER-metric [18] as well as the field equations governing the behavior of the gravitational wave while introducing the soliton solutions from Pomeransky's ISM [10]. We introduce the ER-metric [18] and the three Einstein field equations. Thus, we consider that a four-dimensional spacetime has a symmetry, then giving the existence of two fields of navigating Killing vectors, an axisymmetric Killing vector $\partial/\partial\phi$ and a spatially translational Killing vector $\partial/\partial z$, where the coordinate of the polar angle ϕ and the coordinates z have the ranges $0 \le \phi < 2\pi$ and $-\infty < z < +\infty$. Validating these different hypotheses of symmetry, we start from the general form of the Jordan and Ehlers metric [4, 13, 19]. We eliminate the nonlinear term $\omega = 0$ and obtain the following expressions:

$$ds^2 = e^{2(\gamma-\psi)}(d\rho^2 - dt^2) + \rho^2 e^{-2\psi}\, d\phi^2 + e^{2\psi}\, dz^2, \quad (1)$$

$$\psi_{,tt} - \frac{\psi_{,\rho}}{\rho} - \psi_{,\rho\rho} = 0, \quad (2)$$

$$\gamma_{,\rho} = \rho(\psi_{,t}^2 + \psi_{,\rho}^2), \quad (3)$$

$$\gamma_{,t} = 2\rho\psi_{,t}\,\psi_{,\rho}\,. \quad (4)$$

We note that (ρ, z, ϕ) represents the cylindrical coordinates and t the time. We specify that the different functions ψ and γ depend on ρ and t. In this metric including the Einstein field equations, ψ represents a dynamic degree of freedom of the gravitational field and γ plays the role of the gravitational energy of the system. It is also noted that the previous observables written with comma as subscript denotes the partial derivatives with the associated variables. We introduce, the solutions of Piran and Stark [4] relative to the field equations in the following form:

$$A_+ = 2(\psi_{,t} + \psi_{,\rho}) \quad (5)$$

and

$$B_+ = 2(\psi_{,t} - \psi_{,\rho}). \quad (6)$$

In our investigations, the quantity A_+ represents the amplitude of the explosion wave and B_+ represents the amplitude of the implosion wave. We introduce the wave vectors of explosions and implosions as defined by Piran and Stark [4] in the following form:

$$u = \frac{1}{2}\,(t - \rho) \quad (7)$$

and

$$v = \frac{1}{2}\,(t + \rho). \quad (8)$$

Knowing the different expressions of vectors mentioned above, we simplify the expressions of the amplitudes of the explosive and implosive waves [13, 14] in the following form:

$$A_+ = 2\psi_{,v} \quad (9)$$

and

$$B_+ = 2\psi_{,u}\,. \quad (10)$$

We used these different expressions above to demonstrate the decomposition of the cylindrical gravitational pulse wave into explosion and implosion waves according to the radial ρ and temporal t coordinates, whose physical implications are represented in Figs. 1 and 3 followed with some descriptions in captions. Using the simplified expressions from the amplitudes of the explosion and implosion waves, we rewrite the field equations in the following form:

$$A_{+,u} = \frac{A_+ - B_+}{2\rho} \quad (11)$$

and

$$B_{+,v} = \frac{A_+ - B_+}{2\rho}. \quad (12)$$

We introduce in the Einstein field equations for energy densities, the expression of the amplitudes of the explosion and implosion waves in the following form:

$$\gamma_{,t} = \frac{\rho}{8}\,(A_+^2 - B_+^2) \quad (13)$$

and

$$\gamma_{,\rho} = \frac{\rho}{8}\,(A_+^2 + B_+^2). \quad (14)$$

We specify that the field equations related to the energy density $\gamma_{,t}$ represents the non-gravitational energy density of the wave and $\gamma_{,\rho}$ the gravitational energy density. We note that the expressions of the different energy densities as functions of the different amplitudes A_+ and B_+ of the explosion and implosion waves show that the propagation of the cylindrical gravitational impulse wave is vector of energy, of which Figs. 2

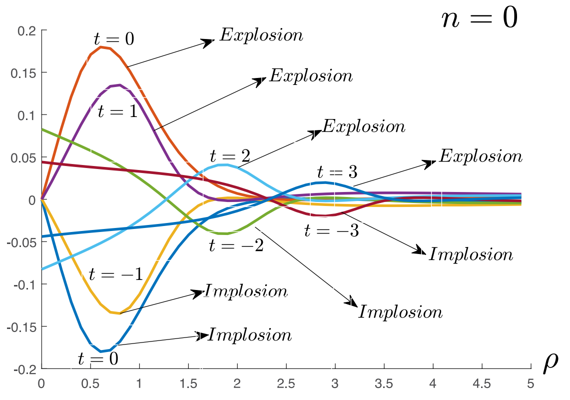

Fig. 1. We observe that the implosion wave B_+ is focused near $\rho = -t$ on the implosion course for negative values of t. While we observe that the explosion wave A_+ is concentrated near $\rho = +t$ when it reexpands out from the axis of symmetry. We use $t = \pm 1, \pm 2, \pm 3$ and $(k, \theta, q) = (2, n\pi/4, 1)(n = 0)$

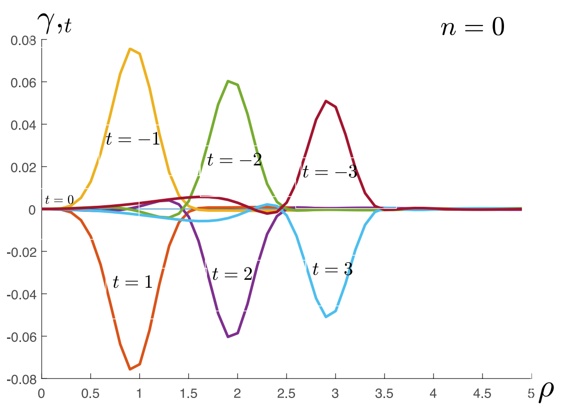

Fig. 2. We have the representation of the energy density with the following contions: $(k, \theta, q) = (4, n\pi/4, 1)(n = 0)$ with $t = \pm 1, \pm 2, \pm 3$. This shows that the gravitational waves are well localized in the spacetime manifold

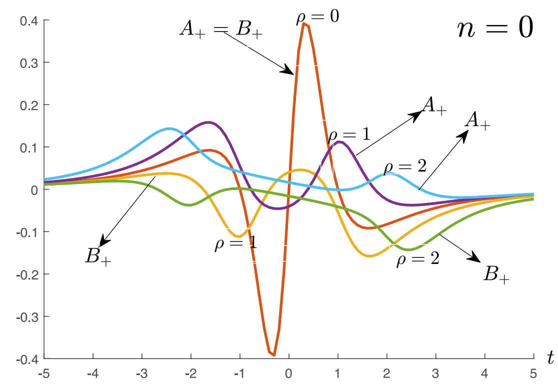

Fig. 3. We show the different behaviors of explosive and implosive waves under the following conditions: $(k, \theta, q) = (2, n\pi/4, 1)(n = 0)$ with $\rho = 0, 1, 2$

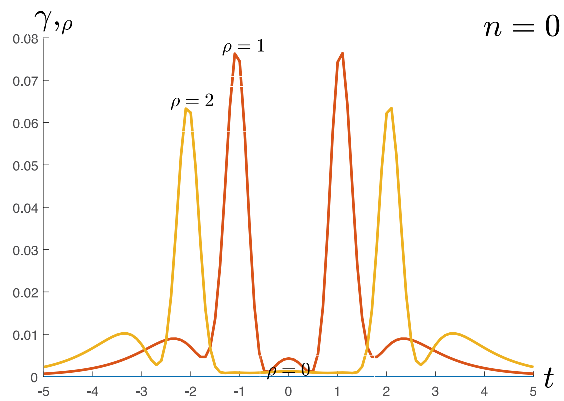

Fig. 4. We have the behavior of the gravitational energy density concerning the following values: $(k, \theta, q) = (4, n\pi/4, 1)(n = 0)$ with $\rho = 0, 1, 2$. This shows the well-localized of the gravitational waves within the spacetime manifold

and 4 are tangible proof. All the equations obtained with Piran and Stark [4] have a major and fundamental interest in the comprehension of the phenomena as mentioned in introduction. We introduce in the expressions of the time t and the radial coordinates ρ, the cartesian coordinates in the following form [13]:

$$t = qxy, \tag{15}$$

$$\rho = q\sqrt{(x^2+1)(y^2-1)}, \tag{16}$$

where q represents a constant. We calculate the following quantities:

$$dt^2 = q^2(y^2\,dx^2 + x^2\,dy^2 + 2xy\,dx\,dy) \tag{17}$$

and

$$d\rho^2 = \frac{q^2}{(x^2+1)(y^2-1)}\left(x^2(y^2-1)^2\,dx^2 + y^2(x^2+1)^2\,dy^2\right) + \frac{q^2}{(x^2+1)(y^2-1)}\left(2xy(x^2+1)(y^2-1)\,dx\,dy\right). \tag{18}$$

From the different quantities calculated previously, we modify the ER-metric [18] into the form below:

$$ds^2 = e^{2\psi}\,dz^2 + \rho^2 e^{-2\psi}\,d\phi^2 + q^2(x^2+y^2)e^{2(\gamma-\psi)}\left(-\frac{dx^2}{x^2+1} + \frac{dy^2}{y^2-1}\right). \tag{19}$$

We determine the functions ψ and γ belonging to the Einstein field equations as well as to the ER-metric [18]. For this, we use the solutions of the cylindrical solitons [10, 13] to construct the different amplitudes of the explosion and implosion wave and we obtain the following relations:

$$e^{2\psi} = \frac{\mathcal{Y}}{\mathcal{X}}, \qquad (20)$$

$$e^{2(\gamma-\psi)} = \frac{\chi}{4096q^6(x^2+y^2)^6}, \qquad (21)$$

where

$$\begin{aligned}\chi = a_i^4(y-1)^2(y+1)^6+2a_i^2(y+1)^2\big(a_r^2(y-1)^2(y+1)^4+\\ +64q^2\big(x^4\big(y(9y-8)+1\big)+2x^2\big(y(y+4)-3\big)y^2+y^6+y^4\big)\big)-\\ -512a_ia_rq^2x(y+1)^2\big(x^2-(y-2)y\big)\big(x^2(2y-1)+y^2\big)+\\ +a_r^4(y-1)^2(y+1)^6+\\ +128a_r^2q^2(y+1)^2\big(2x^6+x^4\big((8-3y)y-1\big)+\\ +2x^2y^2\big(2(y-2)y+3\big)+y^6-y^4\big)+\\ +4096q^2(x^2+y^2)^4 \qquad (22)\end{aligned}$$

and

$$\begin{aligned}\mathcal{Y} = a_i^4(y^2-1)^2+2a_i^2(y^2-1)\big(a_r^2(y^2-1)^3+\\ +64q^2\big(x^4(9y^2-1)+2x^2(y^2+1)y^2+y^6-y^4\big)\big)-\\ -1024a_ia_rq^2x(x^2+1)y(y^2-1)(x-y)(x+y)+\\ +a_r^4(y^2-1)^4+128a_r^2q^2(y^2-1)\big(2x^6+x^4+\\ +(4x^2+1)y^4-3(x^2+2)x^2y^2+y^6\big)+\\ +4096q^4(x^2+y^2)^4. \qquad (23)\end{aligned}$$

In the different expressions obtained, a_i and a_r are real-valued variables. We introduce a relation between the different cartesian and cylindrical variables, and we obtain the following two relations [20].

For $\rho \ll |t|$, we obtain

$$x = t - \frac{\rho^2 t}{2(1+t^2)} + \Theta(\rho^4) \qquad (24)$$

and

$$y = 1 + \frac{\rho^2}{2(1+t^2)} + \Theta(\rho^4). \qquad (25)$$

For $\rho \gg |t|$, we obtain

$$y = \rho + \frac{1-t^2}{2\rho} + \Theta(\rho^{-2}) \qquad (26)$$

and

$$x = \frac{t}{\rho} + \frac{t(t^2-1)}{2\rho^3} + \Theta(\rho^{-4}). \qquad (27)$$

Expressions (24)–(27) are used to highlight the different behaviors of the gravitational pulse wave mentioned above during its propagation in the ER-spacetime [18].

Next, in Sec. 3, we analyze the two soliton solution obtained by calculating the amplitudes of the incoming and outgoing waves. They are assimilated to the explosions and implosions waves from Weber and Wheeler [3], viewpoint taking into account the different energy densities. Following the previous expressions, we observe how gravitational pulse waves would propagate as exploding and imploding waves, as well as different densities of energy in spacetime through a multitude of viewpoints. For a clear understanding, we introduce the different parameters which are the modulus k and the angle θ of the complex parameter $k = |a_r + ia_i| = |a|$, $\theta = \mathrm{Arg}(a)$. In the following analysis, we only consider the case $q = 1$, because the parameter q can be normalized by a scaling of the coordinates. For the investigation of the given orientations, we define the following spacetime: $-5 \le t \le 5$, $-5 \le \rho \le 5$, and $\theta = n\pi/4$ with $n = 0,\ldots,3$. In the following, we aim at investigating detailed behavior of waves propagating near the limits of spacetime, with a particular focus on waves of explosions and implosions as well as energy densities.

1. Timelike infinity. Next we consider the asymptotic behaviors of the waves at late time $t \to \infty$. At $t \to \infty$, the metric behaves as

$$\begin{aligned}ds^2 = \Big(1-\frac{a_r^2}{4t^2}\Big)dz^2 + \rho^2\Big(1+\frac{a_r^2}{4t^2}\Big)d\phi^2 +\\ +\Big(1+\frac{a_r^2}{4t^2}\Big)(d\rho^2 - dt^2). \qquad (28)\end{aligned}$$

This metric allows us to highlight the representation obtained by Weber and Wheeler [3], by associating the two numerical solutions while paying particular interests to the energy density.

2. Spacelike infinity. Let us study the behavior of these gravitational waves when $\rho \to \infty$. At the limit of $\rho \to \infty$, we get a new expression of the metric in the form

$$\begin{aligned}ds^2 \approx \Big(1-\frac{4|a|^2q}{(|a|^2+64q^2)\rho}\Big)dz^2 +\\ +\rho^2\Big(1+\frac{4|a|^2q}{(|a|^2+64q^2)\rho}\Big)d\phi^2 +\\ +\frac{(|a|^2+64q^2)^2}{4096q^2}(d\rho^2-dt^2). \qquad (29)\end{aligned}$$

3. Axis. Now we look at the behavior of the waves on the axis of symmetry $\rho = 0$. Near the axis, the metric behaves as follows:

$$ds^2 \approx \frac{4(q^2+t^2)}{4(q^2+t^2)+(ta_r-qa_i)^2}\,dz^2 + \\ + \rho^2\,\frac{4(q^2+t^2)+(ta_r-qa_i)^2}{4(q^2+t^2)}\,d\phi^2 + \\ + \frac{4(q^2+t^2)+(ta_r-qa_i)^2}{4(q^2+t^2)}\,(d\rho^2-dt^2). \quad (30)$$

This metric allows us to show that in the vicinity of the axis $\rho = 0$, the explosion and implosion waves during their propagation in this region of space are merged. In this case, we study the representation of gravitational waves as well as its gravitational density. We see that when $\rho = 0$, $B_+ = A_+$.

Finally, Sec. 4 is devoted to conclusion and perspectives.

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