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THE ENERGY SPECTRUM OF A NON-IDEAL FERMI GAS

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We have evaluated the energy spectrum and ground state energy of a non-ideal Fermi gas with repulsive interactions, using an expansion in powers of the ratio of the range of the potential to the mean distance apart of the particles (gas approximation). We have obtained the first two terms of the expansion.

INTRODUCTION

IT is well known that in many cases one can consider the excited states of a system of interacting Fermi particles as a gas of elementary excitations — quasiparticles. The energy of a quasiparticle is determined by its momentum in such a way that the energy of the excitation of the system ϵ_s is equal to $\epsilon(p_1) - \epsilon(p_2)$, where $p_1 > p_0 > p_2$ with p_0 the momentum at the Fermi surface. Such a spectrum is called a spectrum of the "Fermi type." A description of a system by means of the method of quasiparticles is exact only in the case of an ideal gas. If there are interactions between the particles, the excited states of the "Fermi type" do not represent the exact stationary states of the systems. This leads to the damping of the quasiparticles.

It was shown in Ref. 1 that it is convenient to apply the methods of quantum field theory to determine the energy spectrum of a system. The energy $\epsilon(p)$ and attenuation $\gamma(p)$ of the quasiparticles can be found as the poles of the analytical continuation of the single-particle Green function $G(p)$. In the present paper we shall apply the methods of

quantum field theory to the problem of a non-ideal Fermi gas in which the interaction between the particles is short range $na^3 \ll 1$ (n is the density of the particles in the system and a the range of the potential), but not necessarily weak. We assume that the radially symmetrical potential $V(r)$ is positive and that the interaction between the particles is not retarded. We expand in powers of the parameter $p_0 f_0$, where f_0 is the real part of the scattering amplitude for small momenta. We shall find the energy spectrum of the system and the ground state energy up to quadratic terms in this parameter. Terms corresponding to higher powers than the cubic can not be expressed by means of two-particle parameters which makes it difficult to obtain them in a general form.* This fact was first remarked on in Ref. 2 in connection with the evaluation of the ground state energy.

1. SINGLE PARTICLE GREEN FUNCTION. THE METHOD OF GRAPHS

It is well known that the single particle Green

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*The author is obliged to E. M. Lifshitz for this comment.

function of a system is given by the equation

$$iG(x - x') = \langle T\{\psi(x)\psi^+(x')S\}\rangle / S_{00}, \qquad (1)$$

where ψ and ψ^+ are taken in the interaction rep-

resentation and where the averaging is performed over the ground state of the non-interacting particles. The S-matrix of the system is in our case given by the equation

$$S \equiv T\left\{\exp\left(-i\int H'dt\right)\right\} = T\left\{\exp\left[-\frac{i}{2}\int V(\mathbf{r}_1 - \mathbf{r}_2)\psi^+(\mathbf{r}_1t)\psi^+(\mathbf{r}_2t)\psi(\mathbf{r}_2t)\psi(\mathbf{r}_1t)\,dv_1dv_2dt\right]\right\} \qquad (2)$$

(we choose our units in such a way that $\hbar = m = 1$). It is convenient to change this definition slightly by adding a dependence on a time t_2 to the dependence on the variable $\mathbf{r}_2$ and by introducing an extra integration over time,

$$S = T\left\{\exp\left[-\frac{i}{2}\int dx_1 dx_2 U(x_1 - x_2)\right.\right.$$
$$\left.\left.\psi^+(x_1)\psi^+(x_2)\psi(x_2)\psi(x_1)\right]\right\}, \qquad (2')$$

where

$$U(x_1 - x_2) = V(\mathbf{r}_1 - \mathbf{r}_2)\delta(t_1 - t_2). \qquad (3)$$

In $(2')$ the integration over the variables x_1 and x_2 is taken over the whole of the infinite four-dimensional space.

If we want to expand the S-matrix of (1) in powers of the interaction U, we must know the average value of the T product of the ψ operators. According to Wick's theorem this T product can be written in the form of a sum of normal products and different connections between operators. To apply the methods of quantum field theory it is necessary to put the average values of the normal products equal to zero. This condition will be fulfilled, if we take ψ in the form

$$\psi(\mathbf{r}) = u(\mathbf{r}) + v^+(\mathbf{r}); \quad u(\mathbf{r}) = V^{-1/2}\sum_{p>p_0} a_p e^{i\mathbf{p}\mathbf{r}},$$
$$v^+(\mathbf{r}) = V^{-1/2}\sum_{p<p_0} a_p e^{i\mathbf{p}\mathbf{r}} \qquad (4)$$

and if we define the normal products as those products in which all operators u and v are on the right, and all operators u^+ and v^+ on the left. In this representation the operators u and v play the role of annihilation operators and the operators u^+ and v^+ the role of creation operators for particles and holes. The average of the N products is equal to zero and the connection of two operators is equal to the Green function of non-interacting particles,

$$\dot{\psi}(x)\dot{\psi}^+(x') = iG_0(x - x') = \langle T\{\psi(x)\psi^+(x')\}\rangle. \qquad (5)$$

In the momentum representation the Green function of non-interacting particles is of the form

$$G_0^{-1}(p) \equiv G_0^{-1}(\mathbf{p}, \varepsilon) = \varepsilon - \varepsilon_p^0 + i\delta\theta(\mathbf{p}), \qquad (6)$$

where $\varepsilon_p^0 = \frac{1}{2}\mathbf{p}^2$ and

$$\theta(\mathbf{p}) = 1 - 2n_p = \begin{cases} 1 & |\mathbf{p}| > p_0 \\ -1 & |\mathbf{p}| < p_0 \end{cases},$$

with n_p is the occupation number of the non-interacting particles in the ground state. We can use the corresponding graphs to assign a definite arrangement of T products to an assembly of connections. The graphs consist of full drawn and dotted lines; each full drawn line corresponds to a particle propagation function $iG_0(x - x')$, or in momentum representation $iG_0(p)$, and each dotted line to an interaction, $iU(x_1 - x_2)$ or in momentum representation $iU(q)$, and at each vertex the law of conservation of the four-dimensional momentum is obeyed, $p_1 - p_2 + q = 0$. Since the interaction does not enter into the normal products, we can restrict ourselves in calculating them to the connections of operators which enter in the same H', that is, to simultaneous operators. To determine those connections we note that in the interaction H' the operators ψ^+ are to the left of the operators ψ so that $G_0(\mathbf{p}, \tau)$ for $\tau = 0$ must necessarily be taken to be $G_0(\mathbf{p}_1, -0) = in_p$.
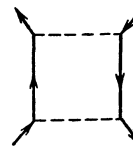

FIG. 1

For our further discussion it is very important that the interaction is not retarded [Eq. (3)]. It is convenient to indicate this absence of retardation by drawing the dotted lines in the graphs horizontally. This representation makes it possible to judge from the graphs the number of particles and holes taking part in the process. The process corresponding to the graph of Fig. 1, for instance, involves one particle and one hole since for each arrangement of vertices one of the lines of 1 or 2 is directed downwards on account of Eq. (3).

2. ESTIMATE OF THE GRAPHS. GAS APPROXIMATION

We introduce now the main part of the energy eigenvalue of the particle $\Sigma(\mathbf{p})$,

$$G^{-1}(p) = G_0^{-1}(p) - \Sigma(p) = \varepsilon - \varepsilon_p^0 - \Sigma(p) . \qquad (7)$$

In the first approximation of perturbation theory $\Sigma(p)$ is determined by the two graphs of Fig. 2. Graph 2a corresponds to a non-exchange scattering by a particle of the Fermi sea (background particle) and graph 2b to an exchange scattering. In the case in which we are interested where $p_0 a \ll 1$ both graphs are equal and are of the order $nV \sim p_0^2 \cdot p_0 a \cdot V_0 a^2$, where V is the Fourier component of the potential for small values of the momentum, and V_0 is the value of the potential inside its range.
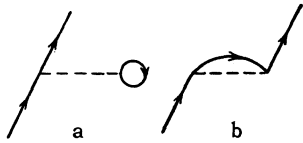

FIG. 2

We can consider now graphs which are more complicated in three different ways.

1. We can increase the number of dotted lines which connect solid lines which are in the same direction (graphs a and b in Fig. 3).

2. We can increase not only the dotted lines of the first kind, but also those which connect solid lines which are in opposite directions (graph c in Fig. 3).

3. We can increase the number of closed loops which are connected by dotted lines to the basic graph [it is well known that unconnected closed loops are eliminated by the denominator in Eq. (1)].
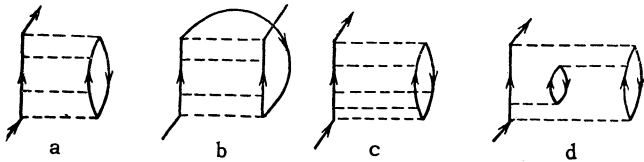

FIG. 3

Let us now estimate the value of these graphs. For graphs of the first kind each additional dotted line adds to Σ a factor $G_0^2 U(q)$ and one integration over the four-dimensional momentum q. As $U(q)$ only depends on q we get after integrating over the fourth component of q an integral of the form

$$\int \frac{dq}{q^2} V(q), \qquad (8)$$

the convergence of which for large values of q is determined by the function $V(q)$. An estimate of this integral leads to Va^{-1} or $V_0 a^2$, i.e., the parameter of the perturbation theory. The collection of graphs of the first kind gives thus the perturbation theory series. For graphs of the second kind, the additional part has the form pictured in Fig. 1. Since one of the two lines corresponds to

the occurrence of a hole, the momentum of which does not exceed the momentum at the Fermi surface, p_0, the integration over q in Eq. (8) will in that case be taken over a limited region of dimensions p_0. We obtain as a result $p_0 V$ or $p_0 a \cdot V_0 a^2$, i.e., graphs of the second kind contain apart from the perturbation theory parameter also an additional "gaseousness" parameter $p_0 a$. The difference between the magnitudes of graphs of the first and of the second kind can be given a simple physical interpretation. Indeed, graphs of the first kind correspond to a further approximation of the perturbation theory in terms of the interaction between two particles, while graphs of the second kind correspond to a further approximation in terms of an interaction between a particle and a hole. The interaction with a hole, however, is essentially an interaction with a background particle so that in the processes corresponding to the graphs 3a and 3b one background particle is taking place, but in the process 3c two background particles. This result can also be extended to graphs of the third kind (3d). Graphs 3c and 3d will be discarded. An exact estimate of these graphs which is not based upon perturbation theory is given in Sec. 5. The result of this estimate shows that graphs of the kind 3c and 3d can necessarily only be considered in the third approximation in terms of the small parameter $p_0 f$.

The graphs of the kind 3a and 3b which determine the energy eigenvalues of the particles in the first and second gas approximation can be given in the form of block diagrams as given in Fig. 4. The
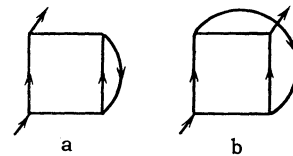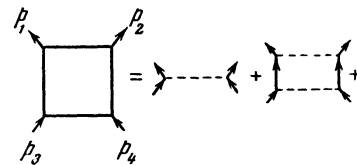

FIG. 4

FIG. 5

square denotes as usual the Feynman diagram assembly describing the interaction of two particles in the "ladder" approximation (Fig. 5). This quantity we shall denote by

$$- i\Gamma (p_1 p_2, \ p_3 p_4)$$

and we shall call it the effective interaction poten-

tial. Graphs 4a and 4b have the same structure as the graphs 2a and 2b corresponding to the first approximation of perturbation theory with one difference, however, namely that the dotted lines corresponding to the Born approximation have been taken into account in our effective potential. The values of the particle energy eigenvalues determined by these graphs is of the form

$$\Sigma(p) = -i \int dp' G_0(p') \Gamma(pp', pp')$$
$$+ i \int dp' G_0(p') \Gamma(pp', p'p). \tag{9}$$

Here and henceforth we shall use the following notation

$$dp = d\mathbf{p} d\varepsilon / 2\pi, \quad d\mathbf{p} = dp_x dp_y dp_z / (2\pi)^3;$$
$$\delta(p) = 2\pi\delta(\mathbf{p})\delta(\varepsilon), \quad \delta(\mathbf{p}) = (2\pi)^3 \delta(p_x) \delta(p_y) \delta(p_z).$$

Equation (9) does not take the presence of particle spin into account. To do that we note first of all that the Green function $G_0(p)$ contains the delta function $\delta_{s,s'}$ (where s and s' are the projections of the spin at the points x and x') which we have omitted, and this means that the particles propagate without change in spin. This result is also still valid in the case where the particles interact during their propagation with other particles, since the potential does not depend on the spin variable. The presence of spin leads thus to the appearance of a factor $\delta_{s_1 s_2} \delta_{s_3 s_4}$ in $\Gamma(p_1 p_2, p_3 p_4)$. The result of summing over the spin variable s' in Eq. (9) is a factor $2s + 1$ for the first term and a factor 1 for the second one. This result has a simple physical meaning. The graph 4b corresponds to an exchange scattering in which one must take into account only those background particles which have a spin the projection of which coincides with the projection of the spin of the impinging particle. In contradistinction to this, graph 4a corresponds a non-exchange scattering process in which all particles in the Fermi sea must be taken into account. If we restrict ourselves to the case of particles of spin $\frac{1}{2}$ we get finally

$$\Sigma(p) = -2i \int dp' G_0(p') \Gamma(pp', pp')$$
$$+ i \int dp' G_0(p') \Gamma(pp', p'p). \tag{9'}$$

3. THE EFFECTIVE INTERACTION POTENTIAL

To determine the effective interaction potential we introduce a function Q which is connected with Γ by the following equation

$$\Gamma(p_1 p_2, p_3 p_4) = \int dq U(q) Q(p_1 - q, p_2 + q, p_3 p_4). \tag{10}$$

The function Q differs from the two-particle Green function K in the "ladder" approximation

by the absence of initial outside lines with momenta p_3 and p_4. We can thus obtain the equation for Q from the corresponding equation for the function K by dividing by the product $iG_0(p_3) iG_0(p_4)$. We have thus finally for $Q' = \delta(p_1 + p_2 - p_3 - p_4) Q$,

$$Q'(p_1 p_2, p_3 p_4) = \delta(p_1 - p_3) \delta(p_2 - p_4)$$
$$+ iG_0(p_1) G_0(p_2) \int dq U(q) Q'(p_1 - q, p_2 + q, p_3 p_4). \tag{11}$$

If we go over to the relative momenta p and p' and the moment of the center of mass g,

$$p = (p_1 - p_2)/2, \quad p' = (p_3 - p_4)/2,$$
$$g = p_1 + p_2 = p_3 + p_4, \tag{12}$$

we get the equation

$$Q(p, p', g) \equiv Q\left(\frac{g}{2} + p, \frac{g}{2} - p; \frac{g}{2} + p', \frac{g}{2} - p'\right)$$
$$= \delta(p - p') \tag{13}$$
$$+ iG_0\left(\frac{g}{2} + p\right) G_0\left(\frac{g}{2} - p\right) \int dq U(q) Q(p - q, p', g).$$

The potential $U(q)$ in Eq. (10) does not depend on the fourth component of q so that it is sufficient to know the function Q integrated over the fourth component of the relative momentum p, ε, in order to determine the effective potential. If we denote that function by $\chi(\mathbf{p}, \mathbf{p}', g)$ and integrate Eq. (13) over ε we find

$$\chi(\mathbf{p}, \mathbf{p}', g)$$
$$- \frac{N(\mathbf{p})}{E - \mathbf{p}^2 + i\delta N(\mathbf{p})} \int d\mathbf{q} V(\mathbf{q}) \chi(\mathbf{p} - \mathbf{q}, \mathbf{p}', g) = \delta(\mathbf{p} - \mathbf{p}'). \tag{14}$$

In Eq. (14) we have $E = g_0 - \frac{1}{4}\mathbf{g}^2$ (where g_0 is the fourth component of g) and the factor $N(\mathbf{p})$ takes the Pauli exclusion principle into account inasfar as it applies to the initial background,

$$N(\mathbf{p}) = 1 - n_{\mathbf{g}/2+\mathbf{p}} - n_{\mathbf{g}/2-\mathbf{p}}. \tag{15}$$

For further calculations it is convenient to consider the effective potential as a function of the relative and total momenta which are defined by Eqs. (12). If we write

$$\Gamma(\mathbf{p}, \mathbf{p}', g) \equiv \Gamma\left(\frac{g}{2} + p, \frac{g}{2} - p; \frac{g}{2} + p', \frac{g}{2} - p'\right),$$

we obtain from (10) the following connection between the effective potential and the function χ,

$$\Gamma(\mathbf{p}, \mathbf{p}', g) = \int d\mathbf{q} V(\mathbf{q}) \chi(\mathbf{p} - \mathbf{q}, \mathbf{p}', g). \tag{16}$$

Equation (14) can not be solved in its general form; our problem is to express the solution of this equation in terms of the scattering amplitudes of the particles. We consider first of all the problem of the scattering in vacuo. In that case $N(\mathbf{p}) \equiv 1$ and Eq. (14) is of the form

$$\chi_0(\mathbf{p}, \mathbf{p}', g) - \frac{1}{E - p^2 + i\delta} \int d\mathbf{q} V(\mathbf{q}) \chi_0(\mathbf{p}, \mathbf{p}', g) = \delta(\mathbf{p} - \mathbf{p}').$$

$$(14')$$

Equation (14'), multiplied by $E - p^2 + i\delta$, is the same as the inhomogeneous Schrödinger equation for the relative motion of two particles. It is therefore easy to express its solution in terms of $\psi_{\mathbf{k}}(\mathbf{p})$, the wave function of the relative motion of particles which are scattered ($\mathbf{k}$ is the relative momentum at infinity),

$$\chi_0(\mathbf{p}, \mathbf{p}', g) = (E - p'^2 + i\delta) \int d\mathbf{k} \frac{\psi_{\mathbf{k}}(\mathbf{p}) \psi_{\mathbf{k}}^*(\mathbf{p}')}{E - k^2 + i\delta}. \quad (17)$$

Let us introduce the scattering amplitude $f(\mathbf{p}, \mathbf{k})$ of the particles, by the equation

$$f(\mathbf{p}, \mathbf{k}) = \int d\mathbf{q} V(\mathbf{q}) \psi_{\mathbf{k}}(\mathbf{p} - \mathbf{q}). \quad (18)$$

This amplitude differs from the usual one by a factor -4π, that is, the usual amplitude is equal to $-f(\mathbf{p}, \mathbf{k})/4\pi$. The wave function $\psi_{\mathbf{k}(\mathbf{p})}$ is connected with the amplitude by the relation

$$\psi_{\mathbf{k}}(\mathbf{p}) = \delta(\mathbf{p} - \mathbf{k}) + \frac{f(\mathbf{p}, \mathbf{k})}{k^2 - p^2 + i\delta}. \quad (19)$$

Using this relation for $\psi_{\mathbf{k}}^*(\mathbf{p}')$ and substituting it into (17) we get

$$\chi_0(\mathbf{p}, \mathbf{p}', g) = \psi_{\mathbf{p}'}(\mathbf{p})$$

$$+ \int d\mathbf{k} \psi_{\mathbf{k}}(\mathbf{p}) f^*(\mathbf{p}', \mathbf{k}) \left\{ \frac{1}{E - k^2 + i\delta} + \frac{1}{k^2 - p'^2 - i\delta} \right\}, \quad (17')$$

and we get for the effective interaction potential of particles in vacuo* which is defined by an equation which is analogous to equation (16)

$$\Gamma_0(\mathbf{p}, \mathbf{p}', g) = f(\mathbf{p}, \mathbf{p}')$$

$$+ \int d\mathbf{k} f(\mathbf{p}, \mathbf{k}) f^*(\mathbf{p}', \mathbf{k}) \left\{ \frac{1}{E - k^2 + i\delta} + \frac{1}{k^2 - p'^2 - i\delta} \right\}. \quad (20)$$

If we use Eq. (II) of the Appendix, we can express Γ_0 also in a slightly different, equivalent form,

$$\Gamma_0(\mathbf{p}, \mathbf{p}', g) = f^*(\mathbf{p}', \mathbf{p})$$

$$+ \int d\mathbf{k} f(\mathbf{p}, \mathbf{k}) f^*(\mathbf{p}', \mathbf{k}) \left\{ \frac{1}{E - k^2 + i\delta} + \frac{1}{k^2 - p^2 + i\delta} \right\}. \quad (20')$$

The effective interaction potential of particles in vacuo is thus equal in first approximation to the scattering amplitude $f(\mathbf{p}, \mathbf{p}')$ or to $f(\mathbf{p}, \mathbf{p}')$.

Going on to the solution of (14), we write it in the form

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*The problems connected with Γ_0 were solved jointly with S. T. Beliaev, who was working simultaneously on the analogous problem of a Bose gas.

$$\chi(\mathbf{p}, \mathbf{p}', g) - \frac{1}{E - p^2 + i\delta} \int d\mathbf{q} V(\mathbf{q}) \chi(\mathbf{p} - \mathbf{q}, \mathbf{p}', \mathbf{q})$$

$$(21)$$

$$= \delta(\mathbf{p} - \mathbf{p}') + \left\{ \frac{N(\mathbf{p})}{E - p^2 + i\delta N(\mathbf{p})} - \frac{1}{E - p^2 + i\delta} \right\} \Gamma(\mathbf{p}, \mathbf{p}', g),$$

where χ_0 is the Green function of the left hand side of this equation. Equation (21) admits thus of the following formal solution,

$$\chi(\mathbf{p}, \mathbf{p}', g) = \chi_0(\mathbf{p}, \mathbf{p}', g)$$

$$+ \int d\mathbf{k} \chi_0(\mathbf{p}, \mathbf{k}, g) \left\{ \frac{N(\mathbf{k})}{E - k^2 + i\delta N(\mathbf{k})} - \frac{1}{E - k^2 + i\delta} \right\} \Gamma(\mathbf{k}, \mathbf{p}', g).$$

If we apply (16) we can go over to an integral equation for the effective potential Γ,

$$\Gamma(\mathbf{p}, \mathbf{p}', g) = \Gamma_0(\mathbf{p}, \mathbf{p}', g) +$$

$$+ \int d\mathbf{k} \Gamma_0(\mathbf{p}, \mathbf{k}, g) \left\{ \frac{N(\mathbf{k})}{E - k^2 + i\delta N(\mathbf{k})} - \frac{1}{E - k^2 + i\delta} \right\} \Gamma(\mathbf{k}, \mathbf{p}', g).$$

$$(22)$$

Equation (22) can be solved by iteration methods since the term involving the integral is small. Indeed, the difference within the braces is different from zero only if $N \neq 1$, i.e., if the variation of k is of the order of p_0. To a first approximation we can substitute the scattering amplitude f for Γ_0. We then get, for $E \sim p_0^2$, the following order-of-magnitude estimate for the term involving the integral in (22):

$$p_0^3 f \cdot p_0^{-2} \Gamma = p_0 f \cdot \Gamma \ll \Gamma.$$

If we now take second-order terms $\ll \Gamma$ into account we can substitute in the integral in (22) $f(\mathbf{p}, \mathbf{k})$ for $\Gamma_0(\mathbf{p}, \mathbf{k}, g)$ and $\Gamma_0(\mathbf{k}, \mathbf{p}', g) \approx f(\mathbf{p}', \mathbf{k})$ for $\Gamma(\mathbf{k}, \mathbf{p}', g)$. We get the following result,

$$\Gamma(\mathbf{p}, \mathbf{p}', g) = \Gamma_0(\mathbf{p}, \mathbf{p}', g)$$

$$+ \int d\mathbf{k} f(\mathbf{p}, \mathbf{k}) f^*(\mathbf{p}', \mathbf{k}) \left\{ \frac{N(\mathbf{k})}{E - k^2 + i\delta N(\mathbf{k})} - \frac{1}{E - k^2 + i\delta} \right\} \quad (23)$$

$$= f(\mathbf{p}, \mathbf{p}') + \int d\mathbf{k} f(\mathbf{p}, \mathbf{k}) f^*(\mathbf{p}', \mathbf{k}) \left\{ \frac{N(\mathbf{k})}{E - k^2 + i\delta N(\mathbf{k})} + \frac{1}{k^2 - p'^2 + i\delta} \right\}.$$

As can be seen from Eq. (23), the effective interaction potential Γ, like Γ_0, is equal in first approximation to the scattering amplitude $f(\mathbf{p}, \mathbf{p}')$. In the second part, the integral converges for values of k^2 of the order of the larger of the quantities E or p'^2. For the energy eigenvalues of the particles $\Sigma(\mathbf{p}, \epsilon)$ in which we are interested, and for values of ϵ near $\frac{1}{2}p^2$, both quantities E and p'^2 are of the order of magnitude p_0^2. The second term in (23) is thus $p_0 f$ times smaller than the first one.

If the momenta $\mathbf{p}$ and $\mathbf{p}'$ have the same absolute magnitude we can write the effective potential in a slightly different form. Using the equation

$$\frac{1}{k^2 - p'^2 + i\delta} = P \frac{1}{k^2 - p^2} - i\pi\delta(k^2 - p'^2)$$

and Eq. (IV) of the Appendix, we get

$$\Gamma(p, p', g) = \text{Re} f(p, p') + \int dk f(p, k) f(p', k) P \frac{1}{k^2 - p'^2}$$
$$+ \int dk f(p, k) f(p', k) \frac{N(k)}{E - k^2 + i\delta N(k)}, \quad (23')$$

where the symbol P indicates the principal part of the integral.

4. THE ENERGY SPECTRUM OF THE SYSTEM. GROUND STATE ENERGY

The main part of the energy eigenvalues of the particles can be written in the form

$$\Sigma(p) = -2i \int \frac{d\varepsilon'}{2\pi} dp' G_0(p') \Gamma(q, q, g)$$
$$+ i \int \frac{d\varepsilon'}{2\pi} dp' G_0(p') \Gamma(q, -q, g),$$

where $q = (p - p')/2$, $g = p + p'$. The first two terms for Γ in (23') do not depend on the fourth component of the momentum so that integrating them over ε' reduces to calculating

$$\int \frac{d\varepsilon'}{2\pi} G_0(p', \varepsilon').$$

This integral is equal to $G_0(p', \tau)$ with $\tau = 0$, i.e., to $G_0(p', -0) = i n_{p'}$, according to the considerations of Sec. 1. The integral over ε' of the last term of (23') is elementary. We get the following results

$$\Sigma(p) = \Sigma_1(p) + \Sigma_2(p), \quad (24)$$

$$\Sigma_1(p) = 2\int dp' n_{p'} \text{Re} f(q, q) - \int dp' n_{p'} \text{Re} f(q, -q), \quad (25)$$

$$\Sigma_2(p) = \int dp' dk \{2|f(q, k)|^2 - f(q, k) f^*(-q, k)\}$$
$$\times \left\{ n_{p'} P \frac{1}{k^2 - q^2} \right. \quad (26)$$
$$- N(k) \frac{N(k) - \theta(p')}{2} \frac{1}{k^2 - q^2 - \varepsilon + \varepsilon_p^0 + i\delta[\theta(p') - N(k)]} \left. \right\}.$$

For wavelengths that are considerably longer than the range a of the potential, the real part of the amplitude will not depend on the momenta. If we therefore restrict our considerations to excitations with momenta p which satisfy the condition

$$pf \ll 1, \quad (27)$$

we can calculate the real part of the constant amplitude. Assuming that

$$\text{Re} f = 4\pi f_0, \quad (28)$$

where f_0 is the real part of the usual scattering amplitude with the opposite sign, and taking f_0 outside the integral sign in (25), we get for the first approximation to the energy eigenvalue

$$\Sigma_1 = 2\pi n f_0. \quad (29)$$

To the first approximation in the gas parameter Σ is thus a real constant quantity. The Green function has in the first approximation the following form,

$$G_1(p, \varepsilon) = 1/[\varepsilon - \varepsilon_p^0 - 2\pi n f_0 + i\delta\theta(p)]. \quad (30)$$

If we introduce instead of ϵ a new variable ϵ',

$$\varepsilon' = \varepsilon - 2\pi n f_0, \quad (31)$$

we can verify that $G_1(p, \epsilon')$ is the same as the Green function of non-interacting particles. We can thus make our calculation more precise by assuming that everywhere instead of $G_0(p, \epsilon)$ the Green function of the first approximation $G_1(p, \epsilon)$ was used. If we make our considerations more precise in this way we must in all equations replace ϵ by ϵ' so that the Green function of the second approximation has the form

$$G_2^{-1}(p, \varepsilon) = \varepsilon - \varepsilon_p^0 - 2\pi n f_0 - \Sigma_2(p, \varepsilon') = \varepsilon' - \varepsilon_p^0 - \Sigma_2(p, \varepsilon'), \quad (32)$$

where $\Sigma_2(p, \epsilon')$ is obtained from (26) by replacing ϵ by ϵ'.

We shall now calculate the energy eigenvalues of the second approximation, Σ_2. In the integral over the variable k in (26), the only appreciable contributions to the integral come from values of k in a region of the order of magnitude q. For excitations with momenta that satisfy condition (27), both arguments of the amplitude are small. The imaginary part of the amplitude can thus be neglected and the real part can be considered to be constant and can be taken outside the integral sign. Using the notation of Eq. (28), we get

$$\text{Re} \Sigma_2 = 16\pi^2 f_0^2 \int dp' dk \left\{ n_{p'} P \frac{1}{k^2 - q^2} \right.$$
$$- N(k) \frac{N(k) - \theta(p')}{2} P \frac{1}{k^2 - q^2 - \varepsilon' + \varepsilon_p^0} \left. \right\}, \quad (26')$$

$$\text{Im} \Sigma_2 = -4\pi^3 f_0^2 \int dp' dk N(k) [N(k)$$
$$- \theta(p')]^2 \delta(k^2 - q^2 - \varepsilon' + \varepsilon_p^0). \quad (26'')$$

The energy spectrum of the system is determined by the poles of the analytical continuation of the Green function (1), or, in our approximation, by the equation

$$\varepsilon_p - \varepsilon_p^0 - 2\pi n f_0 - \Sigma_2(p, \varepsilon_p^0) = 0. \quad (33)$$

After some simple calculations we get for the energy ϵ_p and the damping γ of the quasiparticles the following expressions

$$\frac{\varepsilon_p}{p_0^2} = \frac{1}{2}x^2 + \frac{2}{3\pi}p_0 f_0 + \frac{2}{15\pi^2}p_0^2 f_0^2 \frac{1}{x}\left\{11x + 2x^5 \ln\frac{x^2}{|x^2-1|} - 10(x^2-1)\ln\left|\frac{x+1}{x-1}\right| - (2-x^2)^{5/2}\ln\left|\frac{1+x\sqrt{2-x^2}}{1-x\sqrt{2-x^2}}\right|\right\}, \quad x \leqslant \sqrt{2},$$

(34)

$$\frac{\varepsilon_p}{p_0^2} = \frac{1}{2}x^2 + \frac{2}{3\pi}p_0 f + \frac{2}{15\pi^2}p_0^2 f_0^2 \frac{1}{x}\left\{11x + 2x^5 \ln\frac{x^2}{x^2-1} - 10(x^2-1)\ln\left|\frac{x+1}{x-1}\right| - 2(x^2-2)^{5/2}\cot^{-1}\sqrt{x^2-2}\right\}, \quad x \geqslant \sqrt{2},$$

$$\frac{\gamma}{p_0^2} = -\frac{1}{4\pi}p_0^2 f_0^2(1-x^2)^2,\ x \leqslant 1; \quad \frac{\gamma}{p_0^2} = \frac{1}{15\pi}p_0^2 f_0^2\frac{1}{x}\{5x^2-7+2(2-x^2)^{5/2}\},\ 1\leqslant x\leqslant\sqrt{2}.$$

$$\frac{\gamma}{p_0^2} = \frac{1}{15\pi}p_0^2 f_0^2\frac{5x^2-7}{x}, \quad \sqrt{2}\leqslant x,$$

(35)

where $x = p/p_0$. The expansion (34) in a series in the momenta has for momenta near the Fermi surface the form

$$\frac{\varepsilon_p}{p_0^2} = \frac{1}{2}x^2 + \frac{2}{3\pi}p_0 f_0 + \frac{2}{15\pi^2}p_0^2 f_0^2(11-2\ln 2)$$
$$- \frac{8}{15\pi^2}p_0^2 f_0^2(7\ln 2 - 1)(x-1) + \dots$$

(34')

From Eq. (34') we can obtain the effective mass, m^*, of particles on the Fermi surface,

$$\frac{m^*}{m} = 1 + \frac{\delta}{15\pi^2}(7\ln 2 - 1)p_0^2 f_0^2$$

(36)

and the chemical potential of the system which, as is well known[1] is equal to the energy of the quasiparticles for $p = p_0$,

$$\mu = \frac{1}{2}p_0^2\left\{1 + \frac{4}{3\pi}p_0 f_0 + \frac{4}{15\pi^2}(11-2\ln 2)p_0^2 f_0^2\right\}.$$

(37)

These expressions are the same as the ones obtained by Abrikosov and Khalatnikov.[3] Using the formula $\mu = (\partial E_0/\partial N)_V$, we find for the ground state energy of the system,[4]

$$\frac{E_0}{N} = \frac{3}{10}p_0^2\left\{1 + \frac{10}{9\pi}p_0 f_0 + \frac{4}{21\pi^2}(11-2\ln 2)p_0^2 f_0^2\right\}.$$

(38)

The damping of the quasiparticles γ for momenta p which are nearly equal to p_0 is of the form

$$\gamma = -\frac{1}{\pi}p_0^2 f_0^2(p_0 - p)^2,\ p < p_0,$$
$$\gamma = \frac{1}{\pi}p_0^2 f_0^2(p - p_0)^2,\ p > p_0,$$

(35')

i.e., proportional to the square of the deviation from the Fermi surface.

Let us now consider excitations with large momenta,

$$p_0 \ll p \ll 1/f_0.$$

The quadratic correction term of the quasiparticle energy ε_p is small for such momenta. The imaginary part has the form

$$\gamma = \frac{1}{3\pi}p_0^3 f_0^2 p = \frac{1}{4}np\sigma.$$

If we use the connection between the imaginary part of the scattering amplitude and the total cross sec-

tion σ we find

$$\varepsilon_p - i\gamma \sim \varepsilon_p^0 - 2\pi n f\,(p/2, p/2),$$

(39)

where f is the usual scattering amplitude.

The evaluation of Σ_2 as a function of the variables p and ϵ leads to a very cumbersome expression. The expansion of its real part for the case when $|x - 1| \ll 1$ and $|y| \ll 1$, where

$$y = (\varepsilon' - \varepsilon_p^0)/p_0^2,$$

is of the form

$$\mathrm{Re}\,\Sigma_2(p, \varepsilon) = \frac{1}{\pi^2}p_0^2 f_0^2\left\{\frac{2}{15}(11 - 2\ln 2)\right.$$
$$\left. - \frac{8}{15}(7\ln 2 - 1)(x-1) - 4\ln 2 \cdot y\right\}.$$

(40)

From Eq. (40) we can get the renormalization constant of the Green function Z which is connected, as is well known,[5] with the discontinuity of the momentum distribution function of the particles,

$$Z = n(p_0 - 0) - n(p_0 + 0) = 1 - 2p_0^2 f_0^2\frac{1}{\pi^2}4\ln 2$$

(41)

As far as the imaginary part of Σ_2 as a function of the variables p and ϵ is concerned, there are a number of regions in which $\mathrm{Im}\,\Sigma$ has a different analytical form. We shall show that this function tends to zero at the value $\epsilon' = p_0^2$, independent of the value of the momentum. We write Eq. (26'') in the following form

$$\mathrm{Im}\,\Sigma(p, \varepsilon) = -4\pi^3 f_0^2\int np' dp_1 dp_2\{n_{p'}(1 - n_{p_1})(1 - n_{p_2})$$
$$- (1 - n_{p'})n_{p_1}n_{p_2}\}$$
$$\times \delta\left(\frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 - \frac{1}{2}p'^2 - \varepsilon'\right)\delta(p_1 + p_2 - p' - p).$$

The first term within the braces determines the attenuation of the quasiparticles, and the second one the attenuation of the holes. For $\epsilon' = \frac{1}{2}p_0^2$ each of those terms tends to zero since from the first one it follows that $p'^2 < p_0^2$, $\frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 > p_0^2$, and from the second one that $p'^2 > p_0^2$, $\frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 < p_0^2$, both in violation of the equation $p_1^2 + p_2^2 = p'^2 + p_0^2$. According to the general theory of Green functions of many-body systems, the value for which the imaginary part of the energy eigenvalue tends to zero determines the chemical potential of the system μ

(Ref. 1). The equation

$$\text{Im}\,\Sigma\,(p,\,p_0^2/2) = 0$$

shows that the expression which we have found for $\text{Im}\,\Sigma$ makes it possible to determine the chemical potential only to a first approximation. We can make the result more accurate by including in ϵ' besides $2\pi m f_0$ also the second-order energy correction on the Fermi surface. This method of calculation is used by Beliaev.[6]

We shall now discuss the connection between the results obtained by us and the general theory of a Fermi liquid developed by Landau.[7]

Owing to the monotonic dependence of the quasi-particle energy ϵ_p on the momentum, the quasi-particles fill the Fermi sea up to a limiting momentum p_0. The occupation numbers of the quasi-particles are thus the same as the occupation numbers n_p of the non-interacting particles. This equality is also maintained for states near the ground state. It is easily seen that expressions (25), (26), and (26') for the extra energy of the particles are correct for any distribution of non-interacting particles. To see this it is sufficient to define the operators u and v^+ in Eq. (4) by

$$u = V^{-1/2} \sum_p (1 - n_p)\, a_p e^{ipr}, \quad v^+ = V^{-1/2} \sum_p n_p a_p e^{ipr}, \quad (4')$$

where n_p is the occupation number of the non-interacting particles. In the particular case of the ground state, Eq. (4') is the same as Eq. (4). The sum of expressions (25) and (26') at $\epsilon' = \epsilon_p^0$ gives the energy of the quasiparticles as a functional of the distribution function n_p of the quasiparticles. The variational derivative $\delta\epsilon_p/\delta n_{p'}$ determines the function $f_L(p, p')$ which was introduced by Landau (after a summation over the spins s and s'),

$$f_L(p, p') = 4\pi f_0 + 32\pi^2 f_0^2 \int dk\, n_k \left\{ P \frac{1}{k^2 + pp' - k(p + p')} \right.$$
$$\left. + P \frac{1}{(p' - k)(p' - p)} - P \frac{1}{(p - k)(p' - p)} \right\}. \quad (42)$$

Using Eq. (11) of Ref. 7 we obtain an expression for the effective mass of the particles at the Fermi surface which is the same as expression (36).

5. ESTIMATE OF THE GRAPHS OMITTED. HIGHER APPROXIMATIONS

When we estimated in Sec. 2 the importance of the omitted graphs, we considered an additional interaction with the background particles, corresponding to a single action of the potential $V(q)$ (one additional dotted line). In the case where perturbation theory can not be used this estimate is

incorrect. The Born approximation is the first term, which is large, of a series, the sum of which (the scattering amplitude) is small. To obtain a correct estimate it is necessary to sum the graphs corresponding to all orders of interaction of real gas particles, that is, all ladders of dotted lines connecting two full drawn lines going in the same direction. This sum is equal to the effective interaction potential Γ. To construct the graphs it is thus convenient to use the effective potential Γ (square), and not the potential U (dotted line).

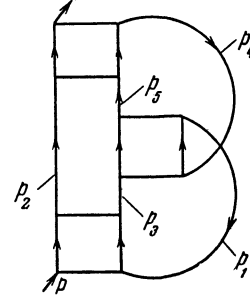Figure 6 illustrates a construction consisting of graphs omitted by us. To estimate the value of

FIG. 6

Σ_3 defined by those graphs it is sufficient to consider Γ in the first gas approximation, $\Gamma \approx f$. We get then

$$\Sigma_3 = i f^3 \int dp_1 dp_2 dp_4 G_0(p_1) G_0(p_2) G_0(p + p_1 - p_2)$$
$$\times G_0(p_4) G_0(p + p_4 - p_2) \quad (43)$$

It is important for us to show that the integral in (43) converges since in that case its value can only depend on the momentum p_0 (we assume that the momentum of the particles is in the neighborhood of p_0) and from dimensional considerations it follows that Σ_3 must be of the order $p_0^5 f^3$. After integrating over the fourth momentum component, there are only two energy denominators left in (43). If we then integrate over the momentum p_2 we get a final quantity which is of the order of a reciprocal momentum. It is essential that the two remaining integrals over p_1 and p_4 be taken over a domain that is bounded by the Fermi surface. Indeed, in our approximation Γ does not depend on the fourth momentum component, that is, the interaction takes place instantaneously. The lines corresponding to the momenta p_1 and $p_3 = p + p_1 - p_2$ thus form a closed loop and one of these lines must correspond to the propagation of a hole (compare the graph of Fig. 1). Since the momentum of a hole is less than the Fermi momentum p_0, one or other of the two conditions, $p_1 < p_0$, or $|p + p_1 - p_2| < p_0$, must be fulfilled. In each of these cases p_1

varies only in a bounded region. The same argument applied to the lines with the momenta p_4 and $p_5 = p + p_4 - p_2$ shows that the domain of integration of p_4 is bounded. We have thus proved the convergence of the integral, and Σ_3 is of the order of magnitude

$$\Sigma_3 \sim p_0^5 f^3 \sim p_0^2 n f^3. \qquad (44)$$

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APPENDIX

We shall derive certain relations to be satisfied by the scattering amplitude f. If we multiply Eq. (18) of the main text by $\psi_k^*(p')$ and integrate over k we find if we take into account that the ψ form a complete set of functions, that

$$V(p - p') = \int dk f(p, k) \psi_k^*(p'),$$

or, if we substitute (19) for $\psi_k^*(p')$,

$$V(p - p') = f(p, p') + \int dk \frac{f(p, k) f^*(p', k)}{k^2 - p'^2 - i\delta}. \qquad (I)$$

If we use the condition that the potential be Hermitian we can obtain from (I) the following formula,

$$f(p, p') - f^*(p', p)$$

$$= \int dk f(p, k) f^*(p', k) \left\{ \frac{1}{k^2 - p^2 + i\delta} - \frac{1}{k^2 - p'^2 - i\delta} \right\}. \qquad (II)$$

In the case where the two vectors p and p' have the same absolute magnitude, the principal values of the integral cancel each other and (II) becomes

$$f(p, p') - f^*(p', p) = -2\pi i \int dk f(p, k) f^*(p', k) \delta(p^2 - k^2). \qquad (III)$$

If we consider scattering in a central field of force, $f(p, p')$ can depend only on p^2 and $(p \cdot p')$, if $p = p'$, so that $f(p, p') = f(p', p)$ and we get from (III)

$$\operatorname{Im} f(p, p') = -\frac{1}{16\pi^2} p \int dn f(p, pn) f^*(p', pn) \qquad (IV)$$

where $|n| = 1$. Equation (IV) contains as a special case the well known relation between the imaginary part of the scattering amplitude at sero angle and the total cross section.

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